Third conference of the International Network for Didactic Research in University Mathematics

INDRUM 2020

September 12-19

Cyberspace-Virtually from Bizerte

TUNISIA

PROCEEDINGS

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INDRUM 2020 is an ERME topic conference

erme
european society for research in mathematics education

INDRUM

كلية العلوم ببنزرت
Faculté des Sciences de Bizerte

جامعة قرطاج
Université de Carthage

IMAG
INSTITUT MONTPELLIERAIN ALEXANDER GROTHENDIECK

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Tunisia

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INDRUM2020 was initially to be held March 27-29 in Bizerte, Tunisia, with the support of the Faculty of Sciences of Bizerte, University of Carthage. Due to the covid-19 worldwide event, it was postponed to September 12-19 and held in the form of an online conference, virtually from Bizerte, with the support avec the IMAG, University of Montpellier.

INDRUM2020 was an ERME Topic Conference.

ERME Topic Conferences (ETC) are conferences organised on a specific research theme or themes related to the work of ERME as presented in associated working groups at CERME conferences. Their aim is to extend the work of the group or groups in specific directions with clear value to the mathematics education research community.

http://www.mathematik.uni-dortmund.de/~erme/

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INDRUM2020 Editorial

Thomas Hausberger[1], Marianna Bosch[2], and Faïza Chellougui[3]

[1]IMAG, Université de Montpellier & CNRS, Montpellier, France, thomas.hausberger@umontpellier.fr; [2]IQS School of Management, Universitat Ramon Llull, Barcelona, Spain, marianna.bosch@iqs.edu; [3]Faculté des Sciences de Bizerte, Université de Carthage, Tunisie, chellouguifaiza@yahoo.fr.

INDRUM2020 was the third conference of the International Network for Didactic Research in University Mathematics. Initiated by an international team of researchers in didactics of mathematics in 2014, INDRUM aims at contributing to the development of research in didactics of mathematics at all levels of tertiary education, with a particular concern for the development of new researchers in the field and the dialogue with mathematicians. After two very successful conferences in 2016 (Montpellier, France) and 2018 (Kristiansand, Norway), the INDRUM Network Scientific Committee (INSC) decided in Kristiansand to pursue the cycle of biennial conferences with a third INDRUM conference to be held on March 27-29 2020 in Bizerte, Tunisia, in association with a side event in honour of Viviane Durand-Guerrier, the INDRUM Network coordinator, for her retirement. This decision followed the application of Tunisia to host the next INDRUM, through the voice of its INSC members, and also considered Viviane Durand-Guerrier's scientific involvement in the development of the Tunisian community of didactics of mathematics.

The INSC nominated in Kristiansand the INDRUM2020 International Programme Committee (IPC) and the Local Organising Committee (LOC), with an intersection to facilitate the coordination of both committees. The IPC was composed of Thomas Hausberger (Montpellier, France) Chair; Marianna Bosch (Barcelona, Spain) Co-chair; Faïza Chellougui (Bizerte, Tunisia); Viviane Durand-Guerrier (Montpellier, France); Imène Ghedamsi (Tunis, Tunisia); Simon Goodchild (Kristiansand, Norway); Reinhard Hochmuth (Hannover, Germany); Elena Nardi (Norwich, United Kingdom); Chris Rasmussen (San Diego, United States); María Trigueros (Mexico City, Mexico). The LOC was composed of Faïza Chellougui (Bizerte, Tunisia) Chair; Rahim Kouki (Tunis, Tunisia) Co-chair; Mahdi Abdeljaouad (Tunis, Tunisia); Sonia Ben Nejma (Bizerte, Tunisia); Béchir Dali (Bizerte, Tunisia); Viviane Durand-Guerrier (Montpellier, France); Imène Ghedamsi (Tunis, Tunisia); Inès Jendoubi (Tunis, Tunisia); Faten Khalloufi (Bizerte, Tunisia); Mahel Mosbah (Tunis, Tunisia).

The first announcement, published in February 2019, communicated the structure of the conference. Similarly, to the two previous INDRUM conferences, themes to be addressed at INDRUM2020 covered teacher and student practices and the teaching and learning of specific mathematical topics at undergraduate and post-graduate level as well as across disciplines. Accepted scientific contributions were to be discussed in four thematic working groups (4h each) after their presentation in two sessions of short communications in parallel (2h). The programme also comprised a poster exhibition and, as a new feature, a workshop for early-career researchers. Last but not least, we were delighted to announce that Carl Winsløw (University of Copenhagen, Denmark) had accepted to give the plenary lecture and that an expert panel discussion on tertiary education in the "digital age" was in preparation. Although the primary language of the conference was English, the linguistic characteristics of the host country were considered, similarly to previous INDRUM conferences. Therefore, authors were offered the possibility to write and present a paper in French or Arabic provided the presenter considered how to address the conference audience in its linguistic diversity through slides or a handout in English. Besides, INDRUM2020 was the third INDRUM conference to be accepted as a Topic Conference by the European Society for Research in Mathematics Education (ERME).

The second announcement was published at the end of April 2019, with further details for the submission. A list of 15 keywords was provided to authors as a means to classify their submission (using two keywords from the list and three optional other keywords) and also to help in the subsequent process of paper allocation to different working groups after the review process.

In response to the call, 47 papers and 4 posters were received. The review process was organised by the chair and co-chair, following principles that were discussed among the IPC. Each paper was thus reviewed by a member of the INSC and by an author of another submission; posters were reviewed by the chair and co-chair. Final decisions in situations where both reviewers had diverging opinions were taken after discussion among the IPC. At the end of the reviewing process, 44 papers and 3 posters were accepted for presentation at the conference. Authors of rejected papers that fell in the scope of the conference were offered the opportunity to resubmit their contribution as a poster. This last step increased the number of accepted posters to 5 in total.

Given the number of accepted contributions and the keywords provided by authors, it was deemed possible and relevant to organise four balanced thematic working groups (TWG). The allocation of papers and posters was proposed by the chair and co-chair, and approved by the IPC. The appointment of TWG co-leaders among INSC members was made by taking into consideration the representation of geographical diversity, gender balance, and the involvement of colleagues who have not yet or recently served as leaders. We were grateful that the appointed INSC members were able to accept our invitation. The third announcement was published in early March 2020 with the following list of groups and names of co-leaders:

TWG1 – Calculus and Analysis: Laura Branchetti (Italy) & María Trigueiros (Mexico)

TWG2 – Mathematics for engineers, Mathematical Modelling, Mathematics and other disciplines: Berta Barquero (Spain) & Nicolas Grenier-Boley (France)

TWG3 – Number Theory, Algebra, Discrete Mathematics, Logic: Viviane Durand-Guerrier (France) & Rolf Biehler (Germany)

TWG4 – Students' and teachers' practices: Irene Biza (United Kingdom) & Imène Ghedamsi (Tunisia)

The names of the panel chair and panellists, appointed by the IPC among conference participants in view of their expertise in the topic of the panel, were also communicated in the third announcement. Pedro Nicolas (Universidad de Murcia, Spain) accepted to chair the panel on tertiary education in the digital age, with Yael Fleischmann (Norwegian University of Science and Technology, Norway), Ghislaine Gueudet (Université de Brest, France) and Said Hadjerrouit (University of Agder, Norway) as speakers. Finally, Elena Nardi (University of East Anglia, United Kingdom) and Carl Winsløw (University of Copenhagen, Denmark) prepared a promising workshop about "starting to write journal articles" for INDRUM early career researchers around two papers published in the INDRUM2016 Special Issue of the International Journal for Research in Undergraduate Mathematics (IJRUME) for which they served as guest-editors. The purpose of the workshop was to share experiences and trigger discussion on what constituted the challenges – and ways to overcome these – of preparing a manuscript for submission to a mathematics education research journal, with a special focus on university mathematics education.

The third announcement included the conference timetable and the conference pre-proceedings. In parallel, the LOC was getting ready to welcome delegates in Bizerte, Tunisia. What happened afterwards was quite unprecedented and led to numerous meetings of the IPC and LOC to take what seemed the best solutions to preserve the spirit of INDRUM in the context of the covid-19 pandemic that much impacted scientific activity and human lives in general.

With the hope that sanitary conditions would allow the conference to be held in the near future, INDRUM2020 was thus first postponed to September 17-19, 2020. In view of the dynamic of the pandemic, it later became apparent that travel restrictions would make it impossible for numerous delegates to attend. A fourth announcement was then published in April 2020 to spread the news that INDRUM2020 will still be held, in the form of an online conference, in the middle of September (12-19). The decision to run INDRUM2020 online was not an easy one as so much effort was invested by the organising committee in Tunisia in preparing to welcome the delegates in Bizerte. We adjust the dates and timetable to take into consideration other academic duties and the delegates' time-zones. In particular, TWG sessions were reduced in comparison with the initial schedule, and the training session for early career researchers was removed. We could then fit the conference in eight days with sessions of no more than two hours per day. We also welcomed newcomers to the online conference, which was accessible upon free registration online.

INDRUM2020 thus took place as a conference in the cyberspace, virtually from Bizerte. 186 delegates from 36 countries (Table 1), going from time zone UTC-9 to UTC+9, registered to the online conference. Up to 120 participants attended the plenary sessions, and an average of 30 participants was present during the TWG discussion sessions. The opening and closing ceremonies were lively thanks to the work of the

Table 1. Participants and countries

COUNTRY	Participants
Andorra	1
Argentina	2
Belgium	5
Brazil	4
Cameroun	1
Canada	3
Chile	5
Colombia	1
Croatia	4
Denmark	1
Etiopia	1
Finland	4
France	40
Germany	12
Greece	1
Indonesia	1
Ireland	4
Israel	3
Italy	5
Japan	5
Lebanon	1
Malta	1
Marroco	1
Mexico	12
New Zeland	1
Norway	8
Peru	5
Portugal	2
Puerto Rico	1
Spain	16
Sweden	1
Switzerland	1
The Netherlands	1
Tunisia	19
UK	4
US	9
	186

LOC who gave delegates a flavour of Tunisia through a slideshow of beautiful landscapes and Tunisian music. Although scientific exchange online cannot reach the quality level of interactions in presence, feedback from delegates and from TWG co-leaders allowed us to conclude that the chosen format of the online conference, the richness of contents that were discussed and the reliability of the video-conference platform made INDRUM2020 a fruitful and enjoyable experience for most participants.

Papers appear in these Proceedings in a version chosen by participants following the optional possibility to upload a final version of their paper after the conference.

Very special thanks are due to the LOC, chaired by Faiza Chellougui, for their tireless work of many months to organise the conference and cope with the unexpected difficulties. We are also grateful for the support offered by Baptiste Chapuisat (IMAG, University of Montpellier) to solve technical aspects of the video-conference system, and to the whole “Tech-team” composed of colleagues from Barcelona and Montpellier who worked in the background to assist in case of technical difficulties. Finally, we are indebted to the University of Montpellier for offering us to use the video-conference license purchased on an institutional basis to provide online courses and webinars for academic activities in the covid-19 context. In these conditions and with the work of all the colleagues who worked unstintingly before, during and after the conference to ensure that participants had a smooth, productive and enjoyable experience, INDRUM2020 was again a success despite of the exceptional circumstances.

Follow-up

The INDRUM2020 closing ceremony was the occasion to report on the strengthening of the INDRUM Network with the enrolment of new colleagues to increase the geographical representation of the INSC and the creation of an INDRUM Network Coordination Group (INCG) to supervise the development of the network. It is now confirmed that ten colleagues from nine countries have joined the INSC to reach a total of 41 members. The INCG (an evolving group) is composed of 8 members: the three previous and the current IPC Chairs and co-Chairs. The composition of the INSC and INCG is published on the INDRUM website hosted by the open archive HAL where all INDRUM proceedings are posted: https://hal.archives-ouvertes.fr/INDRUM.

The forthcoming availability (on April 13, 2021) of the so-called (first) "INDRUM ERME book" was also announced. Edited by Viviane Durand-Guerrier, Reinhard Hochmuth, Elena Nardi and Carl Winsløw, this volume entitled "Research and Development in University Mathematics Education: Overview Produced by the International Network for Research in Didactics of University Mathematics" provides a state-of-the-art synthesis of University Mathematics Education research as exemplified by the works presented and discussed at INDRUM2016 and 2018. A big thanks and congratulation to Viviane Durand-Guerrier who managed the project and to the whole group of editors.

Following this publication and the previous INDRUM2016 Special Issue in IJRUME, we are now delighted to announce that authors of an accepted contribution (paper or poster) in the INDRUM2020 proceedings will be offered the opportunity to publish an expanded, updated or reworked version of their contribution to match the requirements of the following two journals in the context of production of two separate and complementary special issues:

(1) an IJMEST (International Journal of Mathematical Education in Science and Technology) Special Issue guest-edited by the INDRUM2020 chair and co-Chair. We are very grateful to the IJMEST editor-in-chief Colin Foster and associate editor Greg Oates for initiating such a fruitful and promising cooperation with INDRUM for the diffusion of university mathematics education research. This journal will therefore be able to gather both DELTA (series of biennial southern hemisphere symposia on the teaching and learning of undergraduate mathematics and statistics) and INDRUM research papers. We will invite papers of 15-20 pages, written in English, with an aim to publishing approximately ten papers among the best research represented in the INDRUM2020 Proceedings. While aiming at reflecting the thematic richness of the INDRUM2020 programme, we will not commit to a strict representation of the conference structure. We particularly welcome proposals that elaborate and expand the INDRUM2020 submissions' content substantially.

(2) an EpiDEMES (*Épijournal de Didactique et Epistémologie des Mathématiques pour l'Enseignement Supérieur*) Special Issue. This open-access peer-reviewed online journal, founded in 2019, welcomes articles written either in English or French to the

attention of practitioners. It aims at providing a database for the initial and in-service training of higher-education teachers. This second call will therefore complement the first call and highlight the "interface" character of the journal between the community of mathematics education researchers and the community of mathematicians interested in issues related to teaching mathematics in higher education. We warmly thank the editors-in-chief Nicolas Grenier-Boley and Hussein Sabra for engaging this promising collaboration with INDRUM for the dissemination of practice-based INDRUM research.

The deadlines for both Special Issues will be as follows: March 31, 2021: deadline to submit papers; June 15, 2021: decision letters sent to authors; September 1, 2021: deadline for revised manuscripts; October 15, 2021: final decisions. The target is to have both volumes produced in 2021. The official calls for contributions will be sent soon to the authors of INDRUM2020 accepted contributions through the INDRUM mailing list, separately for both projects. We would therefore advise authors who wish to prepare a proposal to select between both options by considering how their INDRUM2020 contribution may best reach the goals above to the best of its potential.

Finally, we are delighted to spread the news that INDRUM2022 will be held in Germany, near Hanover, on a date to be confirmed closer to the time between mid-September and mid-October. The final dates will be decided in June 2021. The local chair is Reinhard Hochmuth, with María Trigueros (Mexico) as chair of the IPC and Berta Barquero (Spain) as co-chair. The INDRUM2022 website https://indrum2022.sciencesconf.org/, which is currently under construction, will open with updated information as soon as possible.

We now invite you to carry on reading this volume, and we hope that the promise of its content will encourage you to consider joining or continuing to be part of the ambitious and stimulating INDRUM enterprise!

Plenary talk

Professional and academic bases of university mathematics teaching for the 21st century: the anthropological approach to practice-based research

Carl Winsløw[1]

[1]University of Copenhagen, winslow@ind.ku.dk

Based on an anthropological approach to the notions of profession and métier within an institution, we show how the label "mathematics" could relate to both notions in the setting of universities. We also propose a finer characterization of segments of the métier. Finally, we revisit some examples of how our own research have addressed different segments so identified. We finally return to the question of how different forms of scholarship interact with the métier and its segments, and formulate a proposal for professionalizing the métier in view of current and future needs.

Keywords: university mathematics teaching, ATD, practice-based research

1. INTRODUCTION

Mathematics as a university discipline has hugely expanded in the 20th century, both at the level of research and at the level of education. This development is inseparable from the interaction of mathematics with other – and equally evolving – disciplines, many of which have not only drawn on, but also contributed to the advances of mathematics both as a field of research and as a matter to be taught. It is neither possible nor helpful to try to draw sharp lines between mathematics and other disciplines. This is so not only for research but also in higher education contexts like Engineering and Natural Sciences.

The growth of mathematics from an institutional point of view is visible in the existence and expansion of mathematics departments in virtually any university type institution. One informal, frequently implicit, characterization of a "university mathematician" is certainly an individual who works at such a department. In mathematics departments, we find a great variety of scholarly activities, often with labels such as "pure mathematics", "applied mathematics", "statistics" and so on, each with further subdivisions and overlaps; and sometimes also "mathematics education", "history of mathematics", "data science" and more. Another informal characterization of a "university mathematician" could be university faculty member with an advanced degree labeled "mathematics", but in practice, this is quite similar as labels often result from the name of departments where they are obtained. In fact, people whose teaching or research are centered on mathematical contents may have other affiliations. We can think, for instance, of a specialist in mathematical education who teaches mathematics methods courses in an education department, or of a statistics researcher employed at a Medical school to teach statistics to future physicians.

From the point of view of university mathematics education, the notion of a *university mathematics teacher* is more relevant than the departmental categorization. Here, labels and descriptions of teaching units can be used to clarify more precisely what is being taught by a given university teacher. We shall pursue this delimitation in the next sections. We also return to the important co-existence and interaction of teaching and research in universities in the case of university mathematics teachers.

2. THE NOTIONS OF *PROFESSION* AND OF *MÉTIER* IN THE FRAMEWORK OF THE ATD

The English language offers many terms to designate a position within an institution that is characterized by a responsibility to carry out certain types of tasks. We shall define use and two of them, *profession* and *métier*, which both have their roots in Latin (via the French language). Our definitions may not correspond to dictionaries that explain common usage. However, they are, at least to some extent, consistent with distinctions made by Chevallard (2017) and Stigler and Hiebert (1999) to discuss the general status of teachers in Western and Asian societies, respectively.

By a *métier,* we mean simply mean the set of positions within an institution defined by a family of types of tasks, as explained above. Members of the métier are individuals holding one of these positions. Naturally, to stay as member of a métier, one will normally need minimal capacity to carry out the involved types of task; there may be no further requirements.

A *profession* is a special case of a métier, in which the praxis, and knowledge on the praxis, is made explicit and shared among its members. The resulting discourse is a logos Λ describing and justifying the use of certain methods or techniques, using a more or less specialized professional discourse. The combined praxis and logos $\mathscr{P} = (\Pi,\Lambda)$ is regulated and developed by the members of the profession (possibly by other agents too). Admission to the profession is strongly linked to sharing $\mathscr{P}$ at some level, obtained through formal training carried out by members of the profession (again, possibly by other agents as well).

In terms of the anthropological theory of the didactic (see e.g. Chevallard, 2019 for details), $\mathscr{P}$ is a collection of *praxeologies*, including both praxis blocks Π_i (types of tasks, techniques) and logos blocks Λ_j (discourse about techniques, and theory to support and justify the discourse). If p belongs to the métier defined by $\Pi = (\Pi_i)$, we can assume some minimal relation $R_I(p, \Pi)$ of p within the institution I at which the métier is exercised. Meanwhile for p to belong to a profession we have further requirements on $R_I(p, \mathscr{P})$, including the role of p in establishing $R_I(p', \mathscr{P})$ for the position p' of newcomer to the profession.

Examples of professions, which are well established in most developed societies, include: lawyers, doctors and engineers. Scientific researchers within specific fields also constitute professions – certainly, disciplines of modern science have extensive and explicit logos blocks, and scientists are trained by other scientists of the same discipline and within the same institution. The case of teachers in, say, primary school is less clear – the development of explicit knowledge about teaching, as well as the training that gives access to it, may only to a very small extent be carried out by primary school teachers.

Considering now the case of university mathematics teachers, many of them *do* belong to professions of research (in a broad range of specialties, as we have seen), although not necessarily, and not necessarily the same. However, one can hardly call the set of university mathematics teachers a profession, as there is little shared and explicit knowledge about how to carry out the types of didactical tasks that characterize this métier, and even less training and development of such shared knowledge that is carried out by the members of the métier. Of course, individual members of the métier have shared their views and principles about the métier (e.g. Halmos, 1985; Krantz, 2015), and in some countries, conferences and committees on university mathematics teaching are being organized with and by members of the métier (e.g. Burn, Appleby & Maher, 1998; Saxe & Braddy, 2025). Such works are usually written to push new agendas which are not widely shared in the profession, and may remain unknown to most members. On the other hand, the interest and willingness to share knowledge on teaching is growing among university mathematics teachers in many institutions and countries. In 1988, a mathematician noted that

> when a mathematician speaks about teaching, colleagues smile tolerantly to one another in the same way family members do when grandpa dribbles his soup down his shirt (Clemens, cited in Krantz, 2015, p. xi).

This is certainly not universally true anymore. Still, university mathematics teaching is hardly a profession in the sense defined above. In the rest of this paper, we try to characterize the métier, and then consider if and how it might become a profession.

3. THE UNIVERSITY MATHEMATICS TEACHER MÉTIER

Mathematical practice and knowledge can itself be modelled in terms of praxeologies ω, and within university institutions U, didactical tasks are all about establishing certain relationships $R_U(\sigma, \omega)$ for individual in a student position σ within U. If the university mathematics teacher position is called τ, a minimal requirement for $R_U(\tau,\omega)$ is certainly that $R_U(\tau,\omega) \supseteq R_U(\sigma,\omega)$. But more is required: σ must have practical knowledge about the didactical praxis of establishing $R_U(\sigma, \omega)$, and if this praxis is called $\Pi(\omega)$, there are thus additional requirements for $R_U(\tau,\Pi(\omega))$ which may be assessed by the extent the praxis of τ actually succeeds to establish $R_U(\sigma, \omega)$. Indeed, university institutions usually have rather explicit and established ways to assess the latter kind of relation, and the observed student performance is frequently also used to assess $R_U(\tau,\Pi(\omega))$. Newcomers to the métier will often have to develop $R_U(\tau,\Pi(\omega))$ more or less through building up personal experience with $\Pi(\omega)$ and possibly drawing on their own experience as students. However, as the student position σ may be different from positions they have themselves held – for instance, in the case of a background as graduate of pure mathematics, who is faced with the praxis of teaching applied mathematics to populations of some other discipline. In this case, both

$R_U(\tau,\Pi(\omega))$ and $R_U(\tau,\omega)$ have to be developed based on $R_U(\tau,\omega')$ where ω' consists of mathematical praxeologies somewhat similar to ω. While all of this appears at first sight a bit theoretical, it all was eminently concrete to someone who, like the author of this paper, started out in the métier with a background as researcher in pure mathematics, faced from day one with the task of teaching mathematics and mathematical biology to future biologists. While the mathematical elements were all very familiar, the full praxeologies to be taught, as well as relevant didactical techniques to do so, were largely to be acquired by the teacher.

One way to characterize the métier is thus in terms of the didactical tasks, closely related to the relationship $R_U(\sigma,\omega)$ to be established, and in particular in terms of the student positions σ and the praxeologies ω concerned. Even when teaching students in positions that the teacher has actually occupied, $R_U(\tau,\omega)$ is evidently of central importance to $R_U(\tau,\Pi(\omega))$. In the frequent absence of external support (specific training) to establish $R_U(\tau,\Pi(\omega))$, and therefore also of shared logos blocks $\Lambda(\omega)$, we see the clear traits of a métier in the establishment and function of the position τ.

Considering that the set of mathematical praxeologies ω_σ to be taught to students in position σ depend largely on σ, a first rough "topology" of the métier can thus be given in terms of the student populations: a teacher is in position τ_σ if she must engage in $\Pi(\omega_\sigma)$. In brief, the métier can be subdivided according to positions τ_σ for which $R_U(\tau_\sigma,\Pi(\omega_\sigma))$ must then satisfy some minimal requirements, more or less assessed trough $R_U(\sigma,\omega_\sigma)$. Still, formal training aiming to support entrance into the position τ_σ is often generic (see Winsløw, Biehler, Jaworski, Rønning & Wavro, to appear), corresponding to the assumption that $R_U(\tau_\sigma,\Pi(\omega_\sigma))$ is not only independent of σ, but also that $\Pi(\omega)$ is independent of ω. The techniques from $\Pi(\omega)$ assumed to be independent of ω are basically *pedagogical* and concern, for instance, how to prepare and conduct interactive lectures on a generic praxeology ω, relate to a generic student independently of her actual position σ, and so on. While any training on this generic practice Π will then also involve some form of logos block Λ, the relation $R_U(\tau_\sigma,(\Pi,\Lambda))$ may indeed fail to establish $R_U(\tau_\sigma,\Pi(\omega_\sigma))$, even when combined with the relationship $R_U(\tau_\sigma,\omega')$ that τ_σ may hold to praxeologies ω' that are somewhat similar to ω_σ, or even include ω_σ.

The access to the métier is, nevertheless, to a large extent based on developing $R_U(\tau_\sigma,\omega')$ through the *mathematical* training of τ_σ, which (in the case of researchers) may be assumed to largely guarantee that $R_U(\tau_\sigma,\omega_\sigma)$ can be established satisfactorily by any person in position τ_σ, irrespectively of the student position σ concerned. This could seem justified at least in case where τ_σ has previously developed $R_U(\sigma,\omega_\sigma)$ successfully. Even in this case, the establishment of $R_U(\tau_\sigma,\Pi(\omega_\sigma))$ remains, and the assumption that some $R_U(\tau_\sigma,(\Pi,\Lambda))$ will suffice to complement $R_U(\sigma,\omega_\sigma)$ is hardly without raising concerns – especially in the frequent case where existing didactical practices $\Pi(\omega_\sigma)$ appear, for students in position σ at large, in need of improvement in some sense. Again, taking measures of $R_U(\sigma,\omega_\sigma)$ as measures of the quality of $\Pi(\omega_\sigma)$ may at best help to realize such a need, while a logos block $\Lambda(\omega_\sigma)$ is needed to identify possible shortcomings of $\Pi(\omega_\sigma)$, to device innovations of $\Pi(\omega_\sigma)$, and to conduct systematic experiments of these innovations – and finally, to share the results in a way that conveys the position τ_σ with professional traits.

Considering again the variety of student populations that university mathematics teaching may address, and the idea that these somehow provide a structure on the métier as a whole, we will draw on an idea initially presented in the thesis of Kim (2015) and further developed by Chevallard (2019). They consider that a human population P at large can be subdivided as $P_1 \cup P_2 \cup P_3$. Here, P_1 consist of people who engage in production of new mathematical knowledge based on a postgraduate degree in some mathematically intense discipline (like physics, pure or applied mathematics, statistics etc.). Population P_2 consist of people who do not engage in production of new mathematics, but whose work is nevertheless crucially based on a postsecondary study of mathematical disciplines (like most secondary level mathematics teachers, but also most engineers, business specialists etc.). Finally, P_3 is "the rest" and certainly the largest portion of P. While there may be some grey zones left from these somewhat informal definitions, we can nevertheless identify corresponding positions σ_i ($i = 1,2,3$) of students at university, who are in some sense preparing for adult life in population P_i. These are not, in general disjoint, at least in early stages of university studies: especially σ_1 and σ_2 may be required to build the same relations to basic mathematical praxeologies from, say, linear algebra, and thus be taught together, depending on the institution U; and even future members of P_3 may at some point face such requirements. At later stages, specializations may be more common. Nevertheless, university mathematics teachers may be roughly assumed to occupy positions τ_i ($i = 1,2,3$) corresponding to the students they face, with both overlaps (as mentioned) and with further specializations (e.g. $\tau_{2,t}$ when the didactical tasks is specialized for the case of future secondary teachers t, $\tau_{1,\mu}$ when the public are future researchers μ in a field of pure mathematics, etc.).

We note here that it is common for individuals at U to occupy certain positions, such as $\tau_{1,\mu}$ and μ, simultaneously. Indeed, the simultaneous occupation of positions as scholar or researcher, and as teacher, is both currently and traditionally considered a hallmark of university institutions – associated with the more radical Humboldtian ideal of *Einheit von Lehre und Forschung* (unity of teaching and scientific research, cf. Madsen & Winsløw, 2009). Before delving further into research based on the subdivisions of university teaching métier that were introduced above, we shall dwell on research into how such double occupancy of teaching and research positions may influence and shape the métier, and in particular how the first position may lend and draw professional traits from the latter.

4. RESEARCH ON THE TEACHING-RESEARCH NEXUS

The position μ as researcher of mathematical sciences (including, for instance, statistics and various fields of applied mathematics), clearly *is* a profession which, from the19th century, has become firmly established at university institutions all over the world. Mathematical knowledge and expertise was of course of great societal important even before, but was often developed with and under the auspices of other disciplines such as astronomy, geodesy, mechanics and (from the 18th century) engineering. However, far into the 20th century most university mathematicians were primarily teachers. According to Tucker (2013, p. 699), who focuses on American universities:

> While in the early 1950's most faculty at doctoral institutions still saw undergraduate teaching as their primary mission, by 1970 that mission had changed with research becoming the primary focus of these faculty.

This reflects a more general development in many (not only American) university institutions over the 20th century (Cuban, 1999): the increasing priority of research over teaching both in the tasks characterizing university faculty positions, and in the selection of individuals to fill those positions. In particular, occupying position τ can be a mere corollary of occupying position μ.

The position μ is clearly what Halmos (1985, p. 400) talks about in the following quote:

> I spent most of a lifetime trying to be a mathematician – and what did I learn? What does it take to be one? I think I know the answer: you have to be born right, you must continually strive to become perfect, you must love mathematics more than anything else, you must work at it hard and without stop, and you must never give up.

The role of commitment and personal ability is evident also in similar descriptions of the research profession by other mathematicians. At first sight, it would look like being a mathematician is merely a personal affair. At the same time, the professional character of the research métier is evident for mathematics for the same reason as in other research areas (system of training and regulating access to position μ are both internal to the métier, as is the system for developing new explicit knowledge relative to the métier).

Halmos, who was a famous textbook author and an eminent lecturer, feels guilty after a day where he taught well but did not do research (ibid., p. 322):

> Despite my great emotional involvement in work, I just hate to start doing it; it's a battle and a wrench every time. Isn't there something I can (must?) do first? Shouldn't I sharpen my pencils, perhaps? (…) Yes, yes. I may not have proved any new theorems today, but at least I explained the law of sines pretty well, and I have earned my keep.

Here, "work" is evidently research, and "sharpening pencils" becomes a metaphor for distractions from μ-tasks – including teaching ("the law of sines"). Certainly, filling τ well does not make up for failures to accomplish the tasks of μ.

Despite this competition for time, it is also a classical idea that teaching and research can somehow resource and inspire each other – in the higher education literature, one speaks of a teaching-research nexus (Neumann, 1992), reflecting that the link between the two is by no means simple or one-way. Of course, we often think about "research based teaching" as teaching that somehow draws on research, but to the extent knowledge is produced by research, this is somewhat trivial. It becomes less trivial if we think of the individual occupant of μ and τ - so that somehow the concrete research activity of the individual influences the same person's teaching. In between, one could conceive of how the teaching of a larger or smaller institution (department or university) is affected by its research activity. Finally, there might be influences from teaching on research at both individual and institutional level, such as including students in research activities.

For the case of mathematics, not many studies exist of the teaching-research nexus. Winsløw and Madsen (2009) modeled it in terms of praxeologies of research and teaching, $\mathscr{P}_R$ and $\mathscr{P}_T$ for short. Each of these centrally include mathematical praxeologies (used or developed during research and teaching) but not be limited to them – for instance, research praxeologies could be broadly conceived to include practices of publishing, funding, communication, etc. In the study, five researchers in pure mathematics were interviewed on teaching practice, their research practice and connections between them (in that order). Not surprisingly, the links between $\mathscr{P}_R$ and $\mathscr{P}_T$ turn out to be rather "indirect", in the sense that the mathematical praxeologies involved in the two are normally quite distant. As one of the informants says (ibid., p. 756):

> How long did it take me before I had an impression of what is going on in the research area that interests me? Well it took 5 years, after I had graduated.... You can't tell a bachelor student what it is about can you?

In other words, the mathematical praxis and logos in undergraduate teaching is very far from that involved in the research of the informant.

At the same time, four out of five informants insist with considerable energy and many examples that considerable indirect links exist at the level of types of task. For instance, this involves the didactical task of constructing challenging assignments for students, and also the similarity between the activities aimed at for students (when working on the assignments) and the research task. While solving teaching tasks, one may discover points that later, maybe by association or further development, can be used in research. Many other links are mentioned to illustrate the experience expressed by one informant: "I feel that I can get things forth and back between the two parts." We can say that the teaching-research nexus is largely implicit and indirect, and it concerns mainly the level of practices which are not the same but somehow similar, while it does reach the level of logos.

The fact that the experience of students in university mathematics is not always very close to mathematical research was emphasized as a motivation for Burton's (2004, p. 27) interview study with 70 researchers of mathematics:

> It was my hope that a gap, between how mathematicians themselves came to know and how they promoted learning in others, if confirmed in the study, would help to explain student disaffection with the discipline, their difficulties in learning it, and the gender bias in those who take it up as a career.

The study itself considers mainly the creative part of research practices, the "enquiry", through which the individual researchers are supposedly learning (new) mathematics. This activity is found to be quite different from what university students are asked to do, for instance because it involves intuition, room for different "thinking styles" and so on. The study does not investigate the experience of university students but relies, for this part, on the author's personal experience. A main value of this study is the detailed analysis of a large and exciting survey of how $\mathscr{P}_R$ is described by people in position μ. Direct observation of these praxeologies is obviously difficult to arrange, which explains the use of interview methods in both of the above-mentioned studies.

Similar methods were used in Misfeldt's (2006) study of the writing practices of researchers in pure and applied mathematics, also in view of informing didactical practices at university to enhance students' learning in this area. However, in this study, interviews were carried out in the presence of material traces of current research of the informants, in the form of pieces of writing. In particular, three distinct modes of writing were found: exploratory writing (often involving more diagrams and the like, than linear text), "first drafts" (attempts to write out proofs etc. in a linear fashion, often by hand) and "article text" (produced with TeX or similar software).

From all three studies, we can see that possible links between students' and researchers' praxeologies are quite indirect, and maybe even missing (Burton's "gap"). The motivation of all three studies is that when university teachers are also researchers one could hope that those teachers could somehow produce a didactic practice that could make the link closer. In fact, much of recent didactical engineering research on university mathematics is more or less explicitly based on this idea: draw on teacher-researchers' experience with research to create a didactical practice that somehow enable students to do mathematics in ways that are similar to mathematical research. It could also be that students are to engage in mathematical activities close to what is found in another professional field or discipline that they will subsequently encounter. This brings us back to the different teacher positions ($\tau_{1,\mu}$, etc.) in the university mathematics teaching métier, and research on controlled experiments with corresponding didactical practices.

5. THE TEACHING MÉTIER BY STUDENT POPULATIONS

We now return to take a closer look at experimental research involving cases of τ_i (i = 1,2) and the didactical praxeologies undertaken by teachers in these positions. We note in passing that in some university type institutions, such as liberal arts colleges in the

US, there is a strong tradition for teaching mathematics to "all" students, including those whose study and professional aims do not strictly need postsecondary mathematics. Such teaching often focuses on historic and otherwise humanistic aspects of the discipline (Fried, 2018). At other universities (including, to my knowledge, most European universities) many students have no mathematics courses, and positions σ_i (and hence τ_i) may not exist at all.

5.1 Research-like situations in undergraduate analysis

A recent survey of undergraduate pure mathematics programmes (Bosch et al., to appear) confirms that their structure offers many similarities, and all programmes analyzed include a calculus-analysis sequence of modules as a central component. The calculus vs. analysis distinction is in general somewhat blurred and contingent upon local conditions. However, a quite common interpretation can be made precise in terms of praxeologies (Winsløw, 2008; Gyöngyösi, Solovej & Winsløw, 2011; Winsløw, 2015; Kondratieva & Winsløw, 2018). Calculus praxeologies ω involve types of tasks concerning functions that are given in closed form, including functions of one or more real variables, real and complex valued sequences, and so on; differential and integral calculus are among the central sectors. In calculus courses, the student relationship $R_U(\sigma,\omega)$ aimed at (and especially, assessed) is often focused on students' mastery of *techniques* corresponding to a well-defined set of type of tasks. Thus, students will learn techniques to evaluate certain integrals, find extremum points of functions in certain situations, etc.; the logos block is on the other hand more informal, compared to analysis courses. These, on the other hand, focus on *theory*, and first analysis courses may merely complement calculus courses with formal theory, and thus in a sense extend $R_U(\sigma,\omega)$; we call this type of extension a *type I transition.* In more advanced Analysis courses, students face type tasks that are formulated in terms of previous logos blocks (e.g. investigate whether this or that function is a complete metric on R^n), thus working with entirely new and generally more "abstract" praxeologies. We call this kind of passage to new praxeologies, in which tasks concern logos from previously developed praxeologies, a *type II transition.*

To support type I transitions, it is evidently crucial to create *links* between familiar praxis from Calculus, and the new theoretical superstructure. One strategy to do so, pursued by Gyöngyösi et al. (2011), is to design student assignments which involve new theoretical material which are explored based on computer supported experiments with objects from Calculus. An example was to "explore the convergence properties of the sequence of functions given by $f_n(x) = 1/[1+\exp(n(2–x))]$". Based on plots in software like Maple*,* the students quickly see the pointwise convergence of (f_n) to a non-continuous function, and infer that the convergence is not uniform. They can also, based on Maple calculations, verify that the limit function has the same integral over any interval as the limit of the integrals of f_n, and hence that uniform convergence is only a sufficient but not a necessary condition for the interchanging limit and integral. Experiments with such designs in a first-year analysis course showed that a "middle group" students effectively improved their results when such assignments were included, while it made no difference for high- and low-performing students.

The transition of type II could be attacked with similar ideas for task design, but naturally going further than the mere illustration and application of new theory. Grønbæk and Winsløw (2007) experimented so-called *thematic projects* which are relatively long assignments that proceed from more closed tasks (of the type found at the end of chapters in text books) towards open questions that require students to device and prove a theoretical result, which is naturally supported by the first parts. Instead of an oral exam based on students' presentation of material directly extracted from a textbook, the students should now present one of their thematic projects. Experimentations over several years with thematic projects in a real analysis course demonstrated a significant increase in students' work, satisfaction and results, as measured by the standard exam; however, the increased work for the course teachers (both to create new assignments and to provide adequate supervision to students) made the format less viable outside of a funded project.

Very similar experiments were made later in the context of a less advanced real analysis course (Gravesen, Grønbæk and Winsløw, 2017). In a project funded by a University of Copenhagen grant to further the connections between research and undergraduate teaching, we defined a number of *research-like* activities, and constructed a collection of exercises that would engage students in some of these (for each exercise). Of course, mathematical research is not limited to "prove that" activities, while these dominate end-of-chapter exercises in many post-calculus textbooks. Among the activities explicitly focused on in this design, students were to use special cases to investigate an abstract hypothesis or question, to formulate a hypothesis for a given question, to formalize relations between two or more results, to produce or validate ε–δ type definitions, etc.

Another idea for more advanced courses, developed by Kondratieva et al. (2018) but yet to be tested at larger scale, is to link calculus praxis with proofs of major theoretical results in analysis. As an example, a student assignment was developed in which the so-called Basel problem (convergence and value of $\Sigma(-1)^n/n^2$) is solved by calculus techniques, and then the same sequence of techniques is used to give an elementary proof of Dirichlet's theorem on Fourier series. The construction of assignments that relate different domains in mathematics, or (as here) basic and more advanced courses in the same domain, is proposed as a strategy for task design research linked to Klein's idea of "Plan B" (cf. Klein, 2016, p. 83).

Perhaps we can formulate two overall conclusions emerging from these and many other experiments with task design that aims to create "research like" situations for undergraduate students:

1 The revised didactic practice $\Pi(\omega)$ can certainly be realized and as a result, more ambitious aims for $R_U(\sigma,\omega)$ are in fact realized;

2 It is much less straightforward to establish $R_U(\tau,\Pi(\omega))$ for the position τ as such (rather than for an individual teacher in privileged circumstances), like when $\Pi(\omega)$ requires time-consuming design (e.g. of new assignments).

We close this section by briefly examining a more famous and generic parallel to (1), which offers in some sense also a counterexample to (2): the so-called *Moore method* (described, for instance, by Halmos, 1985, pp. 255 ff.) to teaching theoretical mathematics. Moore was a professor of mathematics at the University of Texas from 1920 to 1969, and a legendary teacher and doctoral supervisor. Over the course of the past century, his methods of teaching expanded and developed several variations in several North American universities. According to the "Moore method" article on Wikipedia (as it looked in December 2020), dozens of professors and departments use some version of it even today.

The core of Moore's method is to let students (re)construct proofs of given theorems, with no use of books or other sources, but referring only to a handout with Definitions and Theorems carefully prepared by the teacher. The method apparently works for any specific mathematical praxeology or domain, except for the clear focus on formal proof (which is anyway common in almost any post-calculus course in pure mathematics). In that sense the method is a set of pedagogical techniques to teach proof, while the didactical practice $\Pi(\omega)$ comes with the concrete handout for a given set of praxeologies ω. It would be very interesting to investigate the institutional and historic conditions that enabled the success of this approach. It is a rather certain hypothesis that one important condition has been the existence of a well-developed "logos" on the didactical techniques, disseminated in several books written for and by members of the métier. The method has not only been transmitted but also further developed by some of these members (see for instance Chalice, 1995). It appears from some of these writings that not only descriptions of the pedagogical techniques, but also examples of handouts for concrete praxeologies ω, have been disseminated widely. It remains that the method also shares the challenge of design by the teacher, to the extent $\Pi(\omega)$ has to be constructed for a given unit of teaching, in view of concrete student populations and specific praxeologies ω.

According to the literature referred to, the Moore method is found to offer an excellent experience for students in position σ_1, as considered in this section. However, at the undergraduate level, such students usually mix with students in position σ_2, for whom training to prove theorems may not be as important. Apparently, the method works best with advanced courses and hardworking students, who are more or less clearly in position σ_1. It is still remarkable as a case of sustained, explicit development of shared didactical practice by the métier itself, which moreover connects clearly to an important aspect of mathematical research, the construction of formal proofs. It is a fair hypothesis that this implicit or even explicit link between $\mathscr{P}_R$ and $\mathscr{P}_T$ contributed to the success of Moore's way to organize $\mathscr{P}_T$.

We now turn to studies of the métier of teaching mathematics to students in position σ_2, focusing on two variants of this type that are both more common than σ_1 in most university institutions.

5.2 Klein's second discontinuity: the case of teacher students

We first consider the case of $\sigma_{2,t}$ or, for short, σ_t : university students who prepare to become mathematics teachers outside of the university, generally at primary or secondary level. In some universities, the position σ_t is found in specialized programs apart from σ_1. In other universities – including many universities in Europe, particularly when it comes to future secondary mathematics teachers – the two positions are indiscriminate at least in the first years of study. In other words, the same requirements are made for $R_U(\sigma_t,\omega)$ and $R_U(\sigma_1,\omega)$, for a good deal of the mathematical praxeologies ω that are taught at U. In this situation, which appears common in many universities even today, Klein (2016, p. 1) identified a major problem as early as 1908:

> The young university student finds himself, at the outset, confronted with problems, which do not remember, in any particular, the things with which he had been concerned at school. Naturally he forgets all these things quickly and thoroughly. When, after finishing his course of study, he becomes a teacher, he suddenly finds himself expected to teach the traditional elementary mathematics according to school practice; and, since he will be scarcely able, unaided, to discern any connection between this task and his university mathematics, he will soon fell in with the time honoured way of teaching, and his university studies remain only a more or less pleasant memory which has no influence upon his teaching.

Klein identifies these two phases of "forgetting previous mathematics" as the first and second discontinuity. The first is a general problem of transition *from school to university* that has been the subject of much (if not most!) research on university mathematics education, given the struggles in which many students σ find themselves in (irrespectively of future orientation). The second discontinuity, *from university to school*, is specific to σ_t, or rather to the passage

$$R_U(\sigma_t,\omega) \to R_S(t,o)$$

where S is naturally the school institution and t the position as teacher and where o designates school praxeologies. The change of praxeologies correspond to the fact that what Klein calls "traditional elementary mathematics according to school practice" is at best somehow related to mathematical praxeologies ω met at U, "his university mathematics". In fact, Klein also emphasizes that the position t requires not only a relation to o but also the "task" to "teach" it "according to school practice"; a more accurate representation of this passage is thus

$$R_U(\sigma_t,\omega) \to R_S(t, \Pi(o)).$$

If we represent the full story of Klein's unfortunate character, who starts out in the position as school student s, we then get

$$R_S(s,o) \rightarrow R_U(\sigma_t,\omega) \rightarrow R_S(t, \Pi(o))$$

under the assumption that o, "elementary mathematics according to school practice" does not change too much while our friend is at university (which may, in fact, be somewhat incorrect in times of curriculum change).

The point of Klein's book is that universities need to take more responsibility when it comes to enrich $R_U(\sigma_t,\omega)$ with explicit links between ω and o. In fact, most of the text consists of revisiting elements of o – especially within the domains of arithmetic, analysis and geometry – from the "higher standpoint" (as the title says) of ω. The text, indeed, resulted from Klein's own lectures to future teachers during the preceding decades, following his inauguration as professor at the University of Erlangen in 1872. We could represent this effort as an attempt to "smoothen" the second discontinuity by adding a relation to be developed (cf. Barquero and Winsløw, in preparation):

$$R_S(s,o) \rightarrow R_U(\sigma_t,\omega) \rightarrow R_U(\sigma_t, \omega\cup o) \rightarrow R_S(t, \Pi(o))$$

where the subject matter of the "Klein course" is naturally not supposed to be a disjoint union $\omega\cup o$, but to emphasize links and overlaps.

The emergence of Didactics of Mathematics (or mathematics education research, in Anglophone countries) as a scientific discipline, both results and departs from this project, particularly from the sixties onwards. On the one hand, Klein type courses were established at many universities (in Germany, often specialized in domains, labeled Didactics of Analysis and so on; in USA as so-called "capstone courses" which are also offered at the end of several other professional university degrees). Still, the last passage $R_U(\sigma_t, \omega\cup o) \rightarrow R_S(t, \Pi(o))$, may remain somewhat discontinuous, given that $\Pi(o)$ is more than o. In many countries, official systems of "induction" into the teaching métier are offered (see e.g. Britton, Paine & Pimm, 2003) to take care of the passage to the praxis $\Pi(o)$, with more or (often) less attention to the specificity of o. This, in fact, means, that yet another relationship is added to smoothen the second discontinuity, between $R_U(\sigma_t, \omega\cup o)$ and $R_S(t, \Pi(o))$. This may involve both university course units, given by specialists of Didactics of Mathematics or Pedagogy, who may introduce more or less subject specific elements of logos $\Lambda(o)$ related to elements of practice $\Pi(o)$ in school. One could then pose the complete model of mathematics teacher education that exists today, with local variations (such as leaving out entire relations aimed at):

$$R_S(t,o) \rightarrow R_U(\sigma_t,\omega) \rightarrow R_U(\sigma_t, \omega\cup o) \rightarrow R_{U/S}(\sigma_t/t, (\Lambda(o),\Pi(o))) \rightarrow R_S(t, \Pi(o)).$$

More can be said about this last extension, and especially of the frequent absence of logos in the last relationship (see Miyakawa and Winsløw, 2019). However, from the point of university mathematics education, which is assumed in this paper, the second passage $R_U(\sigma_t,\omega) \rightarrow R_U(\sigma_t, \omega\cup o)$ is of special interest, as it concerns university teaching of mathematics. It is still important to bear in mind that this passage is very often followed by training more directly related to $\Pi(o)$.

In this vein, let us first recall the considerable body of research which, beginning with Begle's (1972) first demonstrations that $R_U(\sigma_t,\omega) \rightarrow R_S(t, \Pi(o))$ does not succeed better (in terms of performance of the students of t) simply because ω (measured as numbers of advanced courses taken) was larger. Later studies refined his results and nuanced the view both from mere volume to a closer look at contents. Without going into details that are better explained elsewhere, the following recommendation seems still to be of current, consensual value, at least in the United States:

> Prospective high school teachers of mathematics should be required to complete the equivalent of an undergraduate major in mathematics that include three courses with a primary focus on high school mathematics from an advanced viewpoint (CBMS, 2012, p. 18).

We are thus faced essentially with the proposal of Klein, when it comes to the university responsibility to prepare $R_S(t, \Pi(o))$, in the case of upper secondary school S: establish $R_U(\sigma_t, \omega\cup o)$ with a "primary focus" on o, but linking it to the "advanced standpoint" of ω. It is an important challenge for the university mathematics métier to identify what $R_U(\sigma_t, \omega\cup o)$ could best function as stepping stone towards $R_S(t, \Pi(o))$, and to implement didactical practices that can establish such $R_U(\sigma_t, \omega\cup o)$. The complexity of this task is evident, and probably more acute that in the time of Klein, where very small minorities reached the position s.

To solve this task evidently requires a teacher relation $R_U(\tau, \Pi(\omega\cup o))$ which is not immediately derived from $R_U(\tau,\omega)$, although it *also* involves this relation. But in addition to that, to design $\Pi(\omega\cup o)$, requires a relation $R_U(\tau, R_S(t, \Pi(o)))$, where the complexity is even more evident. Such expertise, on the other hand, is in principle held by the faction of the university mathematics métier who engage in *empirical research* not only on mathematics teaching in secondary school, but also on *secondary mathematics teacher knowledge.* This field, of course, is currently under development, and is only slowly getting specialized enough to capture specific praxeologies o. At any rate, we can summarize this theoretical discussion by agreeing that devising and adjusting courses (three, perhaps?) is an excellent opportunity to combine and mix expertise from both teaching and research in mathematics including but also beyond the classical domains of mathematics. We return to this in Section 6.

As an example of evidence from newer "Klein type" courses, Winsløw and Grønbæk (2014) conducted an analysis of student challenges in such a course at the University of Copenhagen, based on praxeological analysis along the lines outlined above. One of the striking observations was that even $R_U(\sigma_t,\omega)$ may have to be developed in such a course. Within the same context, Barquero et al. (in preparation) will delve further into specific challenges when it comes to students' perception on and challenges with praxeologies $\omega\cup o$ related to the real number system.

The education of future school mathematics teachers may be of special interest to scholars in university mathematics education, as they are often also teacher educators.

However, when it comes to the métier of university mathematics teaching – our object of research – there are other target métiers, if not professions, which are equally important, if not more so. We now turn to a major example: future engineers.

5.3 Authentic Problems of Engineering in first year mathematics

As for teacher education, the role of mathematics in engineering education has been the subject of numerous policy papers. Naturally engineering programs include different specialties and academic levels, whose mathematical needs vary significantly. An engineering student σ_e encounters mathematical praxis and logos in many different settings of the university study, but "mathematics courses" (given by members of the university mathematics métier) appears mostly in the first year or two of undergraduate studies. Whatever mathematical praxeologies ω that σ_e studies then, the aim for $R_U(\sigma_e,\omega)$ is to prepare and facilitate the establishment of relationships of type $R_U(\sigma_e,\varepsilon)$ where ε is some praxeology from engineering courses at large, in which mathematical practices or logos related to those of ω appear.

A main problem for university mathematics education in this context is that the transition (or knowledge transfer) from $R_U(\sigma_e,\omega)$ to $R_U(\sigma_e,\varepsilon)$ is not automatic, even when ω and ε are actually bridged by the expert (or teacher) of both. For instance, in the context of a signal theory course, Hochmuth, Biehler and Schreiber (2015) investigated specific ruptures between the mathematical model of "Dirac impulse" treated (and calculated with) in this course, and the technology associated to functions, limits, and distributions in mathematics courses. The techniques required to solve associated problems in the signal theory course, which involve operating with functions that assume the value ∞ at isolated points, "do not fit with higher mathematics discourses (technologies)". The students have somehow to learn that they should neglect specific aspects from those discourses" (ibid., p. 696).

Another, related problem, concern the specialized métier τ_e of teaching mathematics to students in position σ_e. The classical solution is that the mathematical praxeologies ω to be taught are simply some subset of what is taught to σ_1 (including extensive work on formal logos with proofs etc.). While this model still exists in some countries, it seems to disappear in many places due to problems with students' motivation, attrition and transfer (Pohjolainen et al., 2018). Hernandes-Gomes & González-Martín (2016, 2020) found that teachers' relationship $R_U(\tau_e,\omega)$ with basic calculus topics depended significantly on their scholarly background (which included pure mathematics, mathematics education, mechanical engineering and electrical engineering), and on the teachers' corresponding experiences as undergraduate students. For instance, only the university mathematics teachers with an engineering background had precise ideas about how specific mathematical techniques appear (or do not appear) in the engineering program. In the context of supervising capstone projects, professional experience from engineering institutions outside of the university is also of considerable importance, even when it comes to the ways in which teachers assist students with mathematical techniques. These case studies mainly suggest that different institutional backgrounds offer somewhat different qualities to the position of τ_e.

To improve students' motivation to develop, apart from exam requirements, and also to prepare the transfer of type $R_U(\sigma_e,\omega) \rightarrow R_U(\sigma_e,\varepsilon)$, it is an interesting strategy to integrate some concrete tasks from ε in the mathematics course, which can be used to show the relevance to engineering of techniques and logos from ω. In his doctoral thesis, Wolf (2017) carried out an ambitious project on designing and experimenting *application-oriented exercises* in a first-year mathematics course for students of mechanical engineering. The applications were "authentic" in the sense that problems and data were taken from professional contexts of machine construction. The authenticity was ensured by collaboration with university teachers of engineering.

Schmidt and Winsløw (to appear) investigate a similar, but more longitudinal and entirely spontaneous collaboration pattern, focused on designing *authentic problems of engineering assignments* for a first-year mathematics course with more than 1100 students every year. They describe the explicit principles that have developed, through practice but also from leadership in the position τ_e to facilitate collaboration with scholars of engineering fields, who often produce a first draft of the assignment, which is subsequently revised and implemented by (mostly) mathematics faculty in position τ_e. The assignments appear in a first-year course on calculus, linear algebra and differential equations, and the main challenges for students are thus to be of a mathematical nature, combining several praxeologies taught in the course. Still, the mathematical model is built up from an authentic problem of engineering. Here, "authentic" in that it comes from recent publication in scholarly engineering. To organize the systematic collaboration with institutions of scholarly engineering, informal didactical logos was developed from the position τ_e, adding a professional trait to this specific form of the university mathematics métier.

6. A PROPOSAL FOR THE FUTURE

In the preceding two sections, we have analyzed instances of the variety of positions and knowledge bases that the university mathematics métier is currently based on. Both scholarly knowledge, coming from engaging in research praxeologies, and knowledge built from didactical praxis, contribute to this knowledge basis. In some cases, didactical practice is supplied with more or less strongly developed logos, which often takes on relatively generic forms. Experiments initiated by scholars specializing in university mathematics education research appear to be mostly punctual, while we have also identified instances of more sustained and explicitly framed efforts to develop the métier from positions τ_1 and τ_2. As is the case for mathematics teaching in other institutions, the development of professional – shared, explicit and practice-specific knowledge – remains quite limited and local. Professional journals focusing on university mathematics teaching do exist in some countries like the USA, but even then, there seems to be a considerable distance, in terms of logos and readership, between these and scholarly publications in the field of university mathematics education research (such as the present volume).

Indeed, the university mathematics teaching métier remains, to a large extent, a secondary occupation of various professions of scholarly research in the mathematical sciences, interesting new forms of collaborations emerge especially in the position of type τ_2. Still the formal preparation for occupying such positions seems to be mainly pedagogical, as a complement to the more substantial training for a scholarly profession. Reactions to external reform requirements often take the form of more or less minimal reconstruction of external didactical transpositions at least when it comes to undergraduate programs in pure mathematics (Bosch at al., to appear). In the parts of the métier catering to students in positions σ_2, more significant developments appear, while in all cases, massive challenges with attrition and failure remain evident and perhaps even growing in many universities.

As a result, the *impact problem* for research in university mathematics education is pointed out in several recent syntheses (e.g. Winsløw, Gueudet, Hochmuth and Nardi, 2018, p. 71). The current institutional model separates, largely, such research from the university mathematics teaching métier. To seek impact of an external scholarly field on a métier of teaching implies two risks that are very well known from the teaching métiers at primary and secondary levels in many countries. The first is to continue to fail. The second is to succeed, at least to some extent, but to have merely "robbed teachers of the opportunity to participate in the development of new knowledge about teaching" (Stigler et al., 1999, p. 174).

To avoid these risks, a new nexus between teaching and research seems necessary. Of course, various blends of scholars and teachers have appeared spontaneously under current institutional conditions, as has now and then also appeared in some of the efforts outlined here. However, the vast majority of the professional training of PhDs in mathematical sciences remains totally disjoint from the preparation of PhDs in Didactics of Mathematics, including those focusing on university mathematics education. Why would the latter not include some level of further mathematical education and experience with research in some mathematical domain? And why not include elements of education and research in the didactical domain? Should future teachers of university mathematics not be prepared to engage in (rather than, hypothetically, be mute consumers of) didactical research on university mathematics? What are the institutional and intellectual conditions under which it would be realistic to establish mixed doctoral programs (of various compositions) that could prepare for shared and fruitful professional development of the métier in all its different forms?

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Panel

Tertiary education in the digital age

Yael Fleischmann[1], Ghislaine Gueudet[2], Said Hadjerrouit[3] and Pedro Nicolás[4]

[1]Teknisk-Naturvitenskaplige Universitet, Norway, yael.fleischmann@ntnu.no;
[2]Université de Bretagne Occidentale, France, ghislaine.gueudet.1@univ-rennes1.fr;
[3]Universitetet i Agder, Norway, said.hadjerrouit@uia.no;
[4]Universidad de Murcia, Spain, pedronz@um.es

We address the relationship between digital technologies and tertiary education. For that purpose, we first consider some conceptualisations of the idea of "digital resource" from different points of view, and the evolution of the presence of digital technologies in our lives. Then, we wonder whether digital resources play a particular role in the university, compared to primary or secondary education. We also consider some affordances and constraints of their use. Afterwards, we explore how the idea of "didactic paradigm" provides a framework for the analysis of different possible uses of digital resources. Finally, we report on some experiences about the use of digital technologies motivated by the appearance of the COVID-19.

Keywords: digital resource, tertiary education, mathematics education.

INTRODUCTION

When the panel started to be planned, by December 2019, we could not imagine how trendy the topic of the panel was going to be now, along the year 2020. The aspect of potentials of digital means to organise teaching and learning in a different way have gained increasing attention. Resources and teaching environments that allow avoiding physical presence at universities during the COVID-19 pandemic were (and still are) extensively needed and used. The impact of the pandemic in education and the role played by digital resources to overcome the difficulties and to face the challenges will be one of the issues we will address here. But not the only one.

Digital resources provide both teachers and students with a whole world of possibilities, and so many questions arise concerning their use in the teaching of mathematics at the tertiary level. We have tried to consider some important aspects of this work.

We start the first section by considering the question of what a digital resource is. This is an important question, because the way we conceptualise it strongly relies on the didactic paradigm we assume, and deeply affects the way we analyse different uses of digital materials. Next, we will explain that, as a matter of fact, digital resources seem to be more used at tertiary education than they are at primary or secondary education. We will also provide possible reasons to explain it.

Digital resources enlarge the collection of possibilities for the teachers to present the contents of their courses and for students to engage in these contents. Some of these resources can be provided by the educational institutions, and some others are used by students at their own initiative. However, these resources are not always exempt from problems in their use, either for technical reasons (experienced by teachers and/or students) or for issues concerning the quality of the resources themselves. In relation to this, we will address which are the affordances and constraints of the use of digital resources.

In the context of education, digital resources do not exist by themselves, unrelated to anything else. They are rather placed in the frame of a *didactic paradigm*. In other words, when an educational institution suggests the use of a certain digital device, or when a certain student decides to use it, there is always an underlying set of educational ends and an underlying epistemological model. Both the educational ends and the epistemological model deeply affect what kind of digital resources are to be used, how, and to what purpose. Therefore, we will also make some considerations about digital materials in connection with didactic paradigms.

Finally, as we are not only researchers but also teachers, we will as well consider how the COVID-19 crisis has given rise to the intervention of digital resources in our own practice, and what difficulties or revelations have appeared related to this.

At the end, we present the main ideas in a conclusive section.

WHAT IS A DIGITAL RESOURCE?

We can distinguish between digital resources from a technological and mathematics education point of view. From a technological point of view, the term “digital resource” refers to any resource that is in a digitalised form. Digital resources may include hardware technologies (e.g., a calculator, laptop, or mobile phone), and educational technologies that can be divided into two categories: first, pedagogical software technologies such as Wolfram Alpha, GeoGebra, Stack, or Numbas; and second, generic software technologies that cover a variety of mathematical topics such as Khan Academy, online math video lectures, Massive Open Online Courses (MOOC), Facebook resources, etc.

From the point of view of mathematics education, the term “digital resource” may be conceptualised using the instrumental and documentational approach to didactics (Trouche, 2004; Trouche, Gueudet, & Pepin, 2018). The key notions of the instrumental approach are “instrumentation” and “instrumentalisation”, and the transformation of digital artefacts into instruments. The instrumental approach is most often used with a student perspective.

The documentational approach mainly takes a teacher perspective, but it can also be used to explore students’ use of resources (Hillesund, 2020). This approach to didactics is considered as a further development of the instrumental approach with some more key notions such as document and resource (both digital and non-digital resources). Moreover, with the documentational approach, there is a distinction between educational technologies and digital curriculum resources.

Educational technologies at the tertiary level are studied for a long time (see, e.g., section 5 in the ICMI study dedicated to the teaching and learning of mathematics at the university level, Holton et al., 2001). In recent research, the interest in digital curriculum resources and their use by students has developed. Concerning the conceptualisation of digital curriculum resources, we refer to Pepin, Choppin, Ruthven and Sinclair (2017, p. 647), who contend that:

> It is the attention to sequencing—of grade- or age-level learning topics (all or part of, or of content associated with a particular course of study (e.g., algebra)—so as to cover (all or part of) a curriculum specification, which differentiates Digital Curriculum Resources from other types of digital instructional tools or educational software programmes.

Some studies choose a quantitative approach to this issue. They study what resources are used by students, and for what purposes. Stadler, Bengmark, Thunberg and Winberg (2013) observe that during the secondary-tertiary transition, students in Sweden increasingly use Internet-based resources. This observation can depend on the national context: indeed, in the UK, Anastasakis, Robinson and Lerman (2017) note that students mostly use the resources provided by the institution, with exam-related goals. Also, in the UK, Inglis, Palipana, Trenholm and Ward (2011) investigate the use of three kinds of (optional) resources by students: live lectures, online lectures, Mathematics Support Centres. Interestingly, only a minority of students use more than one resource. The authors conclude that students need guidance for blending different resources.

Recent studies use the documentational approach (Gueudet, Pepin & Trouche, 2012) to investigate the use of resources by students at the university level. This theoretical approach is associated with case studies. Kanwal (2018) studies cases of engineering students working with a learning management system. She observes that the form of the assessment influences their use of resources and concludes that the tasks proposed in a digital curriculum resource must be carefully designed, to lead students to the expected mathematical activity. This coincides with results obtained by Gueudet and Pepin (2018), who observe that some resources are misused by students. The rules of the didactic contract, concerning the use of resources, remain mostly implicit and the actual use by students does not correspond to the use expected by teachers. Pepin and Kock (2019) note that in different courses (Calculus vs Linear Algebra), different kinds of resources are proposed by teachers. Students use institutional resources when they are aligned with examinations. Otherwise, they search themselves for resources, in particular human and social resources.

In terms of the evolution of digital resources in mathematics education, there is a trend. In the beginning, there was a prevalence of visualisation tools, e-learning 2.0, blended and mobile learning, e-assessment systems, programming languages. Later, there seems to be a preference for resources with advanced functionalities such as e-learning 3.0, multi-touch technologies, embodied learning technologies, artificial intelligence-based tutoring tools with feedback, and, most importantly, technologies that connect mathematics education to computational thinking and artificial intelligence, and educational *Internet of Things* aimed at connecting and integrating digital resources into people's everyday life as a guiding principle (Ashton, 2009). Today, students can connect computers, laptops, tablets, and smartphones in mathematics classrooms. Moreover, people can now connect smartwatches, smart devices, cars, and other devices that collect and transfer mathematical data.

More specifically, educational institutions are moving from the early Internet of Things of smart connections to a new phase, one of invisible integration, which results in the disappearance of digital resources in the vision of *ubiquitous computing* formulated by Weiser (1991, pp. 94), who pointed out that

> the most profound technologies are those that disappear. They weave themselves into the fabric of everyday life until they are indistinguishable from it.

In other words, digital resources disappear in a manner that they are in the mode of being the philosopher Heidegger called the *ready-to-handedness* of tools (Heidegger, 1953). This means that the word "digital" starts disappearing from educational terminology, which sooner or later will result in a "post-digital" education (Pandrić, 2018). Are we entering the age of post-digital mathematics education, also partly due to the acceleration of digital teaching in this pandemic period?

PRESENCE OF DIGITAL RESOURCES TO TERTIARY EDUCATION

As a matter of fact, the relevance of digital resources to tertiary education rests, firstly, on the degree and extent to which the resources are used (by mathematicians, mathematics educators, non-specialists such as engineers and biologists, students and teachers from different disciplines), and, secondly, on the integration of digital resources into educational settings. Given these considerations, in practice, it seems that digital resources are more present in tertiary education than they are in primary or secondary education.

Indeed, there seems to be an extensive use and integration of digital resources at the tertiary level across all subjects, combining face-to-face and distance learning, frequently in relation to flipped classroom methods (Pinto & Leite, 2020). In many countries, both teachers and students are now using visualisation and simulation tools, computer-based assessment systems, programming languages to acquire computational thinking skills for mathematical explorations and investigations.

Let us examine some examples of uses of digital resources at the university.

From the point of view of MatRIC, the Centre for Research, Innovation and Coordination of Mathematics Teaching, which is a learning community working for excellence in teaching mathematics in Norwegian universities, digital resources are very relevant for the study processes at tertiary level. Many MatRIC-driven activities at the University of Agder and other universities in Norway reveal the relevance of digital resources for modelling activities, simulation, visualisation, and assessment, etc. Likewise, the new activities aiming at digital mathematics teaching show the relevance of digital resources for inquiry-based mathematics education and online mathematics teaching and learning. Moreover, MatRIC has developed *Drop-in*, an additional digital resource that offers extra help and guidance to support students who are working with challenging mathematical tasks.

The use of programming language for mathematical investigations at university has been studied for several years in the context of the MICA course (Mathematics Integrated with Computers and Applications, see, e.g. Buteau & Muller 2010) at Brock University in Canada. In this course, mathematics majors and future mathematics teachers learn and use programming for mathematical investigations "like mathematicians". A programming language falls within the artefact definition given by Rabardel (1995). Nevertheless, compared, for example, with Digital Geometry Systems, it is clearly of a different nature. For this reason, studying instrumental geneses linked with the use of a programming language when solving mathematical problems can lead to identifying new kinds of schemes, and deepen our understanding of the relations between computer science knowledge and mathematical knowledge. Along with their use of the programming language, students develop an instrument. This instrument associates the programming artefact with different kinds of schemes, in particular, what we call "p+m-schemes" where knowledge about programming and about mathematics are strongly associated (see Gueudet, Buteau, Muller, Mgonbelo & Sacristan, 2020).

Also, the relevance of digital resources can be seen in mathematics education for non-specialists. Indeed, in many mathematics courses, mathematical objects are not clearly linked to objects in the "real world", are not used to create models of "real systems". Rather, there seems to be an emphasis on understanding the behaviour of those objects regardless of the properties of the "exterior world". However, in mathematics courses of other disciplines (e.g. engineering, biology, etc.), mathematics is expected to provide, via modelling, useful information about certain systems appearing in nature, in real-life. Digital resources for engineering students and other non-specialists seem to be of help in this task. Engineers on the workplace use computers and software, and their studies have to prepare them for this use. Nevertheless, engineering students sometimes use technology as a black box, allowing them to obtain a solution without understanding the mathematics behind (e.g. Kanwal, 2020). Recent research has advanced our understanding of this complex issue. Drawing on the concept of "techno-mathematical literacies" (Kent, Bakker, Hoyles & Noss, 2005), defined as combinations of mathematical, Internet of Things and workplace-specific competencies, van der Waal, Bakker and Drijvers (2017) identified seven categories of techno-mathematical literacies for working engineers. Drawing on this work, they implemented and evaluated inquiry-based teaching where engineering students were invited to present and comment on their use of software (van der Waal, Bakker & Drijvers, 2019). This use was then collectively discussed. The authors evidenced that this kind of teaching can support the development of techno-mathematical literacies for future engineers.

A possible explanation for the increasing use of digital resources at the tertiary level is that university mathematics students are given more responsibility for learning than in school mathematics (Hillesund, 2020). This also entails that, while universities provide students with digital resources, it is up to them to decide how to use the resources. Indeed, on tertiary level students are usually expected to have, to a certain extent, the competence to work on their own, since the scheduling of tertiary education mostly just dedicates a relatively minor percentage of the total required learning time to supervision by teaching staff. Depending on a variety of contextual factors, students can use other digital resources that are freely available online, some of them are related and other unrelated to the ones used in the university courses across many disciplines. Among those factors, we can find technicalities of the resource, familiarity with the resource, availability of time and human resources (teachers, peers, etc.), exam situations and mandatory tasks, etc. External digital resources beyond the ones used at the university may include video resources, e-books, simulation and visualisation tools, games, videos used in flipped classrooms, MOOCs, collaborative distance learning environments, etc. University students can make their own decisions as to whether to use external digital resources and to what extent and purpose, particularly with examinations and compulsory assignments in mind (Hillesund, 2020). Thus, it is sometimes required to find and work with adequate learning materials, but also, to some degree, to distinguish between good and bad materials that can be found and to take responsibility for learning outcomes, and these requirements are increasing with the ascending educational level. Thus, in comparison to primary or secondary education, the potential for individual use of digital learning materials and the self-reliant use of those materials appears to be higher at the tertiary educational level.

AFFORDANCES AND CONSTRAINTS OF THE USE OF DIGITAL RESOURCES

Digital learning resources can be used to enable students to follow new learning trajectories and to change the way students engage in the learning of mathematics (Sacristán et al., 2010).

The flexibility digital resources provide concerning the pace, order and organisation of learning can allow students for developing their own, individually preferred learning routines, strategies and schedules according to their personal needs (Gold, Fleischmann, Mai, Biehler & Kempen, in press).

Digital learning materials can be used to support understanding certain mathematical concepts (in the sense of Tall & Vinner, 1981) by offering a variety of different representations, or by providing detailed feedback. Software environments like GeoGebra (Hohenwarter & Jones, 2007), STACK (Sangwin, 2003) and many others are available, and guide students through the challenging process of change of representational register (Duval, 2006), using dynamic illustrations and the opportunity to actively construct mathematical objects digitally, for example in geometry. There also exist elaborated learning environments and teaching concepts where these technical means are implemented in tertiary education (Kinnear, 2019; Biehler, Fleischmann & Gold, 2018).

Evaluations of the students' learning behaviour when using digital resources are pointing in the direction that, when students work independently with a comprehensive digital mathematics bridging course material, they mainly concentrate on solving tasks rather than working through theory (Fleischmann, Kempen, Biehler, Gold & Mai, 2019). However, thanks to their technical format, digital materials can here support the (mental) linkage between theory and application by providing quick access to the relevant passages and offering detailed feedback on the solutions entered by the student.

Having said that, teaching mathematics in a technology-based environment rests on a combination of several factors: the characteristics of the digital resource, teachers' digital competencies, students' mathematical knowledge background and digital skills, the subject curriculum, the topic to be taught, the discipline (mathematics, engineering, biology, etc.), the learning goal, and most importantly the affordances and constraints that emerge in mathematics educational contexts at the tertiary level.

Therefore, the following question seems appropriate: what affordances and constraints emerge from the interactions between users (teachers/students) and digital resources in mathematics educational contexts?

The types of affordances and constraints that emerge from the interaction between users and digital resources at various levels depending on many factors highlighted above. More specifically, affordances result from the characteristics of the resources and the way users interact with them in an educational context. In other words, affordances involve both the knowledge background of the users and the features of the resources (Hadjerrouit, 2020). Thus, affordances and constraints are not intrinsic properties of the digital resource or users (teachers, students), but rather properties of the whole conglomerate formed by the digital resource, the teachers, and the students, all together.

The affordance issue has two didactical consequences. Firstly, using digital resources for teaching mathematics require good technological, didactical skills and mathematical competencies to foster the emergence of affordances and minimise constraints. Secondly, teachers need to develop a reflective attitude towards the use of digital resources and consider both affordances and constraints in designing mathematical tasks.

In the paragraph above, we had in mind digital resources typically provided to the students by the teaching institution, like a certain programming language or a specific applet suggested by the teacher to represent mathematical objects or to make symbolic calculations. But, of course, those are not the only kind of digital tools. The variety of digital learning resources used at current tertiary level goes from complete online courses, which are available for students, focusing mostly on the study entrance phase (Biehler, Fleischmann, Gold & Mai, 2017; Kinnear, 2019), over online platforms used for communication between students and teachers by most universities, up to a broad and constantly growing collection of medial resources that can be found online with public access. These online materials include forums where mathematics is discussed informally, websites that offer calculations and visualisations, such as Wolfram Alpha, and online videos uploaded mainly on YouTube. In particular, these online videos gained increasing popularity within students (Acuña-Soto, Liern & Pérez-Gladish, 2020), but the fact that the quality concerning contents is sometimes questionable and, like for most online resources, outside of the control of teaching staff at universities, also contains a risk for the education of students using these media. One can identify high potential in the offers of digital means, and the fact that it becomes technically easier to create and provide new materials online leads to the highly desirable opportunity that also unconventional approaches are followed and find their audience. On the other hand, there are also risks and challenges associated with educational use of digital resources. Suitable learning materials must be identified, and reliable criteria for the quality of these materials must be at hand (Hadjerrouit, 2010). Moreover, working with some medial formats, such as videos, can lead to an "illusion of understanding" that might come with the consumption of these materials, that does not necessarily enable students to think and solve problems themselves (Schwartz, 2013).

Essential for the teaching and learning of mathematics is the communication between the teachers and the students, and digital resources can constitute valuable means to support this communication process. While the platforms used by many universities offer channels to provide *learning materials*, to ask questions (possibly also anonymously, which might lower the barrier to do so for some students), also the exchange of *feedback* on the teaching and the learning can be supported by digital environments to the profit for teachers and learners. Data provided by the digital learning environment concerning the work behaviour of the students can help to adjust the teaching (Reinholz, Bradfield & Apkarian, 2019). In the other direction, studies show that students appreciate getting digital feedback on their work via an oral commentary that is recorded in the form of video-based feedback (Robinson, Loch & Croft, 2015; Grove & Good, 2020).

DIGITAL RESOURCES IN CONNECTION TO SPECIFIC DIDACTIC PARADIGMS

It would be interesting to consider whether the inclusion of digital resources is an actual innovation rather than just a variation of the means (Lindmeier, 2018). Moreover, it could be clarifying to regard digital resources in education (and, actually, also about analogue resources, like handbooks, blackboards, chalks, notebooks, etc.), as means to achieve certain ends in the framework of an explicit *didactic paradigm* (Gascón & Nicolás, 2019, 2020, 2021) for the teaching of mathematics. The idea of *didactic paradigm* (assumed by an educational institution) comprises both the assumed *educational purposes* and the assumed *epistemological model* of mathematics. The educational ends are the answers to the question "Which is the purpose of teaching mathematics?”. The epistemological model is the answer to the question “What is to know mathematics?”, which is closely related to the question “What is to teach mathematics?”. Only under the premises stipulated by a given didactic paradigm we can provide arguments for one or another use of a given digital resource.

For instance, if, according to our epistemological model of mathematics, students have to do a considerable amount of empirical work to construct mathematical knowledge (for instance, discovering by themselves counterexamples to their conjectures), like in the Theory of Didactic Situations (Brousseau, 1997), then some software (to represent functions, to deal with statistics, etc.) would be very helpful to support this work.

One of the possible purposes of teaching mathematics (or teaching, in general) could be to promote a receptive attitude towards posing and answering objective questions about the world in a rational way. This is what in the anthropological theory of the didactic (Chevallard, 2006) has been called the *paradigm of questioning the world*. The typical means proposed by that theory to design and describe study processes within that paradigm are the so-called *study and research paths*. Those are “paths” that start with a meaningful *generating question*, to which the students are supposed to provide a “suitable” answer. Normally, one of the few clauses of the didactic contract in those study and research paths is that the teacher is not going to say any possible answer(s), and students are allowed to do whatever to find an answer. And, of course, they have to show some kind of argument to defend the suitability of that answer. If the students use the freedom provided by the didactic contract in the study and research paths, it is quite reasonable to suppose that students are going to use digital resources (search engines, applets, programming languages, etc.) to look for an answer to the generating question. Thus, in the paradigm of questioning the world, digital technologies are implicitly regarded as normal ready-made objects to be used along a path from a question to a corresponding answer.

For instance, if one of the *purposes of teaching mathematics* is to get the students kind of familiar with the mathematical activity, like in the Inquiry-Based Learning approach, then one could use some specific software ‘for mathematicians’, and also allow the students to look for solutions to their problems on the Internet, as the mathematicians do themselves. Digital resources today have become more sophisticated and have the potential to be more than tools to perform tasks faster than by hand and paper-pencil techniques. Digital resources are equipped with interactive graphical user interfaces, making students able to participate more actively with the help of different forms of feedback. Hence, from a didactical point of view, it appears that there is a shift from teacher-centred to a more student-centred pedagogical approach to mathematics education. A good example is mathematics flipped classrooms, which, according to some versions, it takes a student-centred approach to learn at the university level, as studied in (Fredriksen, 2020) with a group of engineering students. The shift consists of moving away from a teacher-centred model, where the teacher is the main source of instruction towards a student-centred approach, where in-class time is used for exploring topics gained from out-of-class video watching, creating rich learning opportunities among students. This is just one example of the impact of current digital resources on instruction. Other examples of student-centred approaches using advanced digital resources are linked to e-assessment systems with formative feedback such as Numbas and Stack, programming languages with interactive user interfaces such as Python and MATLAB, visualisation and simulation tools such as SimReal (Hadjerrouit, 2020; Hadjerrouit & Gautestad, 2019).

Of course, one of the purposes of many degrees at university is to prepare students for certain professions. As some of those professions make extensive use of digital resources (for instance, engineering), their study becomes an essential goal in these degrees. Some study and research paths have been designed and implemented with future engineers (e.g. Florensa, Bosch, Gascon and Mata, 2016, Quéré, 2019), and so they use a generating question anchored in working engineers' practices. For example, for future chemistry engineers, Quéré (2019) proposed a study and research path in statistics starting with the question: "In the pharmaceutical industry, how do you make sure that the product (medicine) meets the dosage on the package?". The students developed sub-questions linked with statistical tests and studied these questions, using in particular statistical software. While the use of educational technologies in study and research paths addressed to future engineers is not a central aim, it is frequently present and plays an important role. Indeed, there are indications that task design for digital resources will be crucial in technological-based mathematics courses at the tertiary level (Leung & Baccaglini, 2017). Task design is important due to the challenges faced by many engineering students in considering mathematics as detached from real-life applications (Fredriksen, 2020). Designing realistic mathematical tasks will thus become crucial in mathematics education across many disciplines.

COVID-19 AND DIGITAL RESOURCES

This last part of the paper will be devoted to reporting on the teaching experiences of two of the four authors during the 2020 lockdown. Needless to say, those reports are not intended to be scientific conclusions. Nevertheless, they can still be of interests, as testimonies of an era in which the implementation of digital resources is being accelerated due to the COVID-19 crisis. In (Clark-Wilson, Robutti & Thomas, 2020) the reader can find further interesting reflection on teaching with technology during the COVID-19 period, mainly concerned with secondary school.

Due to the lockdown of NTNU in the middle of March 2020, Yael Fleischmann had to switch from a traditional attendance-based lecture (with about 120 participants, on Euclidian and hyperbolic geometry) in the middle of the Norwegian spring semester to a teaching format that was completely realised online. Using the screencast software "Explain everything", she decided to record her lectures and upload them for the students twice per week. The software provides a digital "whiteboard" where one can write and record a voice-over explanation simultaneously Additionally, she arranged real-time online meetings with the students, using the universities digital learning- and communication-platform "Blackboard", where students could ask questions and discuss tasks with the lecturer (Yael). As a consequence of this shift of format of the lectures, she noticed that students, even if there were several channels to do this, hesitated to ask questions. As the lecturer, Yael also missed the feedback of the students concerning their level of understanding for the contents she was explaining, as she usually had gotten during the live lectures. Yael also noticed that speaking to a tablet computer instead of an audience influenced the formerly quite lively atmosphere of the lecture heavily. To tackle these challenges, she decided to include something that she called “sound mystery” into the lectures. That was a sound, for example, the music theme of the computer game “Tetris”, that was played sometimes during the recorded lecture, and students were asked to identify the sound and send her the answer. The motivation for this unconventional step was to motivate and provoke reactions of the students, and hereby to lower their barrier to get in touch with her as the lecturer. Indeed, this worked well and together with the answer to the “sound mystery”, students started to send questions concerning the contents of the lecture and also feedback on the format and style, which was very helpful for her as the lecturer. From the times and numbers of reactions, it was also possible to estimate by how many students and when the videos were watched (which is data that the universities’ platform cannot provide to the lecturer). Part of the feedback on the lecture was that students expressed very different needs concerning the new learning situation during the lockdown. While some students appreciated the flexibility given by the opportunity to watch the videos at any time and pace, others expressed their need for a given time schedule to stay motivated and disciplined. Yael tried to address this by providing videos of approximately the length of the former live lectures twice per week shortly before the former lecture hours. Students also expressed a high level of insecurity, in particular regarding the exam that was planned for this course and had to be taken as a digital home exam in consequence of the pandemic circumstances. Here, it was essential to communicate also minor decisions and developments concerning organisational and administrational details very frequently. It must also be said that the development of an exam that was supposed to be written by the students at home, with all possible (online-) sources and communication channels available for the students during the examination time, was a particularly hard challenge, and also the grading of an exam where different students did make use of these unconventional opportunities to dramatically different extents was demanding. It yet remains as an open question of what kind of digital resources could possibly be developed and used in the future to allow appropriate assessment under circumstances that do not allow physical presence.

During the first lockdown in France, Ghislaine Gueudet taught to prospective secondary school mathematics teachers, and to students in educational research. Both kinds of students were engaged in Master degrees; the groups were between 15 and 35 students. Most activities Ghislaine proposed were asynchronous: she offered resources and tasks on a Moodle platform, the students uploaded their work, she corrected it and sent it back. She also organised some synchronous activities but noticed it was very difficult for some students to have access to the video-conference platform. Finally, the best solution to communicate with them was by using cell phones and WhatsApp.

The most important lesson Ghislaine learned was linked with the observation of the difficult situation of some students. The research in mathematics education already addresses for a long time the issue of equity. Concerning the use of digital resources, Forgasz, Vale and Ursini (2010) noted that while the issue of access was important, some research has evidenced that digital resources can contribute to offer equitable learning opportunities. The study of equity issues at the university level is very active. For example, Adiredja and Andrews-Larson (2017) describe the evolution of research towards an increased interest in socio-political issues. They present in particular research that addresses the impact of social discourses and institutional contexts on the negotiations of power and identity in postsecondary mathematics.

Nevertheless, the research they cite in their synthesis does not consider the difficulties raised or the opportunities created by the use of digital resources at the tertiary level. In this time of crisis, maybe the most important question concerning digital resources at the tertiary level is: "How should digital resources be used at tertiary level for fostering equity?"

CONCLUSIONS

There are different conceptualisations of the notion of *digital resource*. Here, beyond the technical one, we have examined those provided by the instrumental and the documentational approach. We have also pointed out that digital resources are considered each timeless noteworthy orthopaedic tools to help the study, as they are becoming more and more transparent and integrated into the normal current life of students.

Sometimes the use made by students of digital resources is essentially rejected, or looked with suspicion, or at least not taken into account, by some didactic paradigms. For instance, some regard the teacher and the notebook (and perhaps some web pages specifically designed for the corresponding course, as if they were *digital notebooks*) as the only sources of information for students. This seems to be the case more frequently in primary and secondary education. In tertiary education students are given more autonomy and more responsibility of their own learning, and, at this point, digital resources (those provided by the university, or any others) seem to be a usual way to support their study.

However, even if the use of digital technologies enlarges both teachers' and students' affordances, it also presents certain constraints, either due to technical reasons (for instance, the complexity of a certain applet) or to the reliability of the service provided by the technology itself. Anyway, those affordances and constraints are never due just to the digital materials or the teacher or the student, but rather to the indivisible system formed by these three components. Not to mention the importance of the content to be studied and the way it is related to the digital resources at stake.

Typically, digital resources do not provide by themselves new ends of education. Instead, they enlarge the collection of means to achieve those ends. Indeed, digital technology offers new possibilities for the design and management of study processes, both for teachers and students. Those new possibilities concern many aspects of study processes: representation of mathematical objects, feedback in the resolution of tasks, etc.

There are didactic paradigms that incorporate digital resources themselves like objects of study (for instance, in degrees devoted to the preparation for a profession in which those resources are of typical use), or even objects of study and, at the same time, means for the achievement of further ends (like in the case of a programming language for the design of algorithms that use some theorems in a course on Numerical Analysis).

Also, concerning certain didactic paradigms, even when a digital resource is not the object of study, it can be considered as an already-made object that students can use at any time along the study process. Actually, digital materials are very likely to be used by students if they have to carry out research in order to provide an answer to a question, and they are allowed to use any means. This is supposed to be the case in many didactic paradigms laying under the broad label of *inquiry-based learning*.

The COVID-19 crisis has forced the imposition of online teaching. This change of scenario has revealed sometimes the economic and social inequality that exists in a single group of students. This is a reality which underlies educational institutions, and that may affect students' commitment to their own education. Also, a teaching proposal completely interfered by digital technologies, with no physical presence, seems to experience defective communication between teachers and students. Moreover, a sudden and unexpected switch to a distance learning regime also entails deep problems in assessments, and not only those concerning cheating prevention.

As we have seen, tertiary education in the digital age is full of challenges and complex issues, some of which have been pointed out and taken under consideration in this work.

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TWG1: Calculus and Analysis

TWG1 report: Calculus and Analysis

Laura Branchetti[1] and Marìa Trigueros[2]

[1]Department of Mathematical, physical and computer science, University of Parma, laura.branchetti@unipr.it;

[2]Instituto Tecnológico Autónomo de México, trigue@itam.mx.

INTRODUCTION

In the Thematic Working Group about Calculus and Analysis (TWG1) 11 papers were presented in two sessions, followed by discussions in subgroups; between 20 and 27 people participated in each presentation and discussion session. Focusing on the topics and the didactical issues raised by the authors in this group, we observed that both classical and new issues were addressed as compared with the main themes discussed in previous conferences (Trigueros, Bridoux, O'Shea and Branchetti, in press). Thus, we organized the papers into two groups, the first concerning the school-university transition and involving basic concepts of Calculus and the second concerning advanced mathematical topics and new research-based approaches to their teaching in advanced courses, including links with more basic concepts. Within each group we identified two subgroups of papers about similar topics and we separated the participants into two rooms, assigned by us to authors and freely chosen by every virtual participant, so to encourage fruitful discussions about issues of common interest between the authors of the presented papers and the participants.

DISCUSSIONS DURING THE CONFERENCES

Group 1

In a first analysis, two common aspects of the 5 papers concerning basic concepts of Calculus were found. The first subgroup's studies were related to basic key concepts at the transition from secondary school to university (3 papers), in particular: functions, limits and sequences (M. Flores Gonzales et al., F. Khalloufi-Mouha) and different approaches to the definition of continuity, using a topological approach (L. Branchetti et al.). The second subgroup focused on integrals (2 papers), one concerned with Riemann sums (I. Akrouti) and the other with the introduction of a different approach to the teaching of integrals using Klein's plan B (G. Planchon et al.).

The first subgroup faced discussion topics related to the transition to university, including a perspective on advanced courses. Discussion initiated the provoking question: Whether and, in any case, why is it meaningful to make a distinction between functions and sequences, since sequences are functions? An interesting debate about choices in different countries and the importance of discussing this distinction with respect to continuous functions emerged. Then discussion turned to an inquiry about a study on recursive sequences and how students compute limits using general theorems on sequences and limits and their relation to the teaching of Calculus. Participants considered the topic as pertinent for the community. Attention turned to trigonometric functions, and the differences involved in their presentation when introduced in courses either at the transition to university or at the transition to advanced mathematics, going from angles/arcs to power series and differential equations. The importance of relating both definitions was underlined as well as the need for further studies.

The notion of local routine, such as routines to determine continuity and differentiability at a given point of simple trigonometric functions or their composition was used in one paper in a way that was not convincing for all the participants, so the question: "Is using limits of a function and its algebraic expression to investigate its continuity at a point "local"? was discussed and the group stressed the need to check these terms carefully to avoid misinterpretations.

Discussion of different theoretical approaches to the introduction of Calculus' basic concepts, in particular the topological approach, brought about the observation that it allows to exploit the graphical register with a good connection to definitions in some tasks, so students might find it helpful. It was also observed that it would be interesting, but not easy, to find a university course where analytical and topological definitions are mixed. The need to adopt a sort of internal "interdisciplinarity" within the different branches of mathematics and in teaching also emerged and it was recognized that nowadays there are too many barriers between different fields.

In the second subgroup, the discussion focused on the teaching of integrals. The notion of integral and students' understanding of the fundamental theorem of Analysis and its relation to the notion of area was discussed together with the need of a new possible praxeology for integration. Interesting issues were addressed such as students' approaches to tasks about integrals and its relation to didactical contract; students' conceptions of integral, with a particular attention to its meaning (the role of the concept of area and measures in understanding integration) and the need to find ways to relate these ideas to their teaching and to formalize students' ideas.

The sub-group discussed students' tendency to conceive "calculate" as "to find the exact value" and to think of it as different from "approximate" in some task. Such an issue was examined in terms of its possibility to lead to different approaches to tasks depending on how they are formulated and also on the didactical contract: either looking for an exact value or expressing the integral as a limit of Riemann sums with an unknown "f". Also, those cases where functions are presented only by their graph and the influence of students' possibility to use graphing calculators or computers during a test were addressed. A relevant discussion took place about the nature of the four conceptions of integral (primitive, area, sum, approximation), participants posed the question: are they concepts or processes? Another issue discussed in the sub-group concerned the praxeologies to teach integrals that should be presented to prospective teachers to connect students' ideas to the different theories of integration (Riemann, Lebesgue, etc.) at the school-university transition. Another relevant point was: How can the formalization of the area be concretely handled with university students? As measures (area) are only formalized in a measure theory course at university, an intermediate theory to establish a link with the secondary school integral should be introduced. Should it be focused more on proposing a new didactical engineering or on a strong mathematical basis of integration? In response, it emerged the need to provide students with new theoretically grounded praxeologies starting from their ideas and followed by a reconceptualization of basic concepts to introduce the concept of area, while making the theory behind them explicit.

Group 2

In one room, papers related to the role of representations in the transition to university mathematics and to more advanced mathematical courses were selected (3 papers): the use of local approximation tools to study functions (F. Belaj Amor); the use of diagrams in proving tasks (K. Gallagher & N. Engelke Infante) and a review concerning research on the calculus of two variable functions (R. Martínez-Planell & M. Trigueros). The other room focused on the teaching of advanced mathematical concepts and the role of intuition in this endeavour (3 papers): students' concept images and example spaces concerning continuity and differentiability (E. Lankeit & R. Biehler); the learning potential of including advanced mathematical ideas into more elementary courses (R. Hochmuth) and intuition and discourse in the teaching of the complex path integral (E. Hanke). The main ideas of the discussions follow.

In the first group participants discussed a difficulty found in the transition to university, namely the fact that when students are taking the first Calculus courses at university, they tend to continue using what they learnt in secondary school instead of new methods. Participants agreed that this may be possibly due to the need to provide opportunities for students to reflect on relations and differences between methods learnt before and the new ones, since this is not frequently addressed in most university courses. Students' difficulties with Taylor series were described as due to a rupture occurring between the ideas of derivative as the slope of the tangent at a point and derivative as a way to find a linear approximation to a curve around a point. The fact that these two ideas remain compartmentalized may be a major cause of students' difficulties. It emerged the need to clarify the role of Taylor series in problem solving when it is introduced.

When confronted with new abstract mathematical concepts, university students may face a new transition. According to research findings, the transition from one variable to two variables calculus involves a reconstruction on the part of the students of their previous knowledge that is new to them to take into account changes and similarities involved when learning new types of functions. Two ideas were brought forward firstly this transition could be approached using different registers of semiotic representation, and secondly, pedagogical means such as teamwork might be useful to facilitate this transition. A question was raised about the need of reconstructing previous knowledge when introducing multi-variable functions, or other functions where mathematical objects are impossible to draw or imagine. It led to a general discussion about knowledge reconstruction. From this discussion it emerged that "very simple" ideas, such as the use of the same symbolism for different mathematical objects, need to be deeply discussed to grasp the meaning of those abstract concepts. It was also commented that, for example, discussing with students how to define the slope could be a way for them to understand that "direction" plays a role in the construction of derivatives of multi-variable functions. Discussion on how some specific notions change when considered in different contexts led to pointing out the need to give students opportunities to represent them in different registers, or to imagine them as a way to give meaning to those objects.

Another important topic was the design of new strategies to teach advanced mathematics: the participants discussed what choices could be more helpful to introduce students to advanced mathematical concepts. The development of students' intuition and its possible use in proofs presented by an author was considered very interesting by participants. The use of diagrams, letting students manipulate and discuss them with their peers, was debated and questions about how to trace students' gestures as they work emerged together with its relation to the development of intuition. Responses to this question lead to a teaching strategy based on these ideas which offers new possibilities to discover how students imagine things and develop their ideas even when they are asked to work on difficult topology problems and proofs. Students' possibility to explore and discuss while they draw and interact with diagrams received a lot of attention and helped to understand its role in developing key ideas as ideas that convince the prover that the statement is true, but which the prover must also be able to translate and use into a written proof. The importance of letting students try, and possibly fail, if topics are not easily represented using diagrams, was also discussed. Participants considered that it might help in gaining valuable information about mathematical practices, such as why certain objects, processes and relationships cannot be represented well in a drawing.

In the second subgroup, students' repertoire of studied functions was considered to be generally limited to certain classes, so students develop from their courses a concept image about function which includes mostly smooth functions. Approaching other functions, such as those that are "locally flat" was considered to provide opportunities to foster a better approach to functions' properties, particularly their differentiability. Questions such as: What happens if the function studied is not that regular? What can we tell about continuity and differentiability in those cases? were raised and suggestions to use some ideas coming from historical studies about partial differentiability implying differentiability, once the latter is defined in a non-standard way, were found helpful in developing students' intuition about difficult results.

As mathematical concepts become increasingly abstract, they become difficult to visualize and mathematical intuition is not readily available. Discussion about what is intuitive in courses such as complex analysis was considered by participants worthy of attention on the part of researchers. Approaches to intuition like that of Fischbein were recalled, but other perspectives, as the commognitive discursive approach, were suggested as useful to both investigate the nature of intuition and provide tools for teachers to introduce students to difficult ideas. An interesting question arose about how to draw the line between rigorous and intuitive discourses. The use of intuitive arguments to make sense of mathematics was pointed out but comments about the difficulty to define an intuitive discourse also appeared. Participants considered the need to use a variety of approaches to abstract concepts so that students can develop meaningful ideas based on visualization, intuition and creativity. A way proposed to foster students' development of intuition for advanced mathematical ideas consisted in designing tasks that could give students a glimpse of advanced mathematics in basic courses. The group discussed what type of tasks could be used, and how they could be presented to students. Approximation was suggested as a good topic for this purpose since it can be related to questions about the smoothness of a function or be used to discuss continuity for functions where continuity is a real issue.

FINAL REFLECTION

Papers presented during the conference were original in terms of the topics studied. They provided interesting aspects related to the teaching and learning of Calculus and Analysis that opened rich discussions on new topics that need more attention from researchers. Common topics in discussions covered the need to relate basic and advanced concepts through a reconceptualization of basic ideas when introducing advanced topics and to present advanced ideas in basic courses to anticipate meaningful concepts. This may require innovative approaches and the use of novel representations, such as diagrams, that can offer new ways to foster a diversity of teaching strategies to develop a deeper understanding of Analysis and abstract mathematical thinking. Moreover, even if the formal aspects were taken into account, as usual, by several authors, in this conference the issues of visualization, intuition and representation and the need to start from the students' ideas became crucial and were addressed during all the group discussions. It was considered that developing students' creativity and intuition is important also in the learning of advanced topics.

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Conceptions de l’intégrale de Riemann des étudiants à l’entrée en classes préparatoires

Inen Akrouti, Université Virtuelle de Tunis (ISEFC) & Université de Carthage (LaRINa), akroutiinen@yahoo.fr

The purpose of this research is to identify students’ interpretations when solving Riemann integral problems. Thirteen students enrolled in public university (first year of preparatory class) participated in this study. Data was collected from a test that was proposed at the end of integration courses (Semester II). Through detailed analyses, large majority of students consider the Riemann integral as representing area under a curve or the values of an anti-derivative. In the other side, a few number of students use the limit of approximation conception in their responses. However, understanding the integral as a Riemann sum is highly productive for conceptual learning Tall (1992).

Keywords: Teaching and learning of analysis and calculus, teaching and learning of specific topics in university mathematics, Riemann integral.

INRODUCTION

Le concept d’intégrale de Riemann est un concept fondamental de l’analyse réelle. Il se caractérise par une nature multiforme. Cette nature nécessite une attention particulière afin de faire comprendre aux étudiants les idées principales qui le fondent. Une utilisation excessive de certaines interprétations de l’intégrale pourrait limiter son application et son domaine d’efficacité.

Dans une étude antérieure sur les conceptions des étudiants de l’intégrale définie à l’entrée à l’université (Akrouti, 2019b), il nous a été possible de souligner que les étudiants possèdent différentes interprétations, parmi lesquelles nous citons le processus d’approximation de produits infinitésimaux. Cette interprétation est particulièrement utile dans de nombreux contextes mathématiques et physiques. Beaucoup de recherches (Orton, 1983 ; Jones, 2013, 2015) la considèrent comme l'interprétation la plus précieuse qui pourrait donner un sens à l'intégration. En effet, cette interprétation se prête à être imaginée comme la somme des produits de longueurs et de largeurs de rectangles où l'un des facteurs est un infinitésimal ou une « très petite quantité ». Par ailleurs, elle permet à l’étudiant de construire effectivement le processus permettant de retrouver la valeur de l’intégrale définie en mettant en œuvre sa structure sous-jacente. Or malgré son utilité, de nombreux étudiants n’arrivent pas à investir cette interprétation dans leur travail. Ils abandonnent la construction de l’intégrale au profit du calcul de primitive et se lancent dans des procédures algorithmiques.

D’un autre côté, beaucoup de chercheurs (Sealy, 2014 ; Ely, 2017) constatent que les étudiants n'ont pas une compréhension approfondie de l'intégration et qu'ils ne pourraient pas bien faire face à des situations légèrement modifiées. Ils ne peuvent pas reconstruire les techniques dont ils ont besoin : la mémorisation des procédures les rend vulnérables face à un oubli (Akrouti, 2016). Ces chercheurs soulignent également que les étudiants peuvent avoir une connaissance procédurale de l'intégration en termes de techniques, sans une connaissance conceptuelle adéquate des structures sous-jacentes. En effet, la structure de l'intégrale renvoie à des méthodes différentes en fonction de la nature de la fonction à intégrer. Cela pourrait expliquer pourquoi les étudiants semblent confus et ont tendance à exercer leur mémoire plutôt que gérer la situation de manière créative et constructive.

Suite à ce constat, nous avons pensé à proposer deux questions où la procédure de primitive ne fonctionne pas. Notre ambition est d'amener les étudiants à réinvestir le découpage/encadrement pour mettre en œuvre un processus d'approximation. Nous cherchons en particulier à explorer les différentes manières par lesquelles les étudiants conceptualisent l'intégrale dans des situations non standard. Nous envisageons notamment de répondre à la question suivante : de quelles manières répondent les étudiants face à des tâches non standard quand il s'agit de la notion d'intégrale ?

LES CONSIDERATIONS THEORIQUES

Pour aborder notre problématique et cerner les caractéristiques des conceptions des étudiants, nous avons choisi trois outils théoriques : le concept image (Tall &Vinner, 1981), la dialectique processus/objet (Sfard, 1991) et les représentations sémiotiques (Duval, 1993).

La notion de *concept image* (Tall & Vinner, 1981) fait référence à la structure cognitive totale associée à un concept mathématique. Une image « *bien définie* » d'un concept mathématique peut être considérée comme la forme ou la structure finale dans laquelle le concept est logé dans le raisonnement d'un individu. Le concept image peut inclure des idées significatives, comme il peut inclure des idées contraires aux significations et aux définitions formelles du concept. Dans certains cas, le concept image peut différer à divers égards du concept formel défini et accepté par la communauté mathématique en général. Tall et Vinner utilisent l'expression « *The evoked concept image* » pour indiquer les éléments du concept image qui ont été découverts dans les réponses de l'individu. Dans notre cas, le concept d'intégration pourrait être identifié à une aire, au calcul de primitive, à un processus d'approximation ou à autres interprétations.

Le travail de recherche que nous avons entrepris depuis quelques années (Akrouti, 2016, 2018, 2019a et 2019b) nous a permis de souligner l'importance du rôle que pourrait jouer le statut de la notion d'intégrale dans son enseignement/apprentissage. Pour cela, nous considérons la *dialectique processus/objet* (Sfard, 1991) comme deuxième outil théorique utilisé. Le statut processus identifie l'aspect opératoire du concept alors que le statut objet identifie son aspect structural. La théorie de réification (Sfard & Linchevski, 1994) considère que le passage d'une « conception » orientée vers le processus à une « conception » orientée vers l'objet est le moyen par

lequel une entité mathématique sera conceptualisée. Cela sous-tend également la capacité de l'individu à interpréter les représentations symboliques d'un concept à la fois de manière opératoire et structurale. La structure de l'intégrale définie renvoie aux sommes de Riemann par la relation mathématique suivante : $\int_a^b f(x)dx = \lim_\infty \sum_{i=1}^n f(x_i)\Delta x$. Donc le travail de l'étudiant se manifeste dans le basculement de l'intégrale à la somme des aires algébriques (et vice versa), en prenant en compte le passage à la limite, donc en réifiant un processus (infinitésimal) de sommation d'aires. Nous considérons également que le calcul de primitives constitue un aspect opératoire de l'intégrale, qui est en quelque sorte « réifié » dans la formule $\int_a^b f(x)dx = F(b) - F(a)$. Il faut noter que Sfard et Linchevski (1994) ont parlé également du niveau pseudo-structurel des connaissances où l'individu se limite au statut objet dans la réponse sans montrer comment elle est obtenue.
Enfin, notre étude requiert les représentations sémiotiques (Duval, 1993). En effet, les types de traitement et de conversion sont des éléments efficients dans l'apprentissage du concept d'intégrale par la mise en relation qu'ils opèrent entre les différentes formes du concept (dans les registres géométrique, algébrique et numérique) : ces processus requièrent en général une flexibilité cognitive importante et sont de ce fait difficiles pour les étudiants.

CONTEXTE ET OBJECTIF

L'intégrale de Riemann est introduite pour les étudiants en classe préparatoire à partir de la définition suivante : $\int_a^b f(x)dx = \lim\limits_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x$. En tant qu'énoncé mathématique, cette écriture est une représentation symbolique de la relation entre l'intégrale de Riemann et les sommes de Riemann (les sommes de Darboux sont exclues du programme officiel). Elle se base sur la décomposition de l'aire principale sous la forme d'une combinaison d'aires rectangulaires. Cela met en œuvre un processus algébrique et géométrique significatif pour trouver l'intégrale en question. L'aire de chaque rectangle est calculée en multipliant sa hauteur et sa longueur qui sont représentées par $f(x_i)$ et Δx. Le symbole de la somme indique que tous les rectangles doivent être additionnés et combinés pour trouver l'aire totale. Le processus consiste à subdiviser l'intervalle d'intégration autant que l'on souhaite : plus la subdivision est petite, plus l'approximation est meilleure. Donc le contexte dans lequel nous proposons les situations relève de la procédure intégrale : découpage →somme →encadrement →passage à la limite ; cette procédure met en œuvre les trois étapes suivantes :

1) le découpage/additivité : si on découpe un domaine D admettant une aire en deux sous domaines disjoints X et Y, alors X∪Y admet une aire et aire (X∪Y) = aire (X) + aire (Y) ;

2) l'encadrement : pour toute fonction f continue par morceaux, positive, croissante et bornée sur $[a, b]$, on définit les deux sommes suivantes :

$S_n^- = \sum_0^{n-1} \frac{b-a}{n} \min_{t \in [x_i, x_{i+1}]} (\mathrm{f(t)})$ et $S_n^+ = \sum_{n=0}^{n-1} \frac{b-a}{n} \max_{t \in [x_i, x_{i+1}]} (\mathrm{f(t)})$. On a alors $S_n^- \leq \int_a^b f(t)dt \leq S_n^+$;

3) le passage à la limite : lorsque le découpage du domaine D devient de plus en plus petit, l'écart entre les encadrements supérieurs et les encadrements inférieurs devient de plus en plus petit. Ce processus converge vers un réel unique qui est la valeur de l'intégrale en question.

Dans le cadre de ce travail, nous supposons que le processus de résolution d'une tâche considérant l'intégrale en tant que limite d'une somme de Riemann est intrinsèquement différent du processus de recherche d'une intégrale par le calcul d'une primitive et que cette différence peut causer divers niveaux de confusion et de perplexité pour les étudiants (Akrouti, 2016). Il faut noter aussi que rien ne permet de penser qu'utiliser une procédure d'intégration via la recherche de primitive devrait être automatique ou naturelle pour les étudiants, du moins, pas pour les étudiants habitués à essayer de donner un sens à leurs activités mathématiques (Akrouti, 2016). En plus, rien de ce qu'ils font lors du calcul d'une intégrale définie ne serait lié aux idées basées sur les sommes de Riemann (Sealy, 2006).

METHODOLOGIE

Pour aborder cette recherche, nous avons choisi de proposer deux tâches non routinières aux étudiants après le cours d'intégration. Nous avons fait le choix d'écarter les tâches dont la résolution se base sur le Théorème Fondamental de l'analyse (TFA) : ces tâches sont fréquemment rencontrées et se focalisent pour l'essentiel sur l'aspect opérationnel de la notion d'intégrale. Les connaissances interrogées ici sont celles supposées acquises en classe préparatoire. Les questions proposées mettent les étudiants face à l'impossibilité d'utiliser la procédure de primitive. Il faut noter que l'enseignement de l'intégration en classe préparatoire ne prend en charge ni l'interprétation géométrique ni aucune explication métaphorique, sauf peut-être dans les cas les plus simples (Akrouti, 2016). Par ailleurs, la majorité des étudiants n'arrive pas à comprendre la signification de l'ostensif de la somme et comment se fait ce passage d'une quantité continue à une autre discrète. La même chose pour le différentiel dx : il semble être étranger et s'évaporer dans le processus de résolution tout simplement. Nous considérons que le raisonnement fondé sur les sommes de Riemann peut correspondre à la syntaxe de l'expression intégrale d'origine, mais il n'est pas utilisé pour expliquer ou extraire une signification du processus de calcul. Il faut souligner également que nous avons posé quelques questions avant le test aux étudiants participants parmi lesquelles :

- Quelle est la méthode que vous considérez la meilleure pour calculer une intégrale ? (I)
- Pourquoi une intégrale définie pouvait être interprétée comme une aire ? (II)
- Comment interprétez-vous la différence entre la somme de Riemann et l'intégrale définie ? (III)

Le test a été soumis à des étudiants de première année préparatoire à l'Institut Préparatoire aux Ecoles d'Ingénieurs de Tunis (IPEIT) parcours Math-physique (groupe MP8) après le cours d'intégration. Les étudiants ont vu l'intégrale de Riemann, les sommes de Riemann, le TFA et quelques techniques de calcul de primitives. Donc ils sont supposés connaître et maîtriser les propriétés algébriques de l'intégrale de Riemann. Le test a été proposé au mois de mai 2019. Treize étudiants ont participé en donnant des réponses écrites.

La première question a pour objectif de mettre en œuvre la procédure intégrale pour une fonction définie à partir d'un graphique. Elle consiste à calculer l'aire d'une surface non polygonale en encadrant la fonction en jeu par deux fonctions en escalier.

La deuxième question vise à mettre en œuvre la procédure intégrale à partir de l'interprétation graphique de l'expression algébrique d'une fonction continue, croissante et positive. Notre ambition est d'amener les étudiants à utiliser de nouvelles procédures (autre que le calcul de primitive) pour calculer la valeur d'une intégrale.

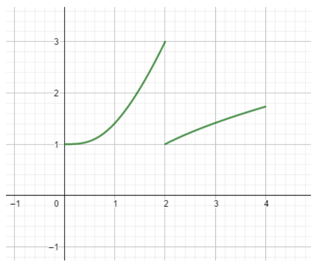
ETUDE EXPERIMENTALE

L'étude expérimentale comporte deux niveaux : analyse a priori et analyse a posteriori.

Analyse a priori

Dans l'analyse a priori, nous procédons de la manière suivante : tout d'abord nous commençons par la description de la question. Puis, nous identifions les procédures de réponse possibles à chaque question. En fin, nous citons les conceptions attendues.

Question 1 : Le graphique ci-dessous est la courbe représentative d'une fonction g. Calculer $\int_0^4 g(x)dx$.



La question est non standard. L'intégrale en jeu est donnée à partir de la courbe représentative de la fonction à intégrer. Les étudiants se retrouvent face à l'aire d'une surface non polygonale. Pour réussir cette tâche, deux procédures sont possibles :

*Dans la première, il s'agit d'encadrer la surface S en question par deux surfaces polygonales, l'une contenue dans S et l'autre contenant S. Cette procédure met en œuvre la conception d'aire. Elle se limite au statut objet de la notion d'aire et par la suite elle met les connaissances dans un niveau pseudo-structurel (Akrouti, 2019a).

**Dans la deuxième, il s'agit de décomposer l'intervalle d'intégration en subdivisions aussi petites que l'on souhaite. La valeur de l'intégrale est encadrée par deux sommes (l'une inférieure, l'autre supérieure) ayant la même limite. La complémentarité processus-objet est ainsi acquise : l'intégrale en jeu (objet) et l'approximation par l'aire des surfaces polygonales subdivisées mettent en œuvre un processus de somme d'aires (processus). Cette procédure se base sur l'interprétation de l'intégrale en tant qu'aire et donc sur la conversion entre le registre graphique et le registre numérique.

Question 2 : Soit f une fonction définie sur [0,2] par $f(x) = \sqrt{1+x^3}$. Calculer $\int_0^2 f(x)dx$.

La question est problématique. Bien que la fonction soit continue et admette une primitive, son expression ne rentre pas sous l'une des formes usuelles des primitives connues. Par ailleurs, elle met les étudiants face à un conflit cognitif : ils connaissent depuis la fin de la scolarité secondaire que toute fonction continue admet une primitive. Les étudiants qui ont gardé une bonne connaissance du cours sont en mesure de pressentir que la recherche d'une primitive n'est pas possible. Donc ils sont susceptibles de se baser sur l'interprétation de l'intégrale en tant qu'aire algébrique. Pour ces étudiants, nous pouvons identifier deux catégories. La première se base sur la procédure de processus d'approximation pour répondre. Cette démarche s'accompagne de conversions entre registres : au départ une conversion (algébrique/graphique), ensuite une conversion (graphique/numérique) pour calculer la valeur de l'intégrale donnée. La complémentarité processus/objet est en jeu ici : l'intégrale de la fonction proposée en tant qu'aire (objet) et le processus d'approximation (processus). Alors que la deuxième catégorie se base sur l'approximation par l'aire d'une surface polygonale. Cette procédure met en œuvre le statut objet et les étudiants sont à un niveau pseudo-structurel. Pour d'autres qui possèdent une conception de primitive, induite en partie par l'institution (Akrouti, 2019a), consistant à restreindre la question à un contexte algébrique, nous attendons qu'ils aillent rechercher une primitive pour pouvoir appliquer le TFA. Ces étudiants pourraient penser aux changements de variables ou à l'intégration par parties : il s'agit d'une conception de primitive qui identifie le calcul intégral à une recherche de primitive.

Analyse a posteriori

Deux étudiants ont répondu à la première question en utilisant la notion d'aire d'une surface polygonale pour calculer la valeur de l'intégrale cherchée. Six étudiants ont donné des réponses en se basant sur la procédure de processus d'approximation et cinq étudiants n'ont pas répondu à la question. Deux procédures de réponses ont été utilisées par les étudiants. La première procédure fait partie de la conception de limite d'approximation et fait appel à la méthode des rectangles : les étudiants qui ont choisi cette démarche n'ont pas pu la mettre en application (Fig. 1). Ils ont simplement décrit la méthode sans calculer effectivement la valeur de l'intégrale. La conception existe mais, elle n’est pas bien développée. Certains étudiants ont cherché une structure algébrique à partir de la représentation graphique donnée (Fig. 2). Il s’agit d’une tentative pour construire une analogie avec la structure algébrique afin de trouver la structure sous-jacente dans une somme de Riemann, où les hauteurs, les longueurs et les sommes de rectangles sont toutes représentées algébriquement. Notons qu'aucun de ces étudiants n'avait invoqué le raisonnement basé sur les sommes de Riemann pour interpréter l’intégrale dans la deuxième question, et que la plupart d’entre eux avaient montré qu'ils ne pouvaient pas le faire dans la première question. Ce qui manquait, c'était la conscience que les procédures de calcul d’une intégrale étaient soumises à un raisonnement géométrique.

La deuxième procédure se base sur l’approximation par l’aire d’une surface polygonale : les étudiants ont décomposé l’aire en somme d’aires de rectangles, de triangles et de trapèzes (Fig. 3). Cette procédure se base sur l’interprétation de l’intégrale en tant qu’une aire algébrique. Cette procédure permet d’approcher la valeur de l’intégrale mais, elle ne permet pas de donner sa valeur exacte.

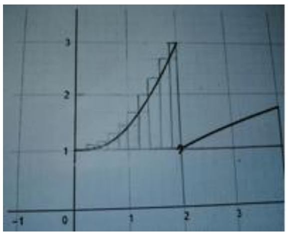


Fig. 1

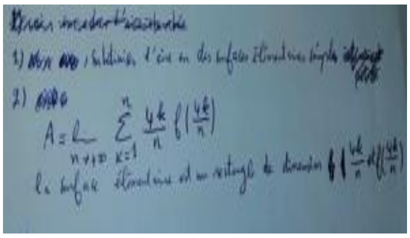


Fig. 2

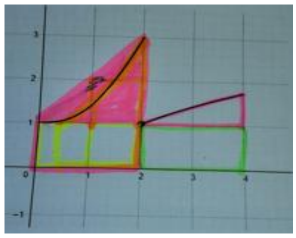


Fig. 3

Pour la deuxième question, sept étudiants ont utilisé la procédure de primitive pour calculer la valeur de l’intégrale proposée. Parmi eux, six étudiants ont choisi la méthode de changement de variables et n’ont pas terminé le calcul parce qu’ils ne sont pas arrivés à trouver la forme d’une fonction usuelle ; un étudiant a choisi l’intégration par parties, il lui semble qu’il s’agit de la recherche d’une primitive de la fonction donnée. Deux étudiants ont choisi de passer à la représentation graphique de la fonction à intégrer et ont utilisé l’aire sous la courbe pour répondre (Fig. 4 et Fig. 5). Le reste des étudiants n’a pas répondu à la question.

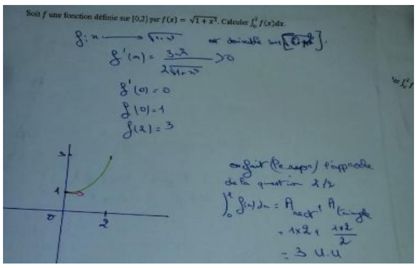


Fig. 4

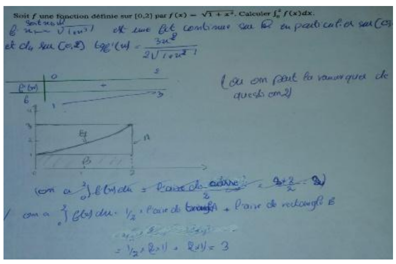


Fig. 5

La majorité des étudiants a choisi de calculer l’intégrale proposée dans la deuxième question par la recherche d’une primitive puis elle a appliqué le TFA. Ce résultat est attendu pour deux raisons. D’une part, pour eux il y a identification entre intégrale et calcul de primitive : pour la majorité d'entre eux, le TFA constitue la définition de l'intégrale et ils n'ont jamais utilisé d'autres méthodes pour calculer une intégrale (en répondant à l'une des questions proposées avant le test : la question I). D'autre part, les étudiants se limitent souvent aux connaissances procédurales nécessaires à la résolution algébrique de certaines intégrales, alors que le recours au graphique est une technique non reconnue. La représentation graphique est généralement utilisée pour interpréter l'intégrale en termes d'aire et non pour la fonder. Il faut noter que lorsqu'on a demandé aux étudiants pourquoi la résolution d'une intégrale définie par le calcul d'une primitive permet de calculer l'aire, aucun des étudiants n'a pu répondre à la question. Quelques étudiants (un étudiant dans la question 1 et deux étudiants dans la question 2 du test) ont utilisé la notion d'aire pour calculer l'intégrale cherchée. Ils l'ont approchée par des aires de surfaces polygonales (rectangle, trapèze, triangle). Bien que ce choix, qui se base sur la conception d'aire, inclue l'idée d'approximation, il est insuffisant pour donner la valeur exacte des intégrales cherchées dans les deux questions.

Aucun des étudiants n'a utilisé un raisonnement fondé sur les sommes de Riemann pour calculer l'intégrale définie de la deuxième question. Pourtant, tous ces étudiants avaient étudié à la fois les sommes de Riemann et l'intégrale définie. Lorsqu'ils ont été interrogés sur les deux notions (la question (III)), ils ont été conscients de l'existence d'une relation, mais aucun ne l'avait articulée. Certains étudiants ont considéré les deux notions comme deux méthodes différentes pour trouver l'aire. D'autres étudiants considèrent les sommes de Riemann comme une méthode d'approximation d'une aire, alors que le TFA permet de calculer sa valeur exacte. Parmi ces étudiants il y a quelques-uns qui ne pensent aux sommes de Riemann que lorsqu'il n'est pas possible de calculer une intégrale « *directement* » (c'est-à-dire par la recherche d'une primitive). La plupart de ces étudiants ont montré que leur connaissance des sommes de Riemann n'a aucune influence sur leur compréhension de l'intégrale définie.

CONCLUSION

La structure du concept d'intégrale est propice à l'utilisation de représentations multiples. A titre d'exemples, il y a des « problèmes de l'aire sous la courbe » qui peuvent être résolus en utilisant des graphiques, des « problèmes des sommes » qui peuvent être résolus dans le registre numérique et des « problèmes des intégrales des fonctions usuelles » qui peuvent être résolus en utilisant des procédures algorithmes se basant sur un raisonnement algébrique.

L'analyse du test nous a permis de souligner que la conception de primitive, qui sous-tend un aspect opératoire de l'intégrale, était la plus utilisée par les étudiants. Ils pensent que les procédures algébriques sont plus utiles dans le processus de résolution. Il faut noter que l'une des raisons qui pourrait les pousser à faire ce choix est la forte présence du raisonnement algébrique tout au long des cours précédents.

Bien que quelques étudiants considèrent que l'utilisation des procédures algébriques est utile pour résoudre quelques problèmes de l'intégrale définie et ne l'est pas dans d'autres (en répondant aux questions que nous avons posées), ils les ont utilisées même dans des problèmes qui devraient être résolus à l'aide d'autres techniques (la question 2 du test). Il reste à savoir pourquoi les étudiants se limitent à utiliser le registre algébrique et pourquoi ils ne réussissent pas à résoudre les problèmes qui nécessitent le travail dans d'autres registres. D'autres étudiants disent, en répondant aux questions posées avant le test, qu'ils n'ont pas rencontré des problèmes sur l'intégrale définie où la résolution nécessite l'utilisation des approches numériques au cours de leur processus d'apprentissage. Ainsi, ils considèrent que les représentations numériques ne sont pas nécessaires au processus de résolution des problèmes.

Beaucoup d'étudiants ont choisi d'autres représentations en répondant à une question particulière sur la représentation convenable pour traiter une tâche issue de l'intégrale définie, alors que, dans le processus de résolution de problèmes, il y avait une accumulation vers un type de représentation unique (algébrique). Cela a entraîné des incohérences entre les représentations utiles pour répondre et celles utilisées effectivement dans la réponse. Le nombre d'étudiants, qui ont basculé d'une représentation vers une autre, était considérablement limité. Cela est peut-être dû à l'enseignement qui s'appuie beaucoup sur les registres de représentations algébriques. D'une façon générale, cette étude nous a permis d'identifier deux types de conceptions :

-Une première conception, utilisée par un ensemble d'étudiants, se base sur le TFA pour calculer la valeur de l'intégrale recherchée. Il s'agit d'une conception de primitive qui se focalise sur un aspect algébrique et met en avant le statut processus. Cela signifie que cette conception perçoit l'intégration comme le processus inverse de la dérivée. En effet, cette conception limite les connaissances à un aspect opérationnel. Les étudiants ayant utilisé cette conception présentent des difficultés lorsqu'il s'agit d'un contexte non routinier comme dans le cas de la question 2 du test.

-Une deuxième conception interprète l'intégrale de Riemann comme une aire. Cette conception pourrait amener à utiliser deux procédures différentes correspondant chacune à un concept image évoqué. La première procédure se base sur l'approximation de l'intégrale recherchée par l'aire d'une surface polygonale ou une somme d'aires de surfaces polygonales. Cette procédure se limite à l'approximation par un nombre limité d'aires de surfaces polygonales ce qui ne permet pas de trouver de bonnes approximations. La deuxième procédure consiste à effectuer des subdivisions très fines à l'intervalle d'intégration pour mettre en œuvre un processus d'approximation puis, à appliquer la limite. Les étudiants, se basant sur cette procédure, utilisent à la fois le statut processus et le statut objet de l'intégrale ce qui permet de développer des connaissances d'ordre structural. Elle met également en œuvre la conversion entre les registres graphique/numérique.

La majorité des étudiants a développé des connaissances qui ne mettent pas en œuvre la structure sous-jacente de l'intégrale de Riemann. Elles sont dans leur majorité des connaissances d'ordre pseudo-structurel (Sfard et Linchevski, 1994). Il faut noter qu'il ne faut pas se limiter, pour la conception d'aire, au cas de fonctions positives. En effet les étudiants pourraient rencontrer des difficultés, lorsqu'il s'agit des fonctions négatives ou qui changent de signes. Les résultats de cette étude montrent également que l'interprétation de l'intégrale en termes d'aire ne pose pas de problèmes chez beaucoup d'étudiants, mais plutôt le recours à la procédure de processus d'approximation était une difficulté considérable.

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Nature et origine des difficultés des étudiants lors de l'utilisation des approximations locales des fonctions en début de l'université : Cas des classes préparatoires aux études d'ingénieurs

Belhaj Amor Fatma

Université Virtuelle de Tunis (ECOTIDI) et Université de Pau et des pays de l'Adour (LMAP), fatma.belhajamor@gmail.com

Résumé : *Cet article porte sur l'enseignement et l'apprentissage du concept d'approximations locales des fonctions. Il vise principalement à étudier la nature et l'origine des difficultés des étudiants lors de la mise en œuvre des outils – développements limités, formule de Taylor-Young - au début du cycle préparatoire aux études d'ingénieurs en Tunisie. Après avoir explicité les grandes lignes des programmes officiel, nous effectuerons une analyse des raisonnements des étudiants confrontés à une situation mathématique visant à évaluer leurs capacités à faire un usage raisonné de leurs connaissances dans le domaine des approximations locales des fonctions au voisinage d'un point et lors de l'étude de leurs comportements asymptotiques.*

Mots clés : *enseignement, apprentissage, raisonnement mathématique, approximations locales, développement limité.*

INTRODUCTION

En Tunisie, au début du cycle préparatoire aux études d'ingénieurs, les programmes en vigueur stipulent que les notions de développement limité, la relation de comparaison des fonctions et la formule de Taylor-Young doivent être enseignées afin de permettre la résolution de problèmes d'approximations locales de fonctions et de modélisations de phénomènes physiques relevant de différents domaines (mécanique, optique, etc.). Dans le cadre de notre mémoire de master (Belhaj Amor, 2016), les investigations conduites dans les domaines de l'histoire des mathématiques, de l'épistémologie, et de la didactique nous ont permis de conclure qu'au début du cycle préparatoire, le concept développement limité n'est pas introduit en tant que nouvelle technique d'approximation locale des fonctions, permettant d'articuler les différents types d'approches (cinématique, graphique, géométrique, analytique, algébrique), afin d'en faire usage dans des domaines intra et extra-mathématiques (Belhaj Amor, ibidem). Ainsi, nos travaux de recherche de master ont mis en évidence un phénomène important le fait que les difficultés éprouvées par les étudiants sont étroitement liées à la difficile conceptualisation des objets d'approximations locales des fonctions à l'entrée dans l'enseignement supérieur. Dans le domaine de l'étude des fonctions, certaines recherches en didactique ont permis d'établir que les difficultés des étudiants sont dues principalement à l'existence de ruptures lors de la transition secondaire/supérieur (Ghedamsi, 2016). Plus précisément, dès le début de l'université, l'approche algébrique fait obstacle à l'entrée dans le champ de l'Analyse et notamment à la conceptualisation des notions locales sur les fonctions (Vandebrouck, 2011). Ainsi lors de l'étude locale des fonctions au début de l'université,

> "…les étudiants traitent algébriquement les équivalents ou les développements limités, donnant très difficilement du sens aux expressions du type o(x) (…). Enfin, les étudiants ne tracent des graphes que quand la question leur est explicitement demandée et ils ne pensent pas spontanément à utiliser cette représentation des fonctions pour faire les raisonnements locaux attendus d'eux." (Vandebrouck, 2011, p.1-2)

Ces précédents travaux n'ont pas ciblé précisément l'analyse didactique des difficultés des étudiants lors de l'étude des approximations locales des fonctions dans la résolution des problèmes intra et extra mathématiques à l'entrée dans le supérieur.

Ces constats nous amènent à nous interroger sur la nature et l'origine des difficultés rencontrées par les étudiants lors de la mise en œuvre des connaissances et des savoirs inhérents aux approximations locales des fonctions en classes préparatoires aux études d'ingénieures tunisiennes (IPEI), dans la section Physique-Chimie.

METHODOLOGIE GENERALE ET CADRES THEORIQUES

Notre travail de recherche porte sur l'enseignement et l'apprentissage des outils d'approximations locales des fonctions. Plus précisément, nous souhaitons étudier les difficultés rencontrées par les étudiants lors de la formulation et l'utilisation des développements limités d'une fonction au voisinage d'un point, de la relation de comparaison des fonctions et de la formule de Taylor-Young.

Notre méthodologie repose sur l'élaboration de trois situations mathématiques - assimilables à des énoncés de problèmes – élaborées en étroite collaboration avec l'enseignante de la classe. Ces situations ont été proposées aux étudiants en avril 2019, dans le cadre d'une évaluation formative et formatrice, en vue de déterminer leurs acquis et leurs difficultés inhérentes aux approximations locales des fonctions.

En collaboration avec l'enseignante, nous avons proposé en classe cette évaluation écrite constituée de trois situations mathématiques à 2 classes, chacune composée de 22 étudiants.

Ce devoir écrit fait suite à l'enseignement des chapitres "*Analyse asymptotique*", "*Intégration*" et "*Séries numériques*". La modalité de passation de l'évaluation est la suivante : chaque étudiant travaille seul afin de produire les réponses aux questions. Les étudiants disposent de 75 minutes pour rédiger leur composition.

Dans cet article, compte-tenu des contraintes éditoriales, nous avons choisi de mettre la focale sur la première situation proposée aux étudiants. D'une part, car elle a été traitée par une majorité d'entre eux, d'autre part par la richesse et la variété des réponses produites.

Nous allons réaliser, dans le cadre de la Théorie des situations didactiques, l'analyse a priori de cette première situation. Ensuite, nous effectuerons l'analyse a posteriori de cette situation ; nous analyserons les productions des étudiants en nous attachant à étudier leurs raisonnements produits en réponse aux questions.

Pour cela, nous adoptons le point de vue de Brousseau et Gibel (2005) qui ont proposé une classification des raisonnements des élèves, en situation de résolution de problèmes, selon leur(s) fonction(s) : organiser sa démarche, décider des connaissances à mobiliser, effectuer un changement de cadre, décider d'un changement de registre, formuler une explication, formuler une justification, interpréter le résultat d'un calcul, contrôler la validité du résultat obtenu.

Comme indiqué dans les travaux de Bloch et Gibel (2011), l'analyse des raisonnements produits par les étudiants nécessite d'avoir recours à une analyse sémiotique afin d'analyser les signes produits, leurs usages et leurs transformations. Nous utiliserons ainsi les notions de registres de représentations sémiotiques (Duval, 1993) afin d'identifier d'éventuels changements de registres.

Pour compléter l'étude des raisonnements, nous effectuerons une analyse en termes de dimensions sémantique et syntaxique (Kouki, Belhaj Amor et Hachaichi, 2016), (Bloch et Gibel, 2011). Dans notre cas, la syntaxe fournit des règles de transformation des expressions analytique et algébrique dans un raisonnement mathématique. Dans certains cas, sa satisfaction nécessite un contrôle sémantique prenant en compte aussi les interprétations et vérifications.

La prise en compte de ces trois composantes du raisonnement (fonction, dimension sémiotique et nature) nous permettra d'analyser les différents types de raisonnements élaborés par les étudiants afin d'identifier la nature et l'origine des erreurs commises dans leurs raisonnements erronés (Gibel, 2018).

Afin de déterminer les connaissances antérieures de l'étudiant sur les concepts d'approximations locales des fonctions, nous commençons par une présentation des programmes officiels de la quatrième année secondaire (section Sciences Expérimentales) et la première année des classes préparatoires (section Physique-Chimie). Ensuite nous conduisons une analyse de notre corpus constitué des 44 productions des étudiants selon deux axes dont le premier se rapporte sur l'étude des fonctionnements et nature des raisonnements. Le deuxième axe est lié à une analyse en termes de répertoire didactique et plus précisément les connaissances et savoirs mobilisables.

PRESENTATION DES PROGRAMMES OFFICIELS

Avant d’analyser les productions des étudiants, nous conduisons une étude des programmes afin de déterminer les éléments du répertoire didactique de la classe, défini par Gibel (2004), qui se décompose de deux types d’objets le "*registre des formules*" qui est la collection des formules et le "*système organisateur*" qui permet d’organiser et d’utiliser ce répertoire didactique (Gibel, 2018).

Etude du programme officiel de la quatrième année du secondaire de la section Sciences expérimentales (SC-E)

Au lycée, l'objectif principal du programme de la quatrième année secondaire (SC-E) confère à l’enseignant une mission principale celle d'aider l'élève à utiliser son répertoire didactique constitué des algorithmes et des procédures faisant appel aux technologies de l’information et la communication (logiciel, calculatrice, etc.) ainsi que l’interprétation des illustrations graphiques. Par ailleurs, l'enseignant est invité à aider l'élève à développer sa démarche de raisonnement par la rédaction et l’explication orale de la résolution des problèmes à travers l’interaction avec ses collègues. Le texte du programme impose le recours aux registres graphique et géométrique par des illustrations graphiques ou par des logiciels pour introduire des nouveaux objets. En recours au contenu du programme lié à nos objets d'étude, les approximations locales des fonctions algébriques et transcendantes sont présentées via les approximations affines des fonctions ainsi que les différentes formes d'équations de tangente obtenues à partir du nombre dérivé. Ainsi, l'enseignement des concepts d'approximations locales des fonctions nécessite l'articulation des différents registres afin de développer le raisonnement de l'élève par la rédaction et l'explication orale d'un raisonnement mathématique.

Etude du programme de la première année Physique-Chimie (PC)

Le texte officiel du programme actuel (2016) de première année PC commence par une introduction précisant la mission de l’enseignant qui se réalise selon deux axes. Le premier est du côté "*objet mathématique*" par la mise en jeu des connaissances antérieures des étudiants et l’intérêt d'introduire des nouvelles notions dans les domaines intra et extra mathématiques. Le deuxième axe est du côté "*raisonnement mathématique*" par la mise en jeu de la vérification des différentes étapes d’une démonstration ou un raisonnement mathématique en utilisant des éléments de la logique, des langages mathématique, des techniques fondamentales de calcul en Analyse, des règles de calcul et d'outils logiciels dans certains cas des situations nécessitant l'explication avec des illustrations graphiques. Par ailleurs, l'objectif principal de l'enseignement du chapitre "*Analyse asymptotique*" est d'amener l'étudiant à maitriser les techniques asymptotiques à travers des calculs asymptotiques simples, la détermination des développements limités des fonctions et la résolution des problèmes que la vérification des propriétés des nouvelles notions et surtout liées à la notion de relation de comparaison. L'enseignant est invité à mobiliser les registres graphique et géométrique par l'utilisation des illustrations graphiques en s'appuyant sur les outils logiciels dans le cas des

> "…situations dont la gestion manuelle ne relèverait que de la technicité seront traitées à l'aide d'outils logiciels" (PO, p.8) [1]

Les savoirs à enseigner dans ce chapitre sont les concepts de relation de comparaison des fonctions, le développement limité et la formule de Taylor-Young afin de déterminer des développements limités usuels. L'objet développement limité est un nouvel outil pour le calcul de l'équivalent et de limite, l'étude locale d'une fonction et de son comportement.

En conclusion, l'objectif principal de ces programmes est de ramener l'élève à la rédaction autonome d'un raisonnement ou d'une démonstration mathématique en articulant les dimensions sémantique et syntaxique par la mobilisation des différents registres graphique, géométrique, analytique, algébrique et numérique.

ANALYSE EXPERIMENTALE

Avant de présenter l'analyse a priori de la première situation selon un plan didactique et l'analyse de notre corpus constitué des 44 productions des étudiants, nous allons commencer par la présentation de l'énoncé de la situation étudiée.

Enoncé de la situation-problème

On considère la fonction f définie par : $f(x)=\frac{x}{x-1}\sqrt{x^2+1}$

1) On s'intéresse à faire une étude locale de f en 0.

a. Donner le $DL_2(0)$.

b. En déduire $f^{'}(0)$ et $f^{''}(0)$.

c. Déterminer l'expression de la tangente Δ à C_f passant par le point $(0, f(0))$; préciser la position de C_f par rapport à Δ.

2) Etude de f en $+\infty$:

a. Montrer que l'on a : $f(x)=x+1+\frac{3}{2x}+\frac{1}{x}\varepsilon(\frac{1}{x})$ avec $\varepsilon(\frac{1}{x})\to 0$ quand $x\to+\infty$

b. Déduire une fonction équivalente à f en $+\infty$.

c. Préciser le comportement de f en $+\infty$.

3) Préciser de même le comportement de f en $-\infty$.

Eléments d'analyse a priori de la situation étudiée

Ce problème a été élaboré afin d'étudier la capacité des étudiants à articuler les différents objets d'approximations locales des fonctions en vue de réaliser l'étude locale des fonctions au voisinage d'un réel et le comportement d'une fonction en $+\infty$ et en $-\infty$, ainsi que l'intérêt de l'objet développement limité en tant que nouvel outil pour résoudre certaines questions traitées au secondaire.

Sur le plan didactique, la situation est assimilable à un problème de réinvestissement des connaissances antérieures relevant des trois chapitres étudiés. Nous pouvons distinguer quatre variables didactiques :

VD_1 : La nature de la fonction à étudier, nous avons choisi de proposer une fonction sous forme d'un produit des fonctions admettant des développements limités usuels.

VD_2 : Les voisinages autours desquels les étudiants doivent déterminer les développements limités de la fonction. Dans l'évaluation, il s'agit d'étudier les développements limités au voisinage de 0, en +∞ et en -∞.

VD_3 : Le choix de l'ordre des développements limités usuels.

VD_4 : La durée laissée aux étudiants pour résoudre les problèmes dans le cadre de l'évaluation.

L'étudiant en s'appuyant sur ses connaissances antérieures va établir une action sur les objets qui est motivée par son répertoire didactique.

Présentation et analyse des principaux résultats expérimentaux

Nous allons présenter les principaux résultats de l'analyse des productions des étudiants de chacune des questions proposées dans cette situation.

Question 1-a

Nous remarquons l'existence des difficultés inhérentes à la justification du raisonnement. La majorité des étudiants ont produit des raisonnements purement syntaxiques. En effet, ils ne contrôlent pas l'ordre auquel chacun des développements limités doit être réalisé. Par ailleurs, l'usage direct de la formule (*) du répertoire didactique de la classe, par certains étudiants dans les étapes de calculus des développements limités de $(x-1)^{-1}$ et $(1+x^2)^{\frac{1}{2}}$, met en évidence l'erreur du signe (-).

(*) $(1+x)^{\alpha} = 1+\alpha x+\dfrac{\alpha(\alpha-1)}{2!}x^2+o(x^2)$

Question 1-b

Certains étudiants ont un raisonnement de nature syntaxique. Ils n'ont pas justifié l'application de la formule de Taylor-Young. En effet, deux étudiants seulement, parmi les 22 admettant des réponses valides, ont vérifié que la fonction f est de classe C^2 sur tout intervalle I un voisinage de 0.

La majorité des étudiants ont des difficultés d'ordre technique. D'un côté, un nombre assez-important d'entre eux ont utilisé une méthode n'ayant aucune relation avec le répertoire didactique de la classe. Ils ont calculé les dérivées successives de développement limité de la fonction f au voisinage de 0 afin de donner les valeurs de $f'(0)$ et $f''(0)$. D'un autre côté, certains étudiants ont présenté directement des valeurs fausses. Ainsi, par leurs réponses erronées, ces étudiants ne donnent pas de sens au développement limité et la formule de Taylor-

Young comme une approximation locale de la fonction f par un polynôme de degré 2 dans le but de déterminer les valeurs de $f'(0)$ et $f''(0)$. Finalement, certains étudiants ont recours à leurs connaissances antérieures, vues au lycée, par l'utilisation de la technique [$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$] pour calculer la valeur de $f'(0)$.

Question 1-c

Nous remarquons que rares sont les étudiants qui ont justifié leur raisonnement. En effet, ils ont écrit directement l'équation de la tangente et sa position par rapport à la courbe représentative de la fonction f. Par ailleurs, certains étudiants ont recours à leurs connaissances anciennes, par l'utilisation des techniques vues au secondaire, pour déterminer l'équation de la tangente à partir du calcul du nombre dérivé, ainsi que sa position par rapport à sa courbe représentative en étudiant le signe de $(f(x)-y)$. En revanche, d'autres étudiants articulent des connaissances du secondaire et du supérieur. Ils ont obtenu l'équation de la tangente par le calcul du nombre dérivé en mettant en valeur le rôle de développement limité pour déterminer sa position par rapport à la courbe C_f.

Question 2-a

La majorité des étudiants mobilisent leurs connaissances antérieures par l'usage direct de la technique de changement de variable et la formule (*) des développements limités. Leur raisonnement est d'un aspect purement syntaxique. Tous les étudiants ont utilisé l'expression de la notion de voisinage [$o(\frac{1}{x})$] au lieu celle de [$\frac{1}{x}\varepsilon(\frac{1}{x})$] donnée dans la question. A partir des raisonnements erronés, nous pouvons identifier plusieurs difficultés liées au signe et à la notion de voisinage. La recherche de développement limité de $f(\frac{1}{x})$ traduit l'existence de difficulté liée à la notion de voisinage. Par ailleurs, certains étudiants font usage de technique de changement de variable et de la formule des développements limités au voisinage de 0 (*) du répertoire didactique de la classe. Mais on relève des erreurs dans les étapes suivantes de leur raisonnement soit par le calcul des produits des développements limités à travers l'élimination du reste, soit par l'écriture de reste $o(x)$ au lieu de $o(\frac{1}{x})$.

Question 2-b

La plupart des étudiants ont un problème de justification de leur raisonnement. Ils donnent directement la fonction équivalente. Par ailleurs, certains étudiants ont justifié leur raisonnement par le calcul des limites des différents termes du développement asymptotique afin d'obtenir la fonction équivalente.

Le calcul de la limite du rapport de $f(x)$ par x, par certains étudiants, traduit la difficulté de l'usage de la technique de supérieur. Les étudiants, par leurs raisonnements erronés, n'ont pas recours à leurs connaissances antérieures pour obtenir une fonction équivalente par la mise en considération des notions de la relation de comparaison et le développement limité d'une fonction en +∞.

Question 2-c

A partir de notre corpus, un quart des étudiants ont un raisonnement de nature syntaxique. La majorité d'entre eux font appel à leur anciennes connaissances de secondaire pour la détermination de l'équation de l'asymptote d'une part et, ils remplacent la fonction f au cours de calcul de limite des fonctions [$f(x)$, $\frac{f(x)}{x}$ et ($f(x)-x$)] soit par son développement asymptotique, soit par sa fonction équivalente, d'autre part. En revanche, un seul étudiant donne l'intérêt de l'objet développement limité en faisant usage de la technique de supérieur afin de décider que la droite d'équation $y = x + 1$ est l'asymptote oblique.

La majorité des étudiants, par leurs raisonnements erronés, ont rencontré des difficultés d'ordre technique liées à leurs connaissances antérieures de secondaire concernant l'étude du comportement d'une fonction. Par ailleurs, d'autres étudiants ont un problème pour employer la formule de fonction équivalente.

Question 3

La majorité des étudiants ont des problèmes à expliquer leur raisonnement ce qui les met en difficulté. En effet, ils donnent directement le développement limité de la fonction f en -∞, puis la fonction équivalente et finalement ils étudient son comportement. Certains d'entre eux considèrent que le développement limité en -∞ est l'opposé de celui en +∞ soit par l'ajout simplement du signe (-) dans cette expression, soit par le changement de variable x par (-x) dans cette représentation analytique, soit par l'étude de la parité de la fonction f. Cette erreur commise est due à l'incompréhension de la notion de développement limité. Par ailleurs, un nombre assez-important d'étudiants a utilisé le même développement limité en +∞ pour étudier le comportement de la fonction f en -∞. D'un autre côté, certains étudiants ont un problème lié à leurs connaissances antérieures du secondaire concernant l'étude du comportement d'une fonction. En effet, ils voient qu'il est suffisant d'étudier la parité de la fonction f ou d'effectuer le calcul de sa limite en -∞ pour préciser son comportement.

Rares sont les étudiants ayant des raisonnements valides. En effet, ils cherchent le développement limité en -∞, puis la fonction équivalente et finalement l'équation de l'asymptote en articulant des techniques de secondaire et du supérieur. Ces étudiants ont un problème de justification de raisonnement.

En conclusion, nous pouvons identifier l'origine et la nature des difficultés des étudiants lors de la résolution des problèmes dans le champ de l'étude des approximations locales des fonctions selon quatre catégories :

-Difficulté d'ordre conceptuel : liée à la notion de voisinage (le reste, changement de variable, etc.). Dans ce cas, l'étudiant est confronté à un problème pour appliquer une formule du répertoire didactique de la classe afin de déterminer le développement limité d'une fonction au voisinage de 0, en $+\infty$ et $-\infty$.

-Difficulté d'ordre technique : l'étudiant utilise soit ses anciennes connaissances et plus précisément, une technique vue au secondaire, soit une méthode fausse n'ayant aucun lien avec le répertoire didactique de classe. Dans ce cas, il néglige le rôle des nouveaux concepts d'approximations locales des fonctions.

-Difficulté d'ordre justificatif : l'étudiant a problème de justification de son raisonnement

-Difficulté d'ordre calculatoire : Dans ce cas, l'erreur est due aux erreurs du calcul. Nous présentons nos résultats dans le tableau ci-dessous :

Difficultés d'ordre	Conceptuel	Technique	Justificatif	Calculatoire
Question 1-a	16	0	24	4
Question 1-b	2	22	20	0
Question 1-c-1	8	8	17	7
Question 1-c-2	15	6	6	2
Question 2-a	34	0	9	1
Question 2-b	21	0	5	1
Question 2-c	15	24	1	1
Question 3	36	5	1	1

Tableau 1 : *Nature et l'origine des difficultés des étudiants*

CONCLUSIONS GENERALES ET PERSPECTIVES

A l'issue de l'étude des programmes, l'enseignement des objets d'approximations locales des fonctions articule les dimensions sémantique et syntaxique par la mobilisation des registres graphique, géométrique, algébrique et analytique afin de ramener l'étudiant à la rédaction autonome d'un raisonnement.

La majorité des étudiants ont construit des raisonnements de nature syntaxique articulant les approches algébrique et analytique. Les difficultés éprouvées par les étudiants sont étroitement liées à la difficile conceptualisation des objets d'approximations locales des fonctions en première année PC.

Notre questionnement nous amène à réfléchir à l'élaboration et la mise en œuvre d'une ingénierie didactique en PC à travers une situation permettant aux étudiants d'élaborer des raisonnements permettant de conjuguer différents cadres et d'articuler différents registres de représentation sémiotique.

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Continuity of real functions in high school: a teaching sequence based on limits and topology

Laura Branchetti[1], Giovanni Calza[1], Sara Martani[1] , and Alberto Saracco[1]

[1]University of Parma, Department of Mathematical, Physical and Computer science, laura.branchetti@unipr.it

It is well known that students have difficulties with the concept of continuity, specifically on points of discontinuity, and concepts like limits and infinity. In Italian textbooks, the continuity of functions is usually defined using limits, while an intuitive characterization of continuous functions is proposed without providing the students with formal tools to use it, like "the graphs of continuous functions can be drawn without lifting the pencil out of the paper". Limits are one of the most complex subjects to learn and are usually introduced in an algorithmic way, without a true comprehension of the subject. We argue that introducing the definition of continuous functions using limits is problematic and we designed and tested a teaching sequence to investigate the potentiality of including a topological approach in high school.

Keywords: Transition to and across university mathematics, Teaching and learning of analysis and calculus, Topology, Concept image, Intuitive models.

INTRODUCTION

Italy has a K13 scholastic system and an introduction to Calculus and Analysis is proposed at the end of high school (18-19 years), in particular in *Liceo Scientifico*. Students are taught continuity, limits and series in different ways in secondary school and university, sometimes with inconsistencies between the two approaches (Trigueros, Bridoux, O'Shea and Branchetti, accepted). Thus there is a typical problem of transition from secondary school to university. It is well known that there are various difficulties with the concept of continuity, specifically on points of discontinuity, limits and infinity. In Italian textbooks, the continuity of functions is usually defined using limits and an intuitive characterization of continuous functions as graphs with no holes, that can be drawn with a pencil without lifting the pencil out of the paper (Bagni, 1994), is proposed in the beginning but never deeply analyzed. On the contrary, after an intuitive approach, traditionally secondary school teaching of continuous functions aims, from the very beginning, to provide the students with the most general and formal conception of continuity, given by a formal epsilon/delta definition. In this approach the intuitive characterization of continuity based on the properties of the graph are considered a crutch to abandon as soon as possible because it can be a source of future mistakes (Bagni, 1994). We wish to problematize such a tradition.

Moreover, Italian textbooks propose problematic characterization of points of discontinuity, since they are usually ambiguous in linking domain, accumulation points, definition of continuity at a point and global continuity of the function. Furthermore, often continuity is presented as a global, instead of local, property and there is an inconsistency with the formal limit approach that leads some students to say, for instance, that to evaluate the global continuity of a function it is necessary to compute the limit in every point of the domain.

We decided to investigate the following research question: whether and how a different approach, using a mixed topological and analytical approach, could result in a better conceptualization of continuity and limits, which yields more students to correctly identify continuous functions and points of discontinuity? We proposed the following teaching sequence: 1. introduce continuity in a topological way, without using limits; 2. introduce the concept of limit; 3. link the two concepts, both formally and with concrete examples.

In this paper, we present the design of a teaching sequence, a general overview of the learning outcomes of an implementation of the sequence in grade 12 and, finally, a comparison with a class of grade 13 who had been taught with the traditional Italian approach. As we will show, a topological approach might provide the students with useful images and methods to classify functions in many cases. This study had promising outcomes: indeed, in the final tests of the grade 12 compared to the initial test of grade 13, we observed fewer students' misconceptions about continuity and limits and a greater ability to manage the semiotic transformation to keep under control in the solving processes. However the connection between limits and continuity was problematic and a refinement of the teaching sequence is necessary.

LITERATURE REVIEW

Two of the most investigated topics in University Mathematics education are the continuity of functions and the difficulties with limits in the undergraduate courses (Trigueros et al., accepted). Tall and Vinner (1981) showed some typical students' conceptions of continuous functions that cause many undergraduate students' difficulties, showing that the effectiveness of their use of definitions, also in simple tasks, is usually far from the expectations. The formal epsilon/delta definition of continuity is powerful enough to evaluate also "pathological" examples, like the Dirichlet function, but it is very far from the students' concept images (Tall & Vinner, 1981) reported in the literature about continuity (Hanke & Schafer, 2017) and many research results show that limits in many cases do not become conceptual tools that improve students' approach to the evaluation of continuity of functions. Moreover, this formal approach is usually not suitably motivated to the students, since it is used only in tasks which would be solvable with a more intuitive approach.

The literature shows that, with the traditional approach, based on intuition on one side and on the formalization based on limits on the other side, the students' concept images about continuous functions are often rooted in specific examples and conflict with the formal definition (Tall & Vinner, 1981). Hanke and Schafer (2017) listed the following seven possible mental images that students use as criteria to discuss the global continuity of a function that are reported in the literature:

I: *"A graph of a continuous function must be connected"*

II: *"The left hand side and right hand side limit at each point must be equal"*

III: *"If you wiggle a bit in x, the values will only wiggle a bit, too"*

IV: *"Each continuous function is differentiable"*

V: *"A continuous function is given by one term and not defined piecewise"*

VI: *"The function continues at each point and does not stop"*

VII: *"I have to check whether the definition of continuity applies at each point".*

For what concerns limits, many researchers showed that this is one of the most complex concepts to learn, and that it is usually introduced in an algorithmic way, without a true comprehension of the subject (Trigueros et al., accepted). Several epistemological and cognitive aspects must be considered in order to face the critical issues that characterize their learning. In particular, considering limits of functions, some aspects have been shown as crucial: the potential and actual conceptions of infinity (Tsamir & Tirosh, 1992) and the difficulties caused by metaphors and some uses of the natural language. Dimarakis and Gagatsis (1997) consider the interactions between the mathematical language and the natural language and note how the expressions "tends to the limit", "approaches" and "converges" are mathematically equivalent, but are not in the everyday language: "approximates" and "tends to", often used as a synonym of "approaching", does not suggest situations related to limits but reinforce a dynamic interpretation. In the case of limits of functions, as is the case for the convergence of sequences, the dynamic conceptions are very resistant (Williams, 1991). Teaching should aim at turning the dynamic representations of students into static conceptions, or at least to scaffold the students' approach to limits making them aware of the relationship between the two aspects (Trigueros et al., accepted).

RESEARCH FRAMEWORK

Often mathematics is identified with the precision of rules and a discipline where concepts can be defined in an accurate way so as to build a rigorous theory based on definitions, formal statements and proofs. However, a large amount of research about mathematics learning argues the necessity of an intuitive and informal base for the concepts to be used by the students as thinking tools and support a formal learning; a pre-existing cognitive structure lies in the mind of every person and, when the student is presented a concept, naturally different personal mental images are evoked, before the formal definition can be accepted. We will use the term *concept image* (Tall & Vinner, 1981) to describe the complete cognitive structure linked to the concept, which includes all mental images as well as related processes and properties. Different stimuli can activate different parts of the concept image; we will call the part of the concept image activated in a precise moment the *evoked concept image*. As the concept image develops, it is not guaranteed it will be coherent; thus when different (and contradictory) parts of it are evoked simultaneously, a sense of confusion emerges. According to Fischbein (1993), while formally there is no difference between accepting a proof and accepting the universality of the assertion, for the pupil the two things usually do not coincide. To pursue the intuitive acceptance of formal reasonings it is necessary to adopt a didactical approach that permits the students to mix and merge different ways of reasoning and make sense of formal statements and proof, connecting them to other kinds of discourses that can activate the intuitive and personal cognition at a different level. According to Lecorre (2016), three types of rationalities are necessary for understanding the learning processes of students in the study of limits:

- *Pragmatic rationality* consists strictly in examining specific cases; there is no attempt to generalize observations.
- *Empirical rationality* is used when a general law is to be obtained; the facts are used to deduce generalizations.
- *Theoretical rationality* begins with theory (theorems, properties, definitions, axioms ...) to establish new properties and theorems.

We relied on this framework to design the activities to introduce limits, encouraging students to connect reasonings of different kinds, bridging empirical and pragmatic rationality with the theoretical one. The notion of representation was also an important reference in the design phase, since representations play a crucial role in the acquisition and the use of the individual's knowledge. As Duval (1995) points out presenting his Theory of Register of Semiotic Representation: "There's no knowledge that can be mobilised by an individual without a representation activity" (p. 15). The main assumptions of the theory are:

1. there are as many different semiotic representations of the same mathematical object, as semiotic registers utilised in mathematics;

2. each different semiotic representation of the same mathematical object does not explicitly state the same properties of the object being represented;

3. the content of semiotic representations must never be confused with the mathematical objects that these represent.

RESEARCH METHODOLOGY

The goal of the research was to check whether and how the didactical approach we designed (including topology as element of the theory, using a methodology oriented to strengthen the students' reasoning with limits intertwining different rationalities and paying attention to semiotic transformations) could help the students to deal with basic tasks about continuity of functions better than the traditional approach of teachers of *Liceo Scientifico* (Scientific High School) and textbooks in Italy.

First of all, we carried out an analysis of Italian high school textbooks. Then we prepared a questionnaire to investigate the students' conceptions in grade 13 (21 students) after a traditional teaching sequence. Then we designed our teaching sequence and, after collecting data about students' conceptions in two classes in grade 12 (38 students), we implemented the teaching sequence with the same students. The design of the teaching sequence was based on: the results from the literature review resumed before, the initial test about students' concept images and the analysis of the audio-recordings lesson by lesson. Finally we carried out a final test, with common tasks in two classes, one in grade 12 and one in grade 13. The questionnaire included open questions and tasks where students were asked to compute limits at the extreme points of the domain and evaluate the continuity of functions, providing explanations. In particular, we collected data about the students' images of continuous functions and limits, their use of concept images of limits and infinity (potential and actual) and the students' ability to manage semiotic transformations in tasks about continuity. Since in grade 13 the students' reasonings had not been clear in some cases, we added in grade 12 one question (task 8, that we discuss later), asking to compute the limits, to state and explain if the function was continuous and then to identify its (possible) points of discontinuity.

We analysed quantitatively the correct/incorrect answers and we compared the initial test carried out in grade 13 after a traditional teaching with the final test in grade 12 in one class who attended our course, in the same school. The results are not statistically significant but informed us about the potentiality of our approach. Then, we analyzed the whole set of data looking for students' concept images and concept definitions of accumulation points and continuity, comparing them with the students' outcomes, to check the consistency and the efficacy of such images and definitions in the students' solving processes. We also checked the students' abilities to manage semiotic representations in a fruitful way. Finally we explored whether and how the students integrated fruitfully limits and the topological approach to continuity.

DIDACTICAL TRANSPOSITION AND ENGINEERING

In Italian High School textbooks it is usually given the following definition:

A function f is said to be continuous in a point x_0 if the limit of f(x) as x goes to x_0 coincides with the value f(x_0).

The problems begin with the negation of the aforementioned definition: a function it is said to be discontinuous at a point x_0 if the function is defined but not continuous there or if the point is an accumulation point of the domain but the function is not defined in x_0, thus calling point of discontinuity also points out of the domain. Textbooks then introduce various types of discontinuity (Fig. 1):

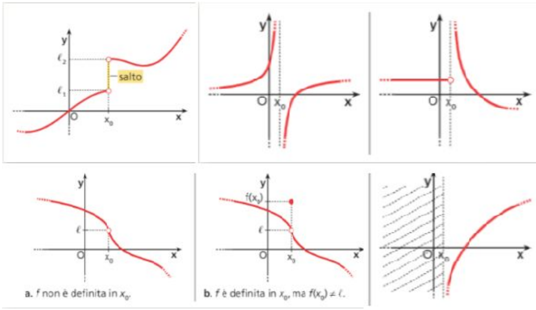

Fig. 1: Discontinuities represented in an Italian textbook

In our teaching sequence, we started carrying out 5 interactive frontal lessons (1 hour each) about continuity from a topological point of view. We decided to introduce continuous functions defined on subsets of R, using a transposition of the definition of connectedness by arcs and neighbourhoods, coming to this definition:

A one real variable function (whose domain $D \subseteq R$ has a finite number of path-connected components) is continuous if the number of the connected (by paths) components of the domain and of the graph are the same.

This definition (path-connected function) is equivalent to that of continuous function in R whenever the domain of the function is locally path-connected and simply-connected (Hanke, 2018). The difference between the two is that this definition, from a didactical point of view, allow to use the graphical representation of the function to evaluate the continuity and to find the possible points of discontinuity (by using a local version of the definition in a neighbourhood of the point) and should help not to confuse a point outside of the domain with a point of discontinuity. Furthermore in this case the negation of continuity is more straightforward. We hypothesized that this approach could reduce the students' misconceptions about continuity and points of discontinuity and provide them with a powerful resource to use facing tasks about continuity of real functions.

Then, we carried out 5 lessons (1 hour each) about infinity, limits and sequences, using a didactical methodology based on the intertwining of rationalities (Lecorre, 2016). We included several examples and definitions from the history, to promote gradually an intuitive acceptance of the formal definitions: Archimedes' *Measure of the circle* and the proof by exhaustion, the Paradox of Achilles and the tortoise, periodic numbers, a graphical representation of the geometric series, an original piece from Cauchy's *Cours d'Analyse* (1821, first definition of limit). To gradually connect a formal definition of limit with the students' concept images of functions, we introduced and discussed with the students some examples of sequences and we showed to the students the limit for n which tends to infinity from the dynamic point of view, using the graphic representation of functions and the numerical representation. Then we moved to a more static approach through the concept of accumulation point.

In the final 2 hours, we matched the two different approaches and we linked the topological idea of continuity with the traditional definition based on limits, discussing with the students about continuity and points of discontinuity using limits, to show to them that the two approaches lead to the same conclusions.

DATA ANALYSIS

Analysing the questionnaires only in terms of correct/incorrect answers, the comparison between the tests in grade 13 and in grade 12 showed that the students in grade 12 had in general better outcomes. The most common error of the 13-grade students was to classify continuous functions as discontinuous. For instance, in grade 13, no one classified the equilateral hyperbola function f(x) = 1/x continuous in its natural domain, while many students in grade 12 answered correctly. The following three tasks concerning continuous functions are good examples of the questions we asked in the questionnaire; 7 and 9 were asked in grade 12 and 13, while 8 was only in the grade 12 test. In task 7 the function is continuous in its domain, while 8 and 9 have points of discontinuity. In task 7 and 8 we also asked to infer the limits at the extreme points of the "natural domains" from the graph.

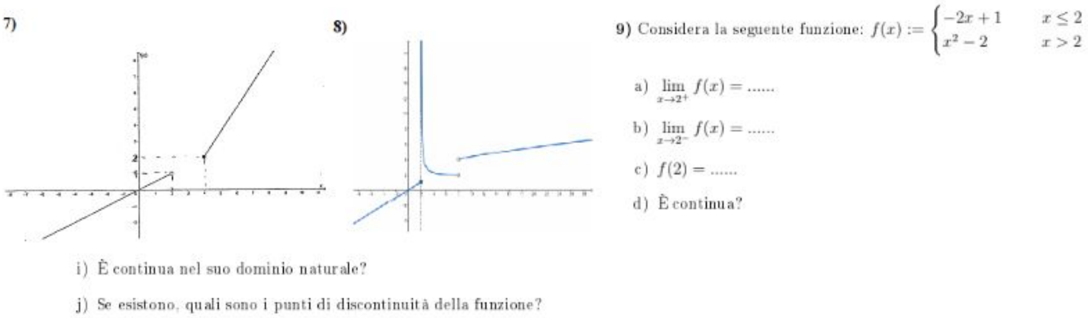

Fig. 2: Questions about continuity and points of discontinuity

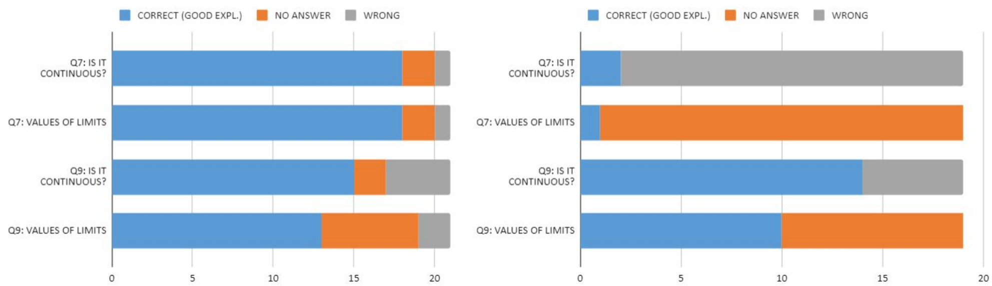

Fig. 3: Students' outcomes in the tasks about continuity in grade 12 (left) and 13

Comparing the answers to tasks 7 and 9 (19 students, grade 13; 21 students, grade 12), it emerged more students in grade 12 answered correctly that a function was continuous (task 7, graphical) rather than discontinuous (task 9, analytical), while the opposite happened in grade 13. Only 2 on 19 in grade 13 recognized the first as a continuous function, while 18 on 21 in grade 12 did it. In task 9 the trend is the opposite, even if the difference is much smaller; 13-grade students who were taught to classify continuity with limits in many cases answered without computing limits.

Looking at the sheets, we observed that while in grade 13 no students made a semiotic transformation from the analytical to the graphical representation in task 9, 13 students drew the graph of the function and 12 of them answered correctly. Even if we did not find evidence of reasonings about limits and continuity, in grade 12 the same students that had problems with discontinuities also made mistakes in the computation of limits (even if there are correct answers and errors with limits).

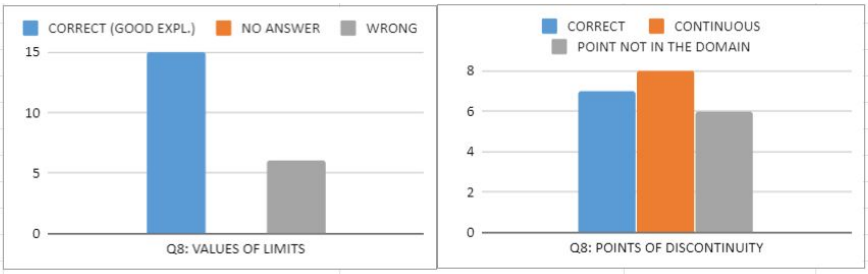

Fig. 4: Students' outcomes in the tasks about discontinuities in grade 12

Task 8, proposed in grade 12, was the most problematic: it had a discontinuity point in 2 and a "false" discontinuity in 8, that was not an element of the domain; 6 students considered 8 a discontinuity point, and 4 of them correctly classified the function in task 9. In these last 4 cases, we are not sure if students carried out a correct reasoning, since they could have classified the function discontinuous even if the point was not in the domain. 8 students considered the function continuous.

We also asked the students the question: "How would you explain in your own words what is a continuous function?". Looking at their answers we identified two further criteria that were not included in the list by Hanke and Schafer (2017):

"A continuous function has R as domain."

"The number of the components of the domain is equal to the number of the components of the graph".

The last criterion, that is the one proposed in the lessons, was expressed by 11 (out of 21) students of grade 12 out and was used effectively to discriminate continuity.

In order to check what kind of conceptualization of limits the same students had developed, before analysing their link to continuity of functions, we analysed the audio-recording of lessons looking bottom-up for emerging relevant students' concept images. A basic recurrent image was that of the asymptote: the limit that "approaches but does not reach a point" was often linked to the concept of asymptote, known to students from grade 11 thanks to the study of hyperbole and homographic functions in analytic geometry. Also the images of the "full and empty cue ball" was often used by the students, for example when they explained what is difference between $0,\overline{9}$ and 1, or to say if a point belonged (full) or not (empty) to the graph of a function. Moreover this image was used to explain if it made sense or not to compute the limit at a point (if the cue ball is empty it makes sense to compute a limit). In both cases the students' tried to recall previous images and use them to create connect limits and functions before our final lesson about the connection limits-continuity. When students were required to connect the concept of limit and that of point of accumulation, inconsistencies appeared between the students' images; many of them did not understand that it did not make sense to calculate the limit for x tending to x_0 if it is not a point of accumulation; moreover, in most cases they said it did not make sense to compute limits at a point in the domain.

Such images did not prevent the students from answering correctly questions about the continuity in task 7, and some students identified correctly discontinuities using the graphs even making errors with limits (1 in task 8, 6 in task 9), but, even if there are no explicit reasonings carried out by the students using limits to identify discontinuities, they seemed quantitatively to affect their answers to determination of the points of discontinuity, in particular where the students had to reason on the graph about the asymptote and the empty-full cue balls. Moreover some students in the questions about limits wrote that the limit of the functions does not exist *since* the function is discontinuous at the point. This point should be explored more.

DISCUSSION AND CONCLUSIONS

As regards the continuity of functions, the comparison between grade 12 and 13 showed that the topological approach was useful to improve the students' use of graphical representations to state the continuity of functions that was at the basis of the better outcomes in grade 12 rather than grade 13. This result encourages us to carry out further experimentations refining the didactical engineering and the data collection in order to grasp in a more accurate way the students' reasonings. However, some students had still problems determining discontinuities; in particular, some students considered points of discontinuities also points not in the domain and in task 8 and 9 classified discontinuous functions as continuous, contrary to the usual trend observed with the traditional approach. These errors seemed to be due to difficulties to visualize the components of the domain, linked to the images of accumulation points, and to classical concept images stressed by Tall & Vinner (1981), like "a jump in the functions implies the function to be not continuous". This result made an important aspect arise: the topological approach is useful if the students are able to count the components of the domain, but in the most relevant cases it requires a good image of accumulation points that is demanding for students; however, this is also a problem of the traditional approach.

Students in general overcame the classical dichotomy potential/actual infinity in most cases and had good outcomes in the questions about limits. However, only in a few cases we observed an explicit connection between limits and points of discontinuity and we were not able to reach clear results about it. It seems that the values of limits are not used by the students, but concept images of limits of functions, built spontaneously by the students recalling previous practices, could have influenced them to face the tasks about continuity, since quantitative data showed a correlation between the answers about limits and the correct identification of discontinuities. The connection between topological continuity and limits should be deeply improved in further experimentations.

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Suites définies par récurrence dans la transition lycée-université : activité et travail mathématique

Macarena Flores González[1], **Fabrice Vandebrouck[2] et Laurent Vivier[3]**

[1, 2, 3] *Université de Paris, France & LDAR*

prof.macarena.flores@gmail.com

Nous présentons une première étude qui permet de rendre compte des ruptures et des continuités dans la transition lycée-université à partir de l'analyse d'une tâche emblématique de cette transition qui vise à étudier une suite définie par récurrence de la forme $u_{n+1}=f(u_n)$. Les outils théoriques utilisés (issus de la théorie des Espaces de Travail Mathématique et de la Théorie de l'Activité) nous permettent de fournir des analyses de la tâche a priori et a posteriori. Nous constatons que les attentes de l'enseignement actuel par rapport à ces suites, se placent sur une analyse algébrisée de la notion. Cela nous amène à proposer des variations dans l'activité et le travail mathématique à atteindre lors de l'étude de ces suites, qui en l'occurrence, pourraient servir à introduire les élèves à l'analyse réelle.

Mots clés : Suites récurrentes, transition lycée-université, activité mathématique, travail mathématique, instruments.

INTRODUCTION

Nous avons comme objectif d'étudier les ruptures et continuités existantes dans la transition lycée-université dans le domaine de l'analyse, à partir des tâches évaluatives qui se trouvent à la fin des cours de l'enseignement secondaire et au début de l'enseignement universitaire. Ces tâches caractéristiques visent à étudier les suites définies par récurrence de la forme $u_{n+1}=f(u_n)$; nous nous intéressons à ses potentialités notamment dans l'étude de la notion de convergence, et analysons leur place dans l'enseignement actuel. D'autre part, l'objet mathématique de suites que nous avons choisi, nous semble être un objet mathématique pertinent pour étudier le passage du *calculus* à l'analyse, car il permet de travailler dans une analyse algébrisée (comme l'étude de variations de fonctions, applications des théorèmes de convergence de suites, etc.), des notions essentielles du début de l'analyse réelle qui sont encore en construction chez les étudiants (comme la limite, la convergence, les nombres réels, etc.), jusqu'aux problèmes que l'on étudie en analyse réelle comme les systèmes dynamiques discrets. Ainsi, concernant l'épistémologie, le choix des suites définies par récurrence comme objet mathématique d'étude dans l'enseignement actuel n'est pas anodin. En effet, ce type de suites a une place épistémologique importante dans le développement du domaine de l'analyse en mathématiques ; d'ailleurs comme le mathématicien D. Perrin[1] l'affirme, un des intérêts d'étudier ces suites repose sur la recherche des points fixes par itérations, ce qui permet de montrer des résultats d'existence en analyse.

En didactique, la transition lycée-université a fait l'objet de nombreux travaux (pour une synthèse, voir Gueudet, 2008). Certains résultats restent encore d'actualité ; tel est le cas de la thèse de Praslon (2000) qui construit des tâches pour cette transition, en prenant en compte le contexte éducatif des deux institutions et qui identifie des micro-ruptures dans cette transition par rapport à la notion de dérivée. D'autres travaux sont également à relever : ceux de Robert concernant l'étude de suites avec les représentations que les étudiants expriment sur la convergence (dynamiques et statiques), et son ingénierie didactique pour établir une représentation de la convergence chez les étudiants (ingénierie reprise par Bridoux (2016)).

Concernant les suites qui nous intéressent, on trouve le travail de Boschet (1982) qui fait une étude au niveau de la première année de l'université. Elle constate que les exercices sur les suites $u_{n+1}=f(u_n)$ sont stéréotypés et utilisent le lexique le plus pauvre en comparaison à d'autres suites. L'auteur remarque que ces suites sont artificiellement conçues pour appliquer certains théorèmes, et que si nous faisons appel aux mêmes connaissances de façon régulière, cela pourrait provoquer une représentation erronée de la convergence.

Plus récemment, Ghedamsi et Fattoum (2018) ont étudié l'évolution des images mentales chez les élèves de 3ème année de secondaire, formées en amont et en aval de la définition de convergence de suites numériques. Elles signalent qu'au moment d'étudier la convergence, les techniques des opérations sur les limites finies de suites « *pourraient renforcer les pratiques d'algébrisation dans le travail des élèves, sans qu'aucun apport significatif dans le processus de conceptualisation de la convergence ne puisse être entrepris* » (Ibid., p. 232). Ce renforcement dans l'algébrisation dont parlent Ghedamsi et Fattoum, nous semble important à prendre en compte et à considérer dans l'étude de la convergence de suites. Ainsi, nous nous demandons : quels sont les attendus de l'institution lycée et de l'institution université en ce qui concerne les suites définies par récurrence $u_{n+1}=f(u_n)$? Et comment l'enseignement actuel dans la transition lycée-université se sert de cet objet mathématique pour conduire les élèves d'une analyse algébrisée vers une entrée dans l'analyse réelle ?

CADRES THEORIQUES

Quand on étudie la transition lycée-université, il s'agit d'un domaine d'étude particulièrement complexe par le nombre de variables à considérer et les problématiques épistémologiques et cognitives, lesquelles supposent des analyses élaborées. Cela nous amène à considérer deux cadres théoriques pour l'analyse de ces deux institutions éducatives et des tâches qu'elles proposent. Nous plaçons au centre de notre étude l'analyse du travail mathématique que chaque institution cherche à développer chez les étudiants, et le travail mathématique que les étudiants développent effectivement. De cette manière, nous utilisons la théorie des Espaces de Travail Mathématique (Kuzniak et al., 2016) qui s'intéresse aux aspects épistémologiques de l'objet mathématique en question (avec l'étude du plan épistémologique en regardant le référentiel théorique, les artefacts mis à disposition et les signes qui interviennent dans le travail de résolution de la tâche) ; mais qui permet aussi de comprendre comment ces composants épistémologiques se mettent en place lorsqu'un sujet les utilise (avec l'étude du plan cognitif en analysant la preuve et le discours mathématique de l'élève, sa construction et la visualisation mathématique de l'objet). Les relations qui existent entre ces deux plans se décrivent à partir des dimensions *sémiotique*, *instrumentale* et *discursive*, et on identifie les plans verticaux [Sem-Dis], [Ins-Dis] et [Sem-Ins] pour caractériser les tensions et relations entre ces dimensions. D'autre part, nous prenons en compte les différents niveaux d'ETM : ETM de référence (permettant d'analyser le travail mathématique visé par l'institution), ETM idoine (qui est une adaptation de l'ETM de référence mis en place par l'enseignant/professeur) et ETM personnel (qui permet de décrire le travail mathématique développé par le sujet). Dans le cadre de la théorie des ETM, Kuzniak, Tanguay et Elia (2016) signalent que les mathématiques enseignées sont en priorité une activité humaine.

Nous faisons l'étude de l'activité humaine et nous approfondissons des aspects de l'individu dans son contexte et d'autres aspects cognitifs grâce à la Théorie de l'Activité (TA) en didactique des mathématiques. La TA se base sur une idée cognitiviste des processus d'enseignement et d'apprentissage, s'appuyant sur les idées des psychologues comme Leontiev et Vygotsky. En didactique des mathématiques, ces travaux ont été enrichis par des didacticiens (pour une synthèse voir Vandebrouck, 2018) permettant de caractériser l'activité mathématique des élèves (ainsi que l'activité des enseignants à partir d'une double approche ergonomique et didactique). Dans cette étude, on s'intéresse aux outils théoriques de la TA qui vont nous permettre de rendre compte de l'activité attendue par l'institution (en analysant la tâche prescrite) et les traces de l'activité effectuée par les élèves (en analysant la tâche effective). Ainsi, nous analysons l'étude de la tâche en prenant en compte : le contexte dans lequel elles sont proposées, les buts, les connaissances (ses adaptations[2] et le niveau de sa mise en fonctionnement) et les types de sous-activités développées par un sujet en activité (*reconnaissance*, *organisation de raisonnement global* et *Traitement*).

Paradigmes de l'analyse

Dans les travaux qui s'intéressent à l'étude du domaine de l'analyse réelle en didactique, nous relevons la notion de *paradigmes de l'analyse* (Montoya et Vivier, 2016). On dégage trois types de paradigmes : « *AI* » qui permet des interprétations provenant de la géométrie, de l'arithmétique ou du monde réel ; « *AII* » qui permet de faire des calculs avec des règles plus ou moins définies et les appliquer sans avoir un travail de pensée critique par rapport à la nature des objets utilisés ; et « *AIII* » où les propriétés et définitions sont bien établies , où on développe un travail en termes d'approximation et voisinage (un travail avec ε), caractérisé par l'implication d'inégalités, de bornes, et du « négligeable ». Ces paradigmes vont nous permettre de différencier et caractériser le type de travail mathématique mis en place.

L'articulation entre les deux cadres théoriques

Nous constatons que les deux théories accordent une importance aux considérations épistémologiques sur les tâches, car la connaissance mathématique joue un rôle indispensable dans l'activité et dans le travail que l'on développe ; ainsi, nous accordons une place prépondérante à la vigilance épistémologique de notre sujet d'étude. Par ailleurs, grâce à une étude précédente (Flores González, 2019), nous avons pu voir que la notion de tâche est un élément didactique essentiel pour analyser le travail et l'activité mathématique (au sens de chacune des théories). D'une part, nous nous centrons sur la notion de « tâche emblématique », qui sert à identifier l'importance des tâches dans le développement et la description du travail mathématique ; ainsi elles doivent « *bénéficier d'une reconnaissance institutionnelle, être utilisées dans les classes ordinaires et doivent permettre de réaliser, au moins potentiellement, un travail mathématique complet*[3] » (Kuzniak et Nechache, 2016). D'autre part, cette tâche nous sert à caractériser l'ETM attendu à la fin du lycée, et l'ETM attendu en première année d'université pour les suites récurrentes $u_{n+1}=f(u_n)$. Enfin, nous cherchons à caractériser la continuité et les ruptures dans l'activité et le travail attendus de l'élève dans la transition lycée-université. En termes théoriques la question qui guidera notre travail sera : Quelles sont les caractéristiques de l'ETM et de l'activité à la fin du lycée et au début de l'université concernant les suites récurrentes $u_{n+1}=f(u_n)$?

METHODOLOGIE

Nous faisons une étude cognitivo-épistémologique de la tâche emblématique grâce aux deux cadres théoriques choisis avec une analyse a priori et posteriori des tâches prescrites. Pour aborder notre question et ainsi approfondir la reconnaissance de ruptures et continuités de cette transition, notre tâche emblématique est construite à partir des tâches d'évaluation de la fin du lycée de l'enseignement scientifique (le Baccalauréat[4] S) et des évaluations du début de l'université (des examens de la fin du premier semestre) en France. Ainsi, nous avons conçu une tâche pour le cours de la fin du secondaire scientifique (FSS) qui prend en compte le modèle proposé par le Baccalauréat mais aussi le contexte évaluatif en début de l'université (DU) à partir de tâches modèles que nous détaillons ci-dessous.

Nos tâches modèles à l'université et à la fin d'études de secondaire

Notre première tâche modèle (Figure 1) a été extraite d'un examen de DU (de 1er année de licence) de l'année 2018. Il s'agit d'une suite de récurrence non linéaire à un terme (homographique). Les connaissances à mettre en place sont principalement du paradigme de l'analyse AII (ou issues d'une analyse réelle algébrisée). De cette tâche nous gardons le choix de la suite pour construire une tâche pour la classe de FSS.

Exercice 5. Math/Math-Info/Physique/CPEI/DLPC/STEP

Soit $\alpha \in]0,1[$ et soit $f : [0,1] \longrightarrow \mathbb{R}$ la fonction définie par $\forall x \in [0,1],\ f(x) = \frac{1}{2-x}$.

On considère une suite $(u_n)_{n\in\mathbb{N}}$ vérifiant les conditions : $u_0 = \alpha$ et $\forall n \in \mathbb{N},\ u_{n+1} = f(u_n) = \frac{1}{2-u_n}$.

1. Montrer que, pour tout $x \in [0,1]$, $f(x) \in [0,1]$. Puis que pour tout $n \in \mathbb{N}$, $u_n \in [0,1]$.
2. Montrer que la suite $(u_n)_{n\in\mathbb{N}}$ est croissante.
3. Montrer que la suite $(u_n)_{n\in\mathbb{N}}$ converge.
4. Quelle est la limite de $(u_n)_{n\in\mathbb{N}}$?

Figure 1 : Tâche d'examen du DU, 2018.

Notre deuxième tâche modèle est un exercice de Baccalauréat de 2016 (Figure 2). Cette tâche se compose de deux parties : La première partie se réfère à l'étude de la fonction $f(x)=x-ln(x^2+1)$ en commençant par la résolution de l'équation $f(x)=x$, puis l'étude des variations de f (grâce au tableau de variations déjà complété dans l'énoncé), et l'étude de la stabilité de l'intervalle [0,1] par f. Dans la deuxième partie de l'exercice, on commence à étudier la suite comme le montre la figure 2 (Partie B).

Partie B

Soit (u_n) la suite définie par $u_0 = 1$ et, pour tout entier naturel n, $u_{n+1} = u_n - \ln\ (u_n^2 + 1)$.

1. Montrer par récurrence que, pour tout entier naturel n, u_n appartient à [0,1].
2. Étudier les variations de la suite (u_n).
3. Montrer que la suite (u_n) est convergente.
4. On note $\boldsymbol{l}$ sa limite, et on admet que $\boldsymbol{l}$ vérifie l'égalité $\boldsymbol{l} = f\ (\boldsymbol{l})$. En déduire la valeur de $\boldsymbol{l}$.

Figure 2 : Exercice 3 Baccalauréat Scientifique Métropole 2016.

Pour adapter la tâche de DU à la classe de FSS, on a adopté la façon dont la tâche est présentée dans le modèle du Baccalauréat, et la suite choisie est celle de la tâche proposée dans l'examen de DU. Pour rester dans le contexte de FSS, on a fait un découpage dans l'étude de la suite, c'est-à-dire que d'abord on étudie la fonction (partie A) et ensuite on fait l'étude de la suite (partie B) (Figure 3). Les changements qui ont été faits se situent principalement au niveau des aides et registres sémiotiques fournis. Par exemple, dans l'énoncé du Baccalauréat la question A.2) fournit le tableau de variations comme aide ; ou encore dans la question B.1) il est écrit « *montrer par récurrence que…* » (ce changement a été fait pour ne pas s'éloigner de la tâche telle qu'elle a été présentée en DU).

PARTIE A : Soit f la fonction définie sur $\mathbb{R} - \{2\}$, par $f(x) = \frac{1}{2-x}$.

1) Résoudre dans $\mathbb{R} - \{2\}$ l'équation : $f(x) = x$.
2) Étudier le sens de variation de f.
3) Montrer que pour tout réel x appartenant à [0,1], $f(x)$ appartient à [0,1].

PARTIE B : Soit α un réel appartenant à]0,1[. On considère la suite (u_n) définie par $u_0 = \alpha$ et, pour tout entier naturel $n, u_{n+1} = f(u_n)$.

1) Montrer que pour tout entier naturel n, u_n appartient [0,1].
2) Étudier le sens de variation de la suite (u_n).
3) Montrer que la suite (u_n) converge.
4) On note l sa limite, et on admet que $l = f\ (l)$. En déduire la valeur de l.

Figure 3 : Tâche adapté à la classe FSS.

Comme nous voulons avoir accès aux ETM personnels et aux activités individuelles, nous analysons des productions des élèves dans les deux institutions éducatives. En ce qui concerne la classe de FSS, nous disposons des productions écrites des 10 élèves qui sont susceptibles de poursuivre des études à l'université (selon l'enseignant). Par rapport aux données de début de l'université, nous disposons de 40 productions d'étudiants de l'examen de DU (Figure 1) proposé en CPEI (cycle de préparatoire aux écoles d'ingénieurs).

Analyse a priori des tâches du lycée et de l'université

Comme exemple, on explique l'analyse a priori fournie avec les deux cadres théoriques choisis de l'énoncé : « *Montrer que pour tout* $n \in \mathbb{N}, u_n \in [0,1]$ » (première partie de l'étude de la suite, nous faisons l'analyse à partir de la stratégie[5] prévue). Il est à remarquer que la façon dont cette tâche a été proposée au lycée et à l'université est différente. Au lycée, la stabilité de l'intervalle par f a été abordée dans la partie A et l'étude de la bonne définition de la suite est présentée dans la partie B de l'énoncé, tandis qu'à l'université, ces deux études ne sont pas séparées.

De la part de la TA, on différencie le contexte de la tâche des deux institutions en question. Pour le lycée, le raisonnement par récurrence a été étudié au début d'année de la classe FSS, et les exercices des suites récurrentes sont présents dans les manuels. Au début de l'université, la démonstration par récurrence n'a été un objet d'étude ni dans les cours magistraux, ni dans les cours de travaux dirigés (TD). Ensuite, nous identifions comme **but de la tâche** le fait de montrer que tous les termes de la suite sont bien définis. En ce qui concerne les **connaissances** à utiliser, elles sont les mêmes pour les deux institutions : raisonnement et preuve par récurrence. Ainsi les **adaptations de ces connaissances** seraient : la reconnaissance de l'utilisation de la preuve par récurrence, l'introduction des étapes classiques de la preuve par récurrence, et l'utilisation du résultat de la question précédente, notamment pour montrer la stabilité [0,1] par f. Le **niveau de mise en fonctionnement** de ces connaissances est *mobilisable*, car on n'indique pas dans l'énoncé d'utiliser le raisonnement par récurrence, mais on dit « montrer que pour tout naturel n … ». Finalement on repère les **sous-activités mathématiques** de *reconnaissance* (du raisonnement par récurrence) et de *traitement* (des données algébriques, leur travail et leurs implications, et les étapes comme traitement de la preuve par récurrence).

De la part de l'ETM personnel, on privilégie la **dimension discursive** en mettant en amont un discours de preuve par récurrence. Cette preuve est ancrée dans le référentiel théorique du sujet et utilise des raisonnements issus de la récursivité et du formalisme liés à ce type de preuve.

À partir de l'analyse a priori de la tâche entière, nous avons pu identifier les continuités et les ruptures dans la transition lycée-université à partir des deux cadres théoriques. Pour les continuités, on garde les connaissances à mettre en place en commun au lycée et à l'université (tel est le cas de la démonstration par récurrence, l'étude de variation de la suite à partir de la méthode $u_{n+1} - u_n$, ou encore le théorème de convergence de la limite monotone). Nous repérons que l'ETM personnel attendu présente presque les mêmes types de signes à utiliser (la plupart dans le registre algébrique), et en général, c'est le traitement de ces signes qui va permettre un discours mathématique cohérent, ainsi que des signes utilisés en tant qu'outils (les artéfacts symboliques) qui vont permettre de résoudre la majorité des tâches. Ainsi, on voit que dans les deux institutions on privilégie un travail mathématique dans la dimension discursive. Du côté de la TA, les aspects en commun aux deux institutions sont pour la plupart les buts des tâches qui seraient les mêmes pour l'étude de la suite (à l'exception de la question 3 pour étudier le sens de variation de la suite).

Concernant les ruptures, du côté des ETM, les plans privilégiés vont changer à partir de la stratégie choisie pour répondre aux tâches. D'autre part, étant donné que la tâche est découpée en deux parties au lycée (étude de la fonction puis étude de la suite), la façon dont la tâche est conçue fait produire des circulations entre les différents plans de l'ETM, tandis qu'à l'université on ne prévoit pas des circulations provoquées par l'énoncé, et cela resterait à la charge de l'élève à partir de la stratégie choisie. En termes de TA, cette rupture se traduit à partir des aides données dans les énoncés des tâches, tel est le cas de la dernière question concernant la limite de (u_n) : On voit qu'au lycée on admet que $l=f(l)$ et puis l'élève doit « *en déduire* » la valeur de l, alors qu'à l'université on demande juste la limite de (u_n). Ce fait produit une rupture au niveau de sous-activités attendues, car pour le cas du lycée il s'agit d'un *traitement* (qui est simple car les élèves doivent juste résoudre l'équation), et à l'université cela va demander des sous-activités de *reconnaissance* et d'*organisation* (l'étudiant doit reconnaître la méthode qui n'est pas donnée dans l'énoncé, et bien organiser ses connaissances pour justifier sa réponse). En termes généraux pour la TA, les ruptures entre le lycée et l'université se décrivent aussi à partir des différences dans les adaptations des connaissances, car à l'université les étudiants ont plus de choix de méthodes pour les différentes tâches. Finalement, la rupture des contextes des tâches est aussi importante : dans le cas de la classe du DU les étudiants trouvent dans la feuille des cours TD, 3 exercices parmi 25 sur les suites $u_{n+1}=f(u_n)$, et le mot récurrence n'est évoqué que pour nommer la suite ; alors qu'au lycée la tâche est présente dans les manuels de la classe de FSS, et la démonstration par récurrence fait un objet d'étude du cours.

Éléments de l'analyse a posteriori et premiers résultats

Dans cette analyse on montrera l'impact des continuités et ruptures dans les productions des étudiants. Concernant les connaissances présentes au lycée et à l'université on voit des changements quantifiables dans la mobilisation de ces connaissances : pour la question 1 liée à la démonstration par récurrence, elle est mobilisée par 50% des élèves au lycée (de 10 élèves au total), tandis qu'à l'université on trouve seulement 22,5 % (même si elle est non formalisée – de 40 élèves au total). Au lycée 80% des étudiants mobilisent le théorème de la limite monotone, alors que le taux est de 37,5% à l'université. Du point de vue des ETM, le plan vertical qui a été privilégié dans le travail mathématique des élèves et des étudiants a été effectivement [Sem-Dis] avec des traitements sémiotiques dans le registre algébrique, et la dimension discursive – cet aspect nous semble important dans les résultats généraux sur la tâche (seulement 3 des 40 étudiants réussissent en DU). Nous y reviendrons dans les conclusions. Du point de vue des ruptures et des aides qui n'ont pas été données, nous relevons le cas de la question 4, car au lycée cette tâche est réussie par 90% des élèves, alors qu'à l'université seulement 7,5 % des étudiants l'ont réussie. Un des éléments qui peut être le plus lié aux ruptures est le contexte de la tâche, car cela va déterminer les différentes adaptations de connaissances disponibles susceptibles d'être utilisées dans les différentes stratégies adoptées. Mais ce contexte va aussi jouer son rôle lors de l'appel aux connaissances, qui ont peut-être été cataloguées comme acquises (comme le cas de la démonstration par récurrence), ce qui expliquerait le décalage de l'utilisation de cette preuve au lycée et à l'université.

Comme erreur répétitive dans les deux institutions, on trouve dans la question 3 l'application du théorème-en-acte « *si f est croissante,* (u_n) *est croissante* » où le domaine de validité est seulement quand $u_n=f(n)$[6] (ce théorème a été appliqué par un 30% d´élèves au lycée et à l'université). D'autre part, la non maitrise du registre algébrique dans l'étude des variations de la suite (question 2) produit des blocages dans le travail mathématique des élèves. Il s'avère qu'aucun étudiant de lycée n'a eu une reconnaissance de l'identité remarquable pour déterminer la croissance de la suite, comme l'exemple de la figure 4 (qui, au-delà de son erreur, fait un travail de décomposition intéressant pour analyser le sens de variation de la suite), et qu'à l'université seulement 12,5% l'ont reconnue.

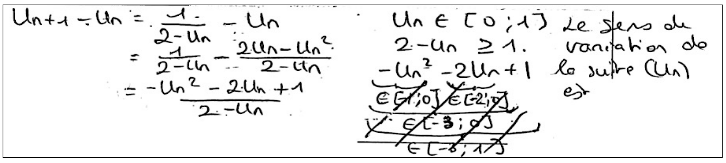

Figure 4 : Production à la question 3 d'un élève de lycée (E1 FFS).

Une dernière erreur fréquente est le fait d'établir des égalités entre l'objet suite et l'objet fonction comme $f(x)=(u_n)$, ou encore entre x et n. À l'université 22,5% des étudiants font cette erreur de façon explicite. On montre la réponse des questions 2, 3 et 4 de l'étudiant E26 (figure 5). D'abord pour le sens de variation de la suite, il détermine sa croissance, mais son tableau de variation montre une confusion dans les objets suite et fonction ; aspect que l'on note aussi dans la question 4 par rapport à l'égalité entre les limites.

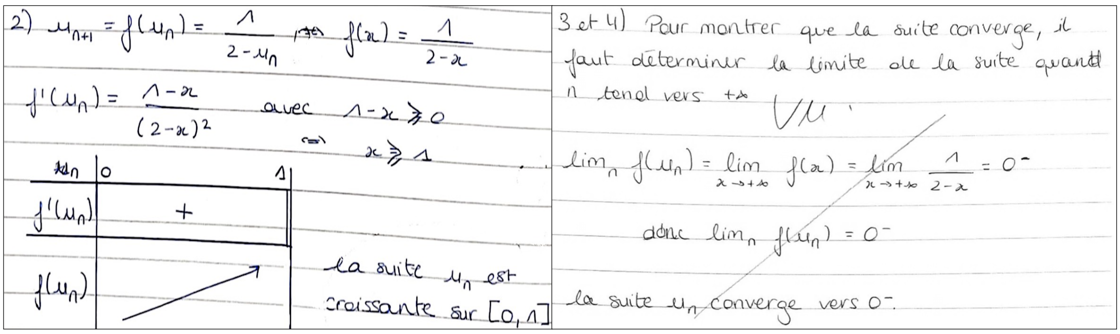

Figure 5 : Production d'un étudiant aux questions 2, 3 et 4 à l'université (E26 DU).

CONCLUSION ET PERSPECTIVES

Grâce à cette étude, nous avons pu obtenir quelques éléments de réponses à nos questions et constater certains effets de cette transition. D'une part, les caractéristiques de l'ETM attendu dans l'étude de ces tâches emblématiques se basent principalement dans le plan [Sem-Dis]. La dimension sémiotique implique surtout des traitements de type algébrique (qui parfois bloquent le travail des élèves) et la dimension discursive implique l'utilisation des méthodes et du référentiel théorique en tant qu'outil.

Les résultats dans les productions des étudiants nous permettent de repérer certaines progressions dans les apprentissages, mais aussi des régressions. Ainsi, cette étude nous montre que l'articulation entre les objets « suite » et « fonction » a une faible cohérence lors de l'étude de suites récurrentes $u_{n+1}=f(u_n)$; et telle que la tâche a été conçue, elle n'exploite pas les potentialités d'une étude contrôlée en autonomie des élèves, avec des notions de l'analyse réelle qui puissent tendre vers le paradigme AIII. Nous pensons que nous devrions appuyer l'apprentissage des élèves à l'entrée de l'université à partir d'un soutien dans la réorganisation des connaissances (qui est nécessaire), en s'appuyant sur des connaissances anciennes pour pouvoir ajouter et organiser des nouvelles. Nous faisons l'hypothèse que cela pourrait être rendu possible par un enrichissement de l'activité et du travail mathématique via l'introduction du paradigme AI (pour aller ensuite vers le paradigme AIII) et en mettant l'accent sur la visualisation de la suite. Pour ce faire, nous comptons d'abord promouvoir un travail mathématique dans le plan [Sem-Ins] à partir d'artefacts (comme des calculatrices ou logiciels) pour que les élèves exploitent les capacités technologiques qu'ils ont développées au lycée. Nous espérons ensuite pouvoir analyser leur pertinence pour évaluer la possibilité de *contrôler* l'activité et le travail mathématique dans la transition lycée-université. Finalement, nous pensons que faire une étude plus approfondie de la convergence (des comparaisons de la vitesse de convergence avec d'autres suites par exemple) et de la distinction de la nature entre suites, pourrait aider à réduire les problèmes et erreurs dans cette transition concernant l'étude des suites définies par récurrence.

NOTES

1. *Sur les suites récurrentes*. Cours préparation au CAPES. https://www.math.u-psud.fr/~perrin/CAPES/analyse/Suites/suites-re%CC%81currentes.pdf
2. A. Robert (1998) identifie 7 types d'adaptations de connaissances : de reconnaissance (A1) ; d'introduction d'intermédiaires (A2) ; des mélanges de plusieurs cadres ou notions, mises en relation ou interprétation, changements de points de vue (A3) ; d'introduction d'étapes par rapport aux calculs ou raisonnements (A4) ; utilisation des questions précédentes dans un problème (A5) ; l'existence de choix (A6) ; et le manque des connaissances nouvelles (A7).
3. Travail mathématique complet : les plans cognitif et épistémologique des ETM sont en relation et les circulations des plans verticaux rendent compte d'une diversité dans le travail.
4. Le baccalauréat est l'examen qui termine les études secondaires en France.

5. Stratégie prévue : *Initialisation :* $u_0 \in [0,1]$. *Hérédité : Soit* $m \in \mathbb{N}$, *on suppose* $u_m \in [0,1]$. *Par résultat de question précédente* $f(u_m) \in [0,1]$, *donc* $u_{m+1} \in [0,1]$. *Conclusion :* $u_n \in [0,1]$ *pour tout* $n \geq 0$).

6. Le théorème valide c'est : « si f est croissante, (u_n) est monotone ».

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Gesture and diagram production as tools for identifying the key idea in topology proving tasks

Keith Gallagher[1] and Nicole Engelke Infante[1]

[1]West Virginia University in Morgantown, WV, USA, kngallagher@mix.wvu.edu

Undergraduates in an introductory topology course participated in a series of study sessions in which they were asked to prove or disprove statements. We present a case study of one student who alternated between using gestures and constructing diagrams when communicating her informal ideas while proving true statements. This cycle was repeated until she identified the key idea of the proof, at which point she began to translate this idea into a formal proof. We observed that the use of gesture combined with diagram modification to explore a heuristic idea supported her identification of the key idea and subsequent completion of a written proof.

Keywords: Teaching and learning of logic, reasoning and proof, Teachers' and students' practices at university level, Topology, Key ideas, Gesture.

INTRODUCTION

Advanced mathematical thinking is often communicated as formal proof. Learning to construct proofs is a critical part of the undergraduate mathematics major curriculum. Many studies have shown that students struggle to construct valid proofs (Weber, 2010; Weber & Alcock, 2004). The concepts involved in these proofs often have multiple representations, both formal and informal. Among the informal representations are gestures and diagrams, which students use to develop and communicate their insights about a problem. These insights can lead to the use of formal notations and logical structures that we see used by the mathematics community. Here, we present a study in which we examine how gesture informs diagram construction and the discovery of the key idea of a proof.

Proof is a form of advanced mathematical discourse, i.e., how we communicate advanced mathematical ideas in written and spoken forms. Research has shown that mathematicians often begin with an image (Carlson & Bloom, 2005; Gallagher & Engelke Infante, 2019; Zazkis, Dubinsky, & Dautermann, 1996). McNeill observed, "…language is inseparable from imagery. The imagery in question is embodied in the gestures that universally and automatically occur with speech. Such gestures are a necessary component of speaking and thinking." (McNeill, 2005, p. 15) Roth (2000) argued that because students are unfamiliar with scientific discourse, their gestures precede their verbal discourse.

Further, Yoon, Thomas, and Dreyfus indicated "gestures are a useful, generative, but potentially undertapped resource for leveraging new insights in advanced levels of mathematics." (2011, pp. 891-892). They advised that students should be provided opportunities to spontaneously create gestures (in small group problem solving sessions) and that instructors model gestures for students in lecture. We created an environment in which students were encouraged to engage in mathematical discourse in order to observe how imagery and gestures may influence the development of formal mathematics. In this paper, we examine how gesture and diagram construction facilitate discovery of the key idea in topology proofs when students are working in small groups.

THEORETICAL PERSPECTIVE

Embodied Cognition

We frame our work by viewing mathematics as a semiotic bundle and taking an embodied cognition perspective. A semiotic bundle is made of signs, such as words (oral and written), gestures, drawings, graphs, and technological devices, that are used by people engaged in a discourse. Arzarello, Paola, Robutti, and Sabena (2009) noted,

> The novelty of the semiotic bundle…is that it allows us to describe the multimodal semiotic activity of subjects in a holistic way as a dynamic production and transformation of various signs and of their relationships. In particular, it properly frames the role of gestures in mathematical activities. (p. 100)

There is a large body of work in cognitive science focused on embodied cognition, which posits that knowledge is shaped by our experiences and interactions with the world around us (Lakoff & Nunez, 2000; Nunez, 2008). Through bodily experiences, such as gesture, our understanding of complex concepts is shaped. Roth (2000) suggested that "…schools (and universities) may be ideal 'laboratories'… to study the genesis of formal discourses (e.g., science and mathematics) and the role gestures play during this development." (p. 1712). While there have been many studies of the effects of gesture on younger students' learning of mathematics (Alibali & Nathan, 2012; Goldin-Meadow, Cook, & Mitchell, 2009), there have been considerably fewer of undergraduate students in advanced mathematics classes. This study documents how students interact, through speech and gesture, with the inscriptions they create as they work to construct proofs.

Similar to Bjuland, Cestari, and Borgersen (2008) and Arzarello, et al. (2009), we examine discourse, gesture, and inscriptions; however, our focus is on students of university age. Roth and McGinn (1998) noted that "When inscriptions are absent from face-to-face encounters, conversational troubles may quickly arise." (p. 43). They further pointed out that representing is a social activity and that interpreting inscriptions is challenging for students. More recently, de Freitas and Sinclair (2011) proposed that diagrams and gesture are intrinsically linked - diagrams are a way to capture gesture. The diagrams that are constructed as a result of gesture are a public inscription that captures embodied (personal) knowledge. Hence, we seek to understand how students (collaboratively) construct inscriptions while determining the key idea of a proof.

Proof: Key Idea

Proof primarily serves two purposes: 1) to convince oneself, and 2) to convince others (Harel & Sowder, 1998). Raman (2003) defined three types of ideas used in proof:

heuristic, procedural, and key. A heuristic idea is based on informal understandings and provides a sense of personal understanding; convincing oneself. In contrast, a procedural idea is based on logic and formal manipulations to provide a sense of conviction; convincing others. Finally, Raman defines, “A key idea is an heuristic idea which one can map to a formal proof with appropriate sense of rigor” (p. 323); convincing oneself and others.

As students learn the discourse of proof, they need to identify and use key ideas. While students discuss the mathematics verbally, they will likely gesture. In turn, these gestures facilitate the construction of diagrams. The resulting diagrams are public inscriptions that capture embodied knowledge. In this study, we start to answer the following question: How does the interplay between gesture and diagram help students identify the key idea when constructing topology proofs?

METHODS

We recruited students from an introductory undergraduate course in point-set topology at a large university in the United States. Participants attended weekly one-hour sessions in which they were asked to complete proof tasks (including “prove” and “disprove” prompts). Participants engaged in nine distinct problem sets over the course of the semester and were encouraged to collaborate on all proof tasks. Only one student, Stacey (a pseudonym), attended all sessions. A qualitative case study methodology (Cohen, Manion, & Morrison, 2011; Yin, 2006) was used to examine how Stacey used gesture and diagrams as she engaged in proof construction tasks.

Sessions were video recorded, and the videos were transcribed and coded for gesture use according to the coding scheme below. We also identified the moment in each session when Stacey verbally expressed the key idea (Raman, 2003) of the proof of the “prove” prompt, in those sessions where this occurred. We coded Stacey’s recognition of the key idea as the moment first moment she vocalized the idea she would eventually turn into a formal proof. This occurred, at most, once per session. The results we present in this paper concentrate on data collected from the “prove” prompts.

The definition of *gesture* varies in the literature, sometimes including all visible body movement including eye gaze and body posture. For this study, we use the definition given by Rasmussen, Stephan, and Allen (2004): “movement made by a hand with a specific form: the hand(s) begins at rest, moves away from the position to create a movement, and then returns to rest” (p. 303), which is adapted from a definition given by Roth (2001) but not as broad as other definitions, see Kendon (2004).

We classified our gestures using McNeill’s dimensions of iconicity, metaphoricity, and deixis (Kendon, 2004; McNeill, 1992, 2005). Iconic gestures are those that have real world objects and actions associated with them while metaphoric gestures are those that are created in the mind to represent something abstract. All representational gestures (i.e., iconic and metaphoric gestures) that referred to mathematical objects such as sets, points, or functions, were coded as metaphoric gestures, since the referents were abstract and not concrete. Deictic gestures are pointing gestures; we further subdivided deictic gestures into static points (using a finger/hand to point to something and not moving it) and tracing points (a gesture that starts as a point but then continues to move to highlight a secondary attribute of the referent, such as tracing the shape of a graph). We agree with McNeill (2005) that these dimensions are not mutually exclusive, and that gestures may contain elements from a mix of dimensions.

RESULTS

We discuss instances of Stacey's gesture use and diagram generation in three proof productions as well as her recognition of the key ideas of those proofs.

During Session 5, Stacey and Tom were asked to prove that, given a subset A of a topological space $(X, \mathcal{T})$, "[I]f for each open set $O \in \mathcal{T}$ we have $A \cap O \neq \emptyset$, then A is dense in X." Stacey began by drawing the diagram in Figure 1 (left).

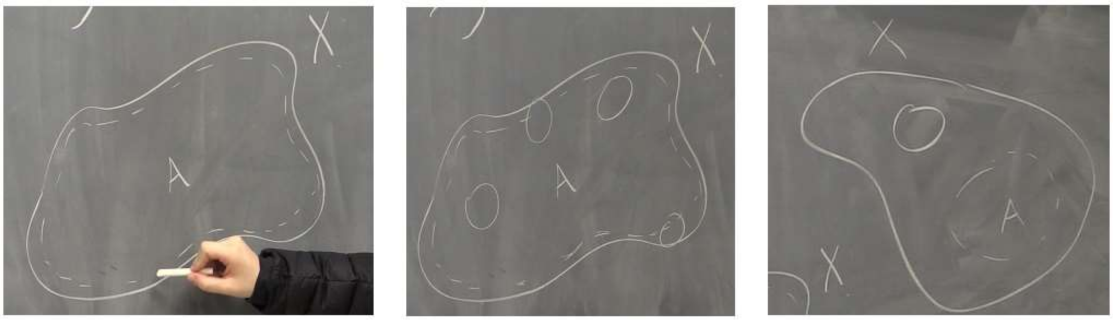

Figure 1: Stacey's evolving diagram.

She then began to explain her thinking (referencing the diagrams in Figure 1):

> I can't really show it with a picture because I can't draw a dashed line over a … solid line, but we have X [static point to the label X (left)] on the outside [tracing point along the boundary of X (left)] and then we have the set A [static point to the label A (left)] which is represented by the dashed [tracing point along the boundary of A (left)], which I wish I could get closer to this [static point to the border of X (left)], but I can't. So, if we had the closure of A [static point to the label A (left)], then it would just be the same as that solid line [tracing point along the border of X (left)]. So then if you take any open set [drawing circles on her diagram (center)] anywhere, there has to be some kind of intersection with A [static point to one of her open sets (center)]. So if it wasn't … if the intersection could be … the empty set [static point to $\emptyset$ in the problem statement] – [draws the diagram in Figure 1 (right)] You've got X here, and A here, and you could have an open set here, and their intersection would be the empty set [recognizes key idea]. But then this closure [static point to the boundary of A (right)] wouldn't be equal to X [static point to the boundary of X (right)]. I get it conceptually I think, but I'm not sure how to prove it.

Notice the alternation between diagram construction/modification and explanation accompanied by gesture production. The diagram gave Stacey a concrete referent to which

Figure 2: Session 5, Stacey reasoning about her diagrams using gestures.

she could point while explaining her thoughts, and the articulation of her ideas led to further modification of the diagram. After a few such alternations, Stacey arrived at the key idea of the proof. Following this excerpt, Stacey and Tom chose to use the method of proof by contradiction and wrote their formal proof.

Stacey's task in Session 7 was to prove that, given a topological space $(X, \mathcal{T})$, "If the sets C, D form a separation of X, and if Y is a connected subspace of X, then either $Y \subseteq C$ or $Y \subseteq D$." As in Session 5, Stacey began by drawing a diagram (Figure 3, left).

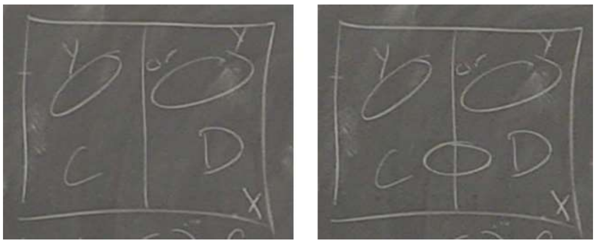

Figure 3: Stacey's diagram for a separation of a topological space.

Referencing Figure 3, she explained,

> If you have X, the ambient space [static point to the boundary of X (left)], and then you have the sets C and D [alternating static points to the left and right rectangles (left)], they form a separation, so that means that they're disjoint [static points to the left rectangle, then the right (left)], so they don't have any of the same elements, and that their union is X [static point to the boundary of X (left)], so that is satisfied for this [alternating static points to left and right rectangles (left)]. And then if Y is connected, which means it's not in these sets [metaphoric gesture indicating two disjoint subsets of Y (Figure 4, left)] that are disjoint whose union is Y, it's just one cohesive set [metaphoric gesture indicating Y as a connected set (Figure 4, right)], then it has to be either in C or in D. It can't be in both, because if it was like that [draws the subset in Figure 3 (right)], it would be disjoint. [recognizes key idea]

Once again, Stacey started her discussion by drawing a diagram. Her subsequent explanation, accompanied by various deictic and metaphoric gestures, led her to modify her diagram and then to vocalize the key idea that she then used to construct her proof in the remaining time.

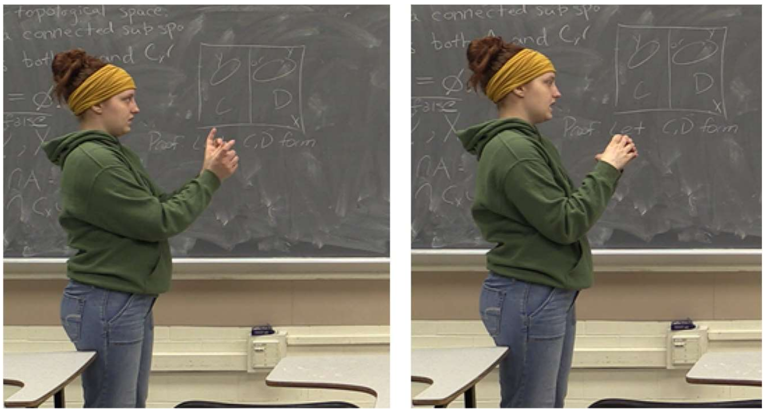

Figure 4: Stacey describing a connected subspace with gestures.

In Session 9, Stacey was asked to prove that, given a compact subspace Y of a Hausdorff space X and a point $x_0 \notin Y$, there exist disjoint open sets U and V such that $x_0 \in U$ and $Y \subseteq V$. Stacey began by extracting a list of details given in the problem statement, and she "wrote it out in symbols to help [her] think of it better," then talked through those details while using static points to highlight each one. It appeared that Stacey did not initially have a heuristic idea for this proof, as she spent a few minutes exploring the implications of her list of given conditions and browsing her textbook for useful hints.

Stacey's examination of her textbook led her to realize that Y must be closed in X, as a compact subspace of a Hausdorff space, a fact which she added to her written list on the board. She then constructed a diagram (Figure 5, left), which she modified as she elaborated:

> The open set U would be the complement of Y in X [writing $C_X(Y)$ (right)], because you know it's open because Y is closed. I don't know if there's a theorem or something that would get me there, but the fact that Y is a compact subspace of something Hausdorff might mean that there exists an open set within it [metaphoric gesture indicating a subset of Y] – oh wait, no. Because Y has to be in V [static point to her inscription, $Y \subseteq V$ (right)]. So we need something like this [draws a larger set containing Y, labels it V (right)]. And then you also need their intersection to be empty. So I guess you can't really take just the complement of Y [static point to the label $C_X(Y)$] because – you can't just take the set Y because it's closed, you need an open set for V. So you need something bigger than Y [tracing point to the boundary of V], but if you take anything bigger than Y, then you're in the complement of it [static point to the label X], and then the intersection isn't empty. So you need something within this complement of Y [static point to $C_X(Y)$] to be the set U.

Stacey used her diagram and her gestures to explain and explore her heuristic idea. In contrast to Sessions 5 and 7, however, her combination of diagram and gesture usage exposed flaws

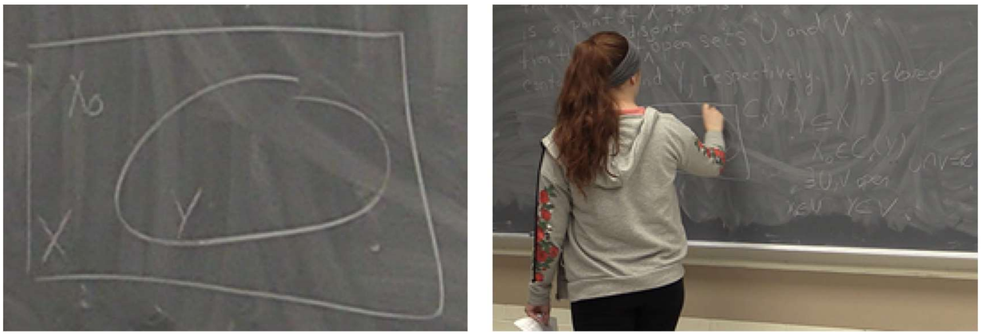

Figure 5: Stacey’s evolving diagram of a Hausdorff space.

in her reasoning. The exposure of these flaws allowed her to modify her understanding of the problem and proceed in search of a new idea. Stacey never identified the key idea of the proof in Session 9, but she continued to explore new ideas through gesture and modifying her diagram to reflect her evolving understanding.

DISCUSSION

The data from Sessions 5 and 7 suggest a sequence of events in Stacey’s recognition of key ideas and subsequent proof constructions. Stacey’s use of gestures and their role in the production of diagrams are critical to her recognition of the key idea (Raman, 2003) that eventually leads to the writing of the formal proof.

The embodied cognition perspective (Lakoff & Nunez, 2000; Nunez, 2008) suggests that internal knowledge is shaped by interactions with the external world. Engaging in discussions of mathematical ideas is a prominent means of interacting with the world. Verbal discourse is enhanced when we use gestures and diagrams to aid in our communication. As Roth and McGinn (1998) noted, communication suffers in the absence of inscriptions. In the data we presented, Stacey began reasoning about her proofs by drawing a diagram to represent her understanding of the problem, giving her ideas physical manifestations. Stacey then gave verbal descriptions accompanied by gestures, and she then used her diagrams to further facilitate communication of her informal mathematical ideas. With her ideas now part of the external world, she was free to interact with them physically, modifying her understanding as she modified her diagrams. The result of this activity was recognition of the key idea of the proof.

Our analysis of Stacey’s proof-writing in Sessions 5 and 7 suggests a pattern in her proving behaviours (see Figure 6). In the following paragraphs, we describe this pattern, and we speculate on the reasons for the observed behaviours.

Stacey’s proof production process began with the formation of a heuristic idea: she started with an informal reason why she believed the statement to be proved might be true. Collaboration with another individual (whether another student or the session’s facilitator) produced a need to communicate that idea externally, resulting in the production of gestures or an inscription (often, a diagram). Though Stacey may have possessed a heuristic idea, she was only able to represent one or two pieces of the idea at a time via gesture alone, and she may only have been able to hold a comparable number of pieces in her working memory. In order to record her gestures, Stacey created or modified a diagram, freeing her to produce new gestures to further explain her idea (de Freitas & Sinclair, 2011). These new gestures could then be added to her diagram, and this cycle repeated. The diagram became a more complete representation of Stacey's heuristic idea as more gestures were captured with each iteration of the cycle.

As the diagram became a more complete representation of Stacey's heuristic idea, she began to see how she might be able to translate her informal ideas into a formal proof. Stacey's heuristic idea became the key idea when she recognized that it should translate into a formal proof. Once this recognition was achieved, Stacey was free to begin construction of the formal proof and the development of the procedural idea, eventually leading to the completion of the proof.

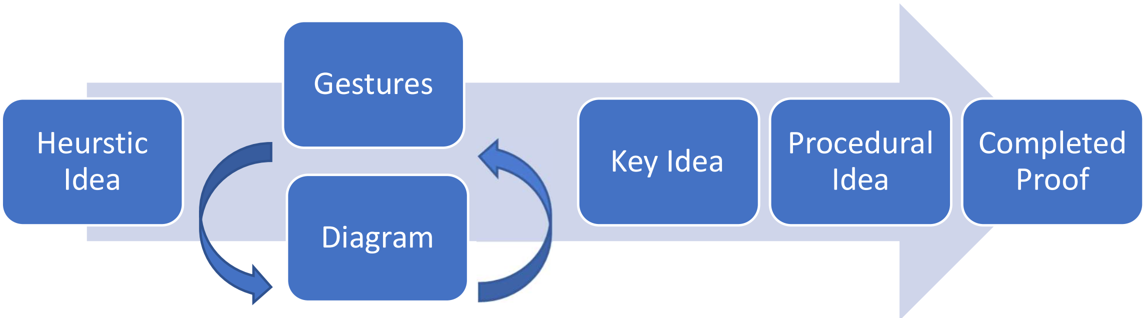

Figure 6: Stacey's observed proof-writing sequence in Sessions 5 and 7.

Though Stacey did not complete a proof in Session 9, we emphasize that Stacey did not identify a key idea during this session. Further, Stacey did not immediately generate a diagram like she did in Sessions 5 and 7. Stacey only drew a diagram after she arrived at an intuitive (albeit inaccurate) idea of why the given statement might have been true, and it was through a combination of gestures, speech, and diagram modification that she recognized the error in her idea. This provides support for the idea that gesture use and diagram modification were integral to Stacey's success in producing proofs.

CONCLUSIONS

This study sought to understand how the interplay between gesture and diagram construction facilitates students' proof writing. We observed that Stacey's engagement in a cycle of using verbal descriptions of her thinking accompanied by gestures and diagram construction led to her success in identifying the key idea of the proof and writing a correct proof. This adds to our knowledge of how gesture and advanced mathematical thinking are linked.

While we chose tasks that we thought would prompt students to draw diagrams and produce gestures, the tasks did not require students to engage in these activities. Future studies could investigate the effect of explicitly prompting students to gesture when explaining their thinking. Additionally, we suggest examining how the instructors of these courses use gesture and how that affects students' gesture use and conceptual understanding. Lastly, we acknowledge that our sample size was small and that to determine the extent of the generalizability of our results additional data are needed. It would further expand our knowledge to observe students working on proofs in other areas of advanced mathematics such as abstract algebra and geometry with a focus on the relationship between gesture and inscriptions.

Proof construction is challenging for students. Our results indicate that the combination of dialogue, gesture use, and diagram construction may be an effective tool to help students translate their informal ideas into formal mathematics. As students transition from the algorithmic, computational mindset of early grades mathematics to advanced mathematics that require more creativity and flexibility, the tools we give them must increase in flexibility as well. Communication is about more than just talk; gestures complement our verbal communication by providing a visual component that may be captured in inscriptions. We concur with Yoon, et al. (2011) that encouraging students to express their mathematical thinking with gesture can help them to be successful in communicating and understanding mathematical ideas.

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Intuitive mathematical discourse about the complex path integral

Erik Hanke[1]

[1]University of Bremen, Faculty of Mathematics and Computer Science, Germany, erik.hanke@uni-bremen.de

Interpretations of the complex path integral are presented as a result from a multi-case study on mathematicians' intuitive understanding of basic notions in complex analysis. The first case shows difficulties of transferring the image of the integral in real analysis as an oriented area to the complex setting, and the second highlights the complex path integral as a tool in complex analysis with formal analogies to path integrals in multivariable calculus. These interpretations are characterised as a type of intuitive mathematical discourse and the examples are analysed from the point of view of substantiation of narratives within the commognitive framework.

Keywords: Teaching and learning of specific topics in university mathematics, teaching and learning of analysis and calculus, commognitive framework, complex analysis, mental images.

INTRODUCTION

Since $\mathbb{C}$ and $\mathbb{R}^2$ are isomorphic as real topological vector spaces, one can try to analyse analytic properties of a continuous complex-valued function $f = u + iv$ by studying analytic properties of the vector field $\boldsymbol{F} = (u, v)$ [1]. For instance, complex differentiability of f at some point is equivalent to real differentiability of $\boldsymbol{F}$ together with the satisfaction of the so-called Cauchy-Riemann equations. However, it is not immediate how to establish an intuitive or geometric understanding of complex path integrals by going back to single- or multivariable calculus.

It is not very present in other textbooks but "visual complex analysis" has been worked out by Needham (1997). In addition, research from university mathematics education about complex analysis, and in particular the complex path integral, is emerging (e.g., Oehrtman, Soto-Johnson, & Hancock, 2019). It thus seems purposeful to investigate intuitive understanding of the complex path integral and other notions in complex analysis more closely, not only for epistemological reasons, but also to intensify our understanding of meaning-making at university level, and to identify interpretations of notions in real analysis that can potentially be expanded for analytic notions which appear beyond first year in mathematics study programmes.

In a larger project, I investigate expert mathematicians' understanding of basic notions of complex analysis. This article continues my investigation of experts' understanding of the complex path integral (Hanke, 2019), and discusses geometric-physical interpretations of the complex path integral and analogies to integrals in real analysis. Next to the enrichment on complex analysis education, I discuss intuitive understanding and mental imagery through a commognitive lens (Sfard, 2008) when discussing experts' *intuitive mathematical discourses*.

Research question

Based on my own engagement in teaching complex analysis, I can say that many students demand for intuitive explanations of complex analytic notions. Acknowledging the expertise of professional mathematicians, it is expedient to ask which kinds of intuitive interpretations arise in experts' thinking about notions in complex analysis, firstly in order to achieve an understanding about proficient usages of these notions, and secondly to study generally how mathematicians at university substantiate their intuitive thinking about notions of the undergraduate curriculum. In this note, I focus on interpretations of the complex path integral expert mathematicians provide when they are explicitly asked to give such:

> How do expert mathematicians interpret the complex path integral and how do they substantiate their interpretations?

PREVIOUS RESEARCH

Definitions and some interpretations of the complex path integral in the literature

The path integral of a continuous complex-valued function $f = u + iv$ on the trace $\mathrm{tr}(\gamma)$ of a piecewise continuously differentiable curve $\gamma: [a, b] \to \mathbb{C}$ can be defined as $\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$ (this is to be interpreted as the sum over the parts where γ is continuously differentiable) (Lang, 1999, ch. III, §2). Then again, it is also possible to extend the definition of the Riemann integral of real-valued functions to the complex setting with Riemann sums of the form $\sum_{k=0}^{n-1} f\big(\gamma(\xi_k)\big)\Delta\gamma_k$ (Polya & Latta, 1974, ch. 5.3) [2]. If one separates $\int_\gamma f(z)\,\mathrm{d}z$ into real and imaginary part, one obtains $\int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_\gamma v\,\mathrm{d}x + u\,\mathrm{d}y$. Furthermore, if γ is simple closed and f is holomorphic on an open neighbourhood of the interior of γ, $\mathrm{int}(\gamma)$, Green's theorem yields that $\int_\gamma f(z)\,\mathrm{d}z$ equals $-\iint_{\mathrm{int}(\gamma)} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\,\mathrm{d}(x, y) + i\iint_{\mathrm{int}(\gamma)} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\,\mathrm{d}(x, y)$ [3].

As a result, one gains three connections to real analysis: The first one extends the definition of the Riemann integral for real-valued functions, the second expresses the complex path integral via real path integrals of second kind, i.e. path integrals for real vector fields, and the third enables to determine the complex path integral for paths on the boundary of an area via area integrals.

Probably the most important fact about the complex path integral is *Cauchy's integral theorem*. One version says that $\int_\gamma f(z)\,\mathrm{d}z$ vanishes if γ is closed and f holomorphic on a simply connected domain which contains $\mathrm{int}(\gamma)$.

Geometric-physical interpretations of path integrals in real analysis assist the interpretation of the complex path integral as well. However, the transfer from the real to the complex setting is more subtle and provides an interpretation dependent on the separation into real and imaginary part. Polya and Latta (1974, ch. 5.1f.) reason that $\int_\gamma \overline{f(z)}\,\mathrm{d}z = \int_\gamma u\,\mathrm{d}x + v\,\mathrm{d}y + i\int_\gamma u\,\mathrm{d}y - v\,\mathrm{d}x = \text{work} + i \cdot \text{flux}$ (note that $\bar{f}$ is on the left side!), where work means the work of $\boldsymbol{F}$ along γ when $\boldsymbol{F}$ is interpreted as a force, and flux means the flux of $\boldsymbol{F}$ across γ when $\boldsymbol{F}$ is interpreted as a current density. If one replaces $\bar{f}$ with f, like Braden (1987) or Needham (1997, ch. 11.II.1), one gets $\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \boldsymbol{w} \cdot \boldsymbol{T}\,\mathrm{d}s + i\int_\gamma \boldsymbol{w} \cdot \boldsymbol{N}\,\mathrm{d}s = \text{work}^* + i \cdot \text{flux}^*$ where work* and flux* are interpreted as above with $\boldsymbol{F}$ replaced by the "Pólya vector field" $\boldsymbol{w} = (u, -v)$ (Braden, 1987, p. 321) [4]. In Braden's terminology, the real part of the complex path integral is the flow of $\boldsymbol{w}$ along γ and the imaginary part is the flux of $\boldsymbol{w}$ across γ. One can see that this geometric-physical interpretation of $\int_\gamma f(z)\,\mathrm{d}z$ involves work and flux of the Pólya vector field $\boldsymbol{w}$, i.e. the vector field associated to the conjugate $\bar{f}$, not $\boldsymbol{F}$.

Moreover, Gluchoff (1991) argues that the complex path integral divided by the length of γ, provided that γ is simple, equals the "average" of the numbers $f(\gamma(t))\gamma'(t)/|\gamma'(t)|$ where t ranges over $[a, b]$, i.e. $\gamma(t)$ ranges over $\mathrm{tr}(\gamma)$. Thus, he generalises the mean value property of integrals of a real-valued functions.

Research on complex path integrals in mathematics education

Research on complex analysis education, besides arithmetic and geometry of complex numbers, is emerging. Oehrtman, Soto-Johnson, and Hancock (2019) present a study on mathematicians' understanding of the complex derivative and complex integration. Their participants could relate the derivative to the idea of "amplitwist", however not always fluently. For integration, the majority of their participants struggled to interpret the complex path integral intuitively, and considered it hard to formulate such an explanation. Two participants mentioned the connection between real and complex path integrals, e.g., one of them formally multiplied $(u + iv)(\mathrm{d}x + i\,\mathrm{d}y) = u\,\mathrm{d}x - v\,\mathrm{d}y + i(v\,\mathrm{d}x + u\,\mathrm{d}y)$ to establish the connection. Only one of the experts provided a more profound personal interpretation. He combined the Riemann sum approach with a story on ship navigation where the captain reconstructs his physically real route on a chart. The function f causes "location-dependent errors" of the original path's segments $\Delta\gamma_k$ in the sense that $f\big(\gamma(\xi_k)\big)\Delta\gamma_k$ is a dilated-rotated version of the original path segment on the chart (Oehrtman, Soto-Johnson, & Hancock, 2019, p. 413). This resembles Needham's (1997, ch. 8.III) interpretation of the Riemann sum approach as a concatenation of rotated-dilated vectors.

BASIC TENETS OF THE COMMOGNITIVE FRAMEWORK

In the *commognitive framework* (Sfard, 2008), of which I can only elucidate the most basic ideas here, thinking as personal communication and communication with others are conceptualised as two sides of the same phenomenon. Mathematics and its various disciplines are seen as special discourses. Objects in mathematical discourses "are, themselves, discursive constructs, and thus constitute a part of the discourse" (Sfard, 2008, p. 129), e.g., the discourses grow recursively when processes, which involve previous discursive or physically perceptible objects, are in turn objectified into new discursive objects. The commognitive framework offers four core categories to analyse mathematical discourses: Word use, narratives, visual mediators, and routines (Sfard, 2008, pp. 129–135). *Word use* refers to the usage of (key) words, keeping in mind that the same words can appear in different discourses. A *narrative* is "any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects", which in formal, literate mathematical discourses are for example definitions or theorems (Sfard, 2008, p. 134). An *endorsed narrative* is a narrative that is considered true by a set of endorsers when rules, which are agreed upon by the endorsers, have been applied to justify that narrative; in other words, an endorsed narrative reflects "the state of affairs" (Sfard, 2008, p. 298). *Visual mediators* are all visible entities that are used in communication, e.g., sketches, or symbols specifically designed for mathematical communication. Finally, *routines* are collections of metarules which govern the actions of the participants of a discourse, the discursants, which are called mathematists in mathematical discourse, rather than the objects of the discourse. For example, exploration routines govern the construction of new narratives. Lavie, Steiner, and Sfard (2019) argue more detailed that discursants may choose their routine performances according to so-called precedent-search-spaces, i.e. communicative situations in which they, or other discursants, participated in a certain manner, which is adapted to the situation at hand.

One form of exploration is *substantiation*. It is "a process through which mathematists become convinced that the narrative can be endorsed" and is "probably the least uniform aspect of mathematical discourses" (Sfard, 2008, p. 231). In formal, literate mathematical discourses, mathematists usually substantiate a definition by checking for its consistency and a theorem with a proof—each of the substantiations in a fashion that is endorsed itself by the group of endorsers. Contrariwise, in personal mathematical discourses, i.e. discourses in which a single mathematist communicates with her- or himself, substantiations can vary considerably (Sfard, 2008, pp. 231–234).

INTUITIVE MATHEMATICAL DISCOURSE

Discourses around discursive objects emerge through narratives, together with metarules, which themselves grow in the discourse. Even the sensitive construct of individual meaning-making can be approached from the commognitive framework. In mathematics education, the ideas of "intuition" or "mental images" became concepts through the narratives created about them, e.g., in the strands of work following Fischbein (1987) or Tall and Vinner (1981). However, it often remains ambiguous what exactly is structured by these concepts and how the gap between cognitive constructs on the one hand and empirical observability on the other hand is bridged. Keeping the discussion on these cognitive constructs in mind, one can look at discourses formed when mathematicians engage in communicating about their personal intuitive understanding of mathematical objects, which I call *intuitive mathematical discourses* here. These include any kinds of visual means or heuristics individuals or communities may use to explain a mathematical notion without requiring rigour.

In the commognitive framework, *understanding* is the "interpretative term used by discursants to assess their own or their interlocutors' ability to follow a given strand or type of communication”, and a “commognitive researcher [...] is interested in the interplay of the participants’ first- and third-person talk about understanding and their object-level discursive activity (Sfard, 2008, p. 302; original highlighting omitted, EH). Here, the focus is on intuitive understanding of a mathematical notion. While examining intuitive mathematical discourses one needs to take into account that discursants shape these discourses by what they consider to belong to their intuitive understanding of a mathematical notion, in the sense of understanding as above. In this context, mental images (German: Vorstellungen [5]) are understood as narratives or combinations of visual mediators and narratives on object- or meta-level which serve as heuristics for communication, such as making explicit their intuitive understanding. Discursants may use them to explain a mathematical notion on a for her or him intuitive level, either for somebody else or for her- or himself. These narratives and visual mediators constitute intuitive mathematical discourses.

Discursants may be unsure whether their own intuitive narratives are in some sense correct or shared by other discursants, or may be afraid of compromising themselves. Thus, the range of endorsement and the substantiations of the narratives a discursant produces in her or his intuitive discourse may vary notably (either within a discursive community, or with respect to what the single discursant expects as agreement from other discursants). An individual’s intuitive mathematical discourse centring on mathematical notions is thus not necessarily about endorsed narratives about mathematical objects per se, like in literate mathematical discourses, but rather about heuristics with which the individual makes sense of these notions. Yet, this can include elements of literate mathematical discourses, e.g., theorems or narratives about related mathematical aspects, of what the individual believes to be endorsable or rejected by other people, or narratives and visual mediators which show the discursants’ struggles to express her- or himself.

The notion of intuitive mathematical discourse is not meant in any prescriptive way. It is an attempt to understand meaning-making from a discursive perspective, which considers individual and interpersonal communication as the same phenomenon, thus bearing theoretical justification for how individual and interpersonal meaning-making through communication can take place in mathematics.

METHOD: PARTICIPANTS AND INTERVIEW QUESTION

Interviews of approximate lengths of 90 to 120 minutes were conducted with expert mathematicians from German mid-size universities, videotaped, transcribed, and all notes written down during the interviews collected. In the beginning of the interviews, I emphasised that my research is about the very personal meaning-making and mental images of mathematicians at university. During the interviews, I asked for the experts’ personal meaning-making of complex differentiation, the complex path integral, and fundamental theorems like Cauchy’s integral theorem or Cauchy’s integral formula. In this article, I draw on data from interviews with two mathematicians: Dirk and Uwe (pseudonyms). They have PhDs and lectured complex analysis for several years.

Here, I focus on interview excerpts on the following question. It was introduced with the geometric interpretation of the integral of a real-valued function as “signed area under the graph”, paraphrased, and handed to the participants printed out (translated from German). Nevertheless, the participants were encouraged to detour from geometrical reasoning in favour of other aspects they deem fertile.

> “Which geometrical meaning does the complex number $\int_\gamma f(z)\,\mathrm{d}z$ for a (piecewise continuously differentiable) path $\gamma\colon [a,b] \to \Omega$ and a continuous function $f\colon \mathrm{tr}(\gamma) \to \mathbb{C}$ have for you?”

RESULTS

The excerpts from the transcripts were translated from German (some filler words have been omitted for readability), and I redrew the figures. (#) indicates the length of a pause, and / indicates unfinished words or interruptions by the interlocutor.

Transferring the “real image”

Dirk rephrases the question and thinks for a long time:

1 Dirk: (13) Uhm, so in strict complex analytical context, what is the meaning? Alright, uhm, (5) if one talks, uhm, about path integrals in vector fields, then it has a physical meaning, but what, what would it be here? (23) Uhm. (7)

Dirk is aware that “path integrals in vector fields” have a physical interpretation but he does not transfer this interpretation to the complex setting. After long silence, the interviewer claims that some people simply consider the complex path integral as a technical tool used for proving in complex analysis. Dirk does not find this satisfying but remains unable to give a geometric description even though he states that he has thought about this before. Therefore the interviewer uses Cauchy’s integral theorem as another stimulus. Dirk says he believes “what this [the stimulus] boils down to” and provides the formula $\int_\gamma f(z)\,\mathrm{d}z = F\big(\gamma(b)\big) - F\big(\gamma(a)\big)$ where F is a primitive of f, and $\gamma(a)$ and $\gamma(b)$ are the start and endpoint of γ. He continues like this:

2 Dirk: Uhm, yes, one can use this [the formula just given] perhaps to he/ help with the imagination [German: Vorstellung], right, but (1) in principle one would like to, uhm, resort to such an image, right [draws Figure 1a]. [incomprehensible: And then] it is, uhm, not an interval in R now, but a path, let’s say, this here is C, right [draws Figure 1b], and, uhm, this is not necessarily helpful, such a picture, since the values are complex [adds a $\mathbb{C}$ to the axis pointing upwards in Figure 1b].

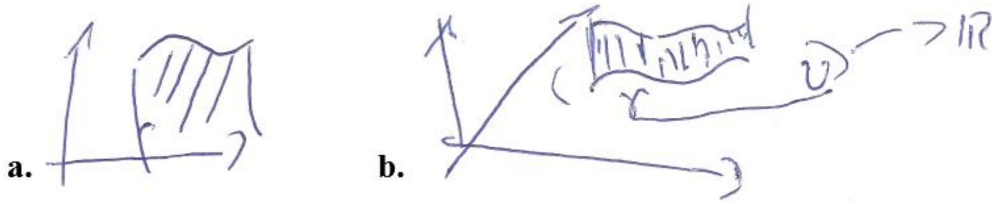

Figure 1. Attempt to transfer “such an image” (a.) to the complex setting (b.)

The narrative involving the primitive is not yet considered as “imagination” but might “help”. This valuation may be the result of the initial question for a geometric meaning. Dirk draws “such a picture” (Figure 1a), common for real integrals, even though he does not say this, and attempts a transfer to the setting where the domain of the function is “a path”. Formally though, the function needs to be defined on the trace of the path, and this seems to be displayed in Figure 1b. However, Dirk’s use of words does not include “function” but “values”. The routine of drawing a sketch and looking for an analogous picture for the complex path integral that builds on a picture for the real integral does not help Dirk for finding geometric meaning of the complex path integral, and he even questions whether such a picture is helpful at all (Hanke, 2019).

Here, except for the narrative $\int_\gamma f(z)\,\mathrm{d}z = F\big(\gamma(b)\big) - F\big(\gamma(a)\big)$, Dirk does not produce an explicit narrative about the complex path integral, and he does not even use any of the words “integral” or “integration” in his immediate reaction to the interview question. Otherwise, his word use of the nouns he uses is object-driven (Sfard, 2008, p. 182) and he produces the visual mediator in Figure 1b while he talks. Dirk’s attempt for a geometric picture does not indicate process-driven understanding of the complex path integral and his drawing seems to hint towards the wish for a static picture, but he does not reach an explicit narrative about a meaning of the complex path integral.

“Path integrals of third kind” as a “tool” in complex analysis

While the interviewer still poses the question, Uwe interrupts to firmly state that the complex path integral has “not any geometric meaning”. Then, he goes on like this:

3	Uwe:	There are path integrals of first, second, and third kind, I like to say. Of first kind is a scalar, uhm, path integral, which, boah, no idea, is especially important for calculating the arc length, where the number one is simply integrated along the path (1) and then there’s path integral of second kind, which is incredibly important for any work along any paths, where one has a scalar product, and then there is the complex path integral, and for this, one does not have any imagination at all at first. There is complex multiplication, so to speak [points to $f(z)\,\mathrm{d}z$]/ This is, so to speak, if you like, f of z is complex multiplied by dz and there one best doesn’t imagine anything at all [giggles]. […] So, I mean, I am of course, of course, only interested in this for holomorphic functions, because that is, that is simply a tool in complex analysis, path integrals. This is nothing more than a tool actually. And, uhm, therefore this is only interesting for holomorphic functions and, well, there one knows the residue theorem and it tells you exactly which image you should have of it, namely: If the path is only passing around isolated singularities of f, (1) I simply have to look at f in the singularities and calculate the residues there, then I also know what this, what this integral means, what comes out of the integral. In the end, this is what the path integral means. The sum of the residues, the weighted one.

Here we find three very explicit narratives which get developed during the interview. Firstly, there is a clear rejection of geometric meaning for the complex path integral and it is granted the status of a tool. Later on, Uwe consolidates that complex path integrals “usually do not have a special meaning in themselves” and serve to evaluate real integrals that “have the meaning with which you [the interviewer] started, that this is an area under a graph”. Consequently, Uwe is not generally indifferent to geometric meaning of mathematical notions, which also becomes evident in other parts of the interview where he argues about the importance of sketching domains of functions and traces of paths. Secondly, there is a clear differentiation between real and complex path integrals. Thirdly, the meaning of the complex path integral is seen in its “outcome”, a weighted sum of residues—which is stated normatively (“image you should have”).

Uwe only considers holomorphic functions to be interesting for integration, which may have singularities the path of integration winds around. Also note that Uwe’s perspective changes: Whereas “one best doesn’t imagine anything at all” for the path integral and “one” has a scalar product in the integrand, he changes to “I” when he restricts the class of functions considered, before he changes back to the normative statement that the weighted sum of residues is the meaning of the path integral. In addition, Uwe’s talk is process-driven at some points: The path is “passing around isolated singularities” and Uwe has to “look at f” and “calculate the residues”. Even though Uwe holds the static image of the complex path integral as weighted sums of residues, the path and he appear as actors in his narratives.

$$\begin{pmatrix} f_1 & -f_2 \\ f_2 & f_1 \end{pmatrix}\begin{pmatrix}\gamma_1' \\ \gamma_2'\end{pmatrix} = \begin{pmatrix} f_1 \\ f_2\end{pmatrix}\gamma_1' + \begin{pmatrix}-f_2\\ f_1\end{pmatrix}\gamma_2' = \left\langle \begin{pmatrix} f \\ if\end{pmatrix}, \gamma' \right\rangle$$

Figure 2. Rewriting $f \cdot \gamma'$ as $\langle (f, if), \gamma' \rangle$ using matrix multiplication

Later, the interviewer points to the apparent similarity that γ' appears both in real and complex path integrals and Uwe counters “Yes, certainly, but here is a complex times and this makes everything a little weird”, which reinforces Uwe’s distinction between real and complex integrals. He identifies complex numbers $a + ib$ with vectors (a, b) and uses this to write the product of the function f and the path γ in terms of matrix multiplication (Figure 2; here is $f = f_1 + if_2$). The integrand $f \cdot \gamma'$ in the complex path integral is transformed into $\langle (f, if), \gamma' \rangle$, which resembles the integrand of a path integral for vector fields that involves the scalar product of the vector field and γ'.

4	Uwe:	[…] and now, uhm, there is the integrability condition for exact, for conservative vector fields, namely that, when I derive the second function [points to if in Figure 2] with respect to the first variable, there comes out the same as when I derive the first function [points to f in Figure 2] with respect to the second variable. (1) And then there comes out exactly d-two f equals i d-one f [writes $\partial_2 f = i\partial_1 f$] and these are exactly the Cauchy-Riemann differential equations.

Here, Uwe substantiates the Cauchy-Riemann equations through a narrative on the equality between the integrand of the complex path integral and an expression that resembles the integrand of a path integral of second kind (Figure 2) on which he applies the “integrability condition” (which stems from vector analysis)—but not from the definition of complex differentiability or the relationship to real total differentiability.

SUMMARY

A discursive perspective on intuitive understanding of mathematical concepts in terms of intuitive mathematical discourse was proposed and used to analyse experts' intuitive understanding of the complex path integral. Although references to real analysis appeared frequently—even if only to emphasise conceptual differences or underline the inappropriateness of "real images"—the reconstructed substantiations of Dirk's and Uwe's narratives about their intuitive understanding of the complex path integral differ considerably. From the excerpts above and the literature we can see that substantiations of the complex path integral include *formal (non-) analogies*: The definition of the complex path integral using Riemann sums is analogous to that of the real Riemann integral as it builds on summation and multiplication of $\mathbb{C}$ instead of $\mathbb{R}$. Similarly, Uwe's *recognition of conceptual similarity and difference* of the product structure of integrands in path integrals substantiates the narrative of the complex path integral as a path integral of third kind: Whereas real path integrals of second kind involve scalar products of vectors in $\mathbb{R}^2$, complex path integrals involve multiplication in $\mathbb{C} \cong \mathbb{R}^2$.

Neither of the two experts here gave a clear geometric image for the complex path integral. However, Dirk tried to *build an analogy* between the geometric meaning of the integral of a real-valued function as an area under the graph and a possible, still to him unbeknownst geometric meaning of the complex path integral. He used *sketching* to look for such a geometric interpretation, which is an instance of a *transfer of known ideas*, since sketches of graphs of functions are useful in real analysis. Unfortunately, his attempt was not successful. Uwe rejected any geometric meaning of the complex path integral. Rather, he substantiated the narrative of the complex path integral as a tool with the fact that real integrals have a geometric meaning and that complex path integrals can help to calculate these—the notion of complex path integral is *valued with its helpfulness*. Lastly, Uwe's *restriction of generality of the prerequisites* (holomorphic functions with the exception of isolated singularities and closed paths) substantiated the narrative of the complex path integral as a weighted sum of residues.

Theorematic images on the complex path integral, i.e. narratives in intuitive mathematical discourses involving propositions, could also be identified: Uwe's narrative about the weighted sum of residues is based on the residue theorem and Dirk's narrative $\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$ shows meaning-making with a version of the "main theorem of calculus".

NOTES

1. A complex-valued function defined on a subset of the complex numbers will be denoted by f, and u (v, respectively) denotes its real part (imaginary part, respectively). Paths in $\mathbb{C}$ are identified with paths in $\mathbb{R}^2$ without notational change.

2. Here $\{a = t_0 < t_1 < \cdots < t_n = b\}$ ranges over the partitions of $[a, b]$, $\Delta\gamma_k = \gamma(t_{k+1}) - \gamma(t_k)$, and $\xi_k \in [t_k, t_{k+1}]$ for every $0 \le k \le n-1$. If γ is piecewise continuously differentiable, this definition agrees with $\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$.

3. The Jordan curve theorem states that the trace of a simple closed curve separates the plane into two connected domains, a bounded region, i.e. the interior $\mathrm{int}(\gamma)$ of γ, and an unbounded region (Apostol, 1971, p. 184).

4. $\boldsymbol{T}$ stands for the tangential vector field given by γ' and $\boldsymbol{N}$ stands for the normal vector field on γ, each in $\mathbb{R}^2$, i.e. $\boldsymbol{N}$ is $\boldsymbol{T}$ turned by $\pi/2$ clockwise (Braden, 1987; Needham, 1997, ch. 11.I.1).

5. Unfortunately, the German word(s) “Vorstellung(en)” do(es) not have a sound English translation; “mental image(s),” “mental imagery,” or “basic idea(s)” come close. I prefer to understand a Vorstellung as an object- or meta-level narrative in an individual’s intuitive mathematical discourse about a mathematical notion, possibly supported by visual mediators, jettisoning the ballast of a cognitive notion.

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Exploring Learning Potentials of Advanced Mathematics

Reinhard Hochmuth

Leibniz University Hannover, Faculty for Mathematics and Physics
hochmuth@idmp.uni-hannover.de

This contribution examines subject-specific potentials of advanced mathematics with regard to the issues transition, rationales and compartmentalization. The Anthropological Theory of Didactics (ATD) is used as a theoretical framework and in the analyses notions from the 4T-model are applied (Bosch & Gascón, 2014). Structural observations in praxeologies terms are illustrated by examples chosen from the presentation of a classical result in Nonlinear Approximation (DeVore, 1998). At the specific focus are aspects for bridging and extending concepts within and across Analysis. Goals of the analyses are to reflect the potential learning gain (beyond the concrete content) by studying advanced mathematics and to lay content-related foundations for pursued teaching innovations.

Keywords: advanced mathematics, praxeologies, compartmentalization, transitions, rationales

INTRODUCTION

The main aim of this contribution is to highlight subject-specific potentials of studying and learning advanced mathematics with research references with regard to aspects of transition, rationales and compartmentalization by means of the presentation of a classical result from Nonlinear Approximation in DeVore (1998; see also the synopsis in the Appendix). At the focus of our considerations are transitions within and across Analysis. For overviews of research on transition aspects in university mathematics education, see Gueudet, Bosch, DiSessa, Kwon and Verschaffel (2016) and Hochmuth, Broley and Nardi (2020). As far as the author knows, very advanced Analysis contents have not yet been didactically explored.

In the introduction we start with a preliminary clarification of the notions transitions, rationales and compartmentalization and pose respective research questions underlying the subsequent analyses. In the course of this contribution, both the notions and the research questions will further be specified against the background of ATD and its 4T model as well as illustrated by examples in the Appendix.

First, we will focus on transitions: Advanced mathematics is based on knowledge from basic lectures. The fact that certain terms and ideas are taught in basic university lectures, such as Analysis and Linear Algebra, and the way they are treated is also due to their importance for advanced mathematics. Depending on the field, advanced mathematics covers content that is relevant for graduates and their professional careers, as well as content that is relevant to current research. This raises the question of how respective transitions and the connection between content of basic lectures and advanced courses might be described. Similarly it may be asked: What are the interrelations between the contents of successive advanced courses? If there are essential relations between contents of advanced courses from different mathematical fields, how can they be characterized, e.g. in praxeological terms? In this paper we subsume those relationships under the key word transition since it expresses the dynamic nature of those relationships regarding learning and knowing. We believe that those transitions are an important part of the rationale of teaching mathematical concepts, and their adoption in learning processes could be seen as an effective possibility to overcome compartmentalized knowledge.

Second, rationales for the treatment of content in basic mathematical courses are sometimes not immediately clear to students. They complain in particular about a lack of relevance and a lack of application (Croft & Grove, 2015, p. 180). In order to counteract this problem, it is proposed to foster context and problem based learning (p. 185). In our opinion, subject related analyses, as presented here, are useful for their subject related design, since relevant rationales can be identified in texts on advanced research related mathematics. Formulated as a question: Are reasons for theoretical discourses and e.g. the introduction of certain terms in context with the considered advanced mathematical content explicated?

Compartmentalization of knowledge means that related knowledge, for example knowledge that belongs to a domain, is composed in separate and not intertwined parts. Mandl, Gruber and Renkl (1993) generally differentiate between “three types of knowledge compartmentalization: compartmentalization of incorrect and correct concepts, compartmentalization of several correct concepts, and compartmentalization of symbol systems and real world entities” (p. 162). In this paper, we focus mainly on the second type. Regarding Analysis and Stochastics, this type has also been considered by Derouet, Planchon, Hausberger and Hochmuth (2018). Particularly relevant in mathematics is another type-2-form of compartmentalization where the knowledge aspects calculi and logic are taught and learned as isolated subject areas and their intertwining is hardly visible in what is actually learned (see for example (Barbé, Bosch, Espinoza, & Gascón, 2005)). Similar to the above, the question arises as to how and to what extent advanced mathematical content can contribute to a reduction of compartmentalization phenomena.

With respect to the mentioned issues, this paper focuses on subject-specific aspects. With the space available here, this can only be done by way of example and sketch. The selected example deals with a classical result by Kahane (1961) from Nonlinear Approximation and its presentation in DeVore (1998), which is mirrored in the Appendix together with the mathematical notions necessary for its understanding. The line numbering in the Appendix allows referring to concrete places in the discussion of the issues.

The current article is structured as follows: In the next section, the ATD notions that are used in the analyses are briefly introduced. Then the issues transition, rationales and compartmentalization are discussed in some detail. Thereby, the ATD notions serve in particular to take a differentiated view of various transitions situations, to present preliminary praxeological and structural insights in a generalized form and to illustrate them by means of concrete passages from the Appendix. It goes without saying that the considerations formulated here in no way aim at completeness with regard to the research questions stated above, nor with regard to the chosen example. Further research ideas are sketched in the outlook.

Another final apology: The analyses refer to numerous mathematical concepts, such as rectifiability, or relations between such concepts, such as limited variation and Riemann-Stieltjes integral or Cea's lemma, without references. This is undoubtedly nasty and happens only for space reasons.

A FEW NOTIONS FROM ATD

ATD (Chevallard, 1999) aims at a precise description of knowledge and its epistemic constitution. The theoretical framework allows explicating institutional specificities of knowledge and related practices in university mathematics (Winslow, Barquero, De Vleeschouwer & Hardy 2014). A basic concept of ATD are praxeologies, which are represented in so called "4T-models (T,τ,θ,Θ)" consisting of a practical and a theoretical or logos block. The practical block $\boldsymbol{\Pi}$ (know-how, "doing math") includes the type of task (T) and the relevant solving techniques (τ). The logos block $\boldsymbol{L}$ (knowledge block, discourse necessary for interpreting and justifying the practical block) covers the technology (θ) explaining and justifying the used technique and the theory (Θ) justifying the underlying technology. In addition, we introduce the symbol PO to denote praxeologies and praxeological essembles in the sense of linked elements from practical and/or logos blocks. The interconnectedness of knowledge is particularly modelled in ATD by means of local and regional mathematical organizations that allow contrasting and integrating practical and epistemological aspects in view of different institutional contexts. Further relations between praxeologies from different institutions can be identified by comparing and contrasting blocks and their elements. In the analyses of this paper, we consider the 4T-model mostly as a heuristic tool for indicating relations and focus in particular on possible relations between praxeologies and their blocks. We consider mathematical areas that play a role in different courses or parts of courses such as Analysis and Nonlinear Approximation as different institutional contexts (represented in particular by standard textbook literature), which are indicated as subscripts of blocks or praxeologies. For example, we use the subscript A for a block or praxeology from Analysis or the subscript NL in the case of Nonlinear Approximation. In the next section, we use the praxeological notions to specify and elaborate on subject specific transitions regarding the piece of advanced mathematics presented in the Appendix.

Transitions

We consider three different transitions: the transition from basic Analysis lectures to the selected content area from Nonlinear Approximation, then from basic Approximation Theory to Nonlinear Approximation, and finally the transition from Nonlinear Approximation to themes of other advanced courses and vice versa. Each of these transition situations is further differentiated with regard to structurally distinct praxeological situations.

From Analysis to Nonlinear Approximation

i. Praxeologies from basic Analysis lectures enter essentially unchanged into praxeologies of nonlinear approximation, i.e.

$$\Pi_A, L_A, PO_A \hookrightarrow PO_{NL},$$

where Π_A, L_A, PO_A denote blocks, parts of them or praxeologies, in the sense of linked elements from practical and/or logos block forms, from Analysis and PO_{NL} denote praxeologies from Nonlinear Approximation. For this case, a large number of locations can be identified in the Appendix. One example is the use of metrics and norms in the mathematization of the idea of distance between objects (A7, A13, A17, A31 etc.). Already in Analysis lectures, important techniques are connected with the triangle inequality, here in particular the use of the triangle inequality itself, in addition, however, also the skillful introduction of third objects (implicit in A44). The latter refers to the logos block linked to the triangle inequality. This also applies to the use of ϵ in connection with limit value considerations and inequalities, which could be seen as a well-known praxeology from Analysis: One deduces a desired inequality up to an arbitrarily chosen $\epsilon > 0$ and can finally conclude the desired inequality (A44-A49).

ii. Practical or logos blocks from Analysis or links of both ($\Pi_A, L_{A,} PO_A$) are used, but supplemented by specific elements from the advanced situation ($\tilde{L}_{NL}$), then linked to these ($\widetilde{PO}_A$) and finally find themselves in praxeologies of Nonlinear Approximation, i.e.

$$\Pi_A, L_A PO_A \& \tilde{L}_{NL} \hookrightarrow \widetilde{PO}_A \hookrightarrow PO_{NL}.$$

In the present context, we would like to refer to the application of the varational semi-norm and the concept of BV as examples (A34-A49). Functions of bounded variation occur in Analysis in the context of rectifiable curves or the Riemann-Stieltjes integral. Here their use in another context is made fruitful and the Analysis-related logos block is extended. This is partly true for the use of the property Hölder-continuity (A13) as well, a notion which is rarely dealt with, although it is possible, within basic Analysis courses (at least in Germany, where Hölder-continuity appears in courses about partial differential equation).

iii. Logos blocks and their praxeologies (L_A, PO_A) from Analysis are integrated into the practice blocks of Nonlinear Approximation and constitute an aspect of a type of tasks in the advanced situation, i.e.

$$L_A, PO_A \hookrightarrow \Pi_{NL}.$$

Examples include the use of the terms average in the context of integral calculus (A23-A24) and the median for a continuous function (A24-A26), which is rarely discussed in an Analysis course, but may instead appear in a beginner's course in stochastics or statistics.

From Basic Approximation Theory to Nonlinear Approximation

Basic Approximation Theory stands for topics which do not have to be treated in specialised courses on Approximation Theory but might be taught in Numeric courses, e.g. in the context of polynomial or spline approximation or in more advanced contexts like Finite Element Methods considering Cea’s lemma for example.

i. Praxeologies from basic Approximation Theory (PO_{BA}) are taken up and supplemented by further questionings and praxeologies ($\widetilde{PO}_{BA}$) and thus constitute a supplemented or completed praxeology of Nonlinear Approximation. As such, however, the supplement could have already been dealt with in the basic approximation theory, but it is rather not, i.e.

$$PO_{BA} \& \widetilde{PO}_{BA} \hookrightarrow PO_{NL}.$$

The saturation results that go beyond the direct estimates are mentioned here as an example (A18-A21). These could already be treated as such in basic courses, but will typically not be treated in introductory or advanced Numeric courses. In the context of Nonlinear Approximation, such results are obviously of particular relevance: From the direct estimates alone one cannot justify the additional gain by Nonlinear Approximation, because it could be that better direct estimates apply to Linear Approximation. The saturation results exclude this in the sense that better estimates, which could apply in principle to certain function classes in single cases, apply here only to the trivial case of constant functions. In this case, the approximation error in the context of piecewise constant functions and linear approximation is zero.

ii. Praxeologies from basic Approximation Theory (PO_{BA}) were again taken up but now supplemented by discourses from Nonlinear Approximation in such a way, that both were integrated to a new praxeology ($\widetilde{PO}_{NL}$), i.e.

$$PO_{BA} \& PO_{NL} \hookrightarrow \widetilde{PO}_{NL}.$$

Following the considerations in i., one could think here of the Nonlinear Approximation results discussed at the end of the Appendix. (A50-A53 in combination with A18-A21)

From Nonlinear Approximation to another Advanced Course and vice versa

Nonlinear approximation is known to be at the intersection between Numeric and various advanced areas of Analysis, such as the Theory of Function Spaces or Interpolation Theory. Accordingly, Nonlinear Approximation praxeologies can be found as aspects of a praxeology from another area just mentioned and vice versa.

i. Praxeologies of Nonlinear Approximation constitute an aspect (of the practice block, the logos block, or both) of a praxeology of another advanced mathematical domain (PO_{AAC}), i.e.

$$PO_{NL} \hookrightarrow PO_{AAC}.$$

Nonlinear Approximation and its praxeologies play a role in the advanced Theory of Function Spaces, for example in the study of characterizations and embeddings of function spaces (Hochmuth, 2002), then of course in the study of adaptive numerical methods , the related regularity theory of partial differential equations and integral equations (DeVore 1998), but also in Stochastics (Kerkyacharian & Picard, 2000).

ii. Praxeologies of another mathematical domain (PO_{AAC}) arise as aspects (of practical blocks, logos blocks or both) of a praxeology in Nonlinear Approximation, i.e.

$$PO_{AAC} \hookrightarrow PO_{NL}.$$

Pertinent examples such as the Theory of Function Spaces (for inherent function related characterizations of approximation orders) or Interpolation Theory (ditto) have already been mentioned.

RATIONALES

Rationales are rather extensively explicated by DeVore (1998). This is, of course, in line with the type of publication: The series *Acta Numerica* as one of its goals wants to inform advanced students and researchers on all levels, in particular those with another special field, about specific underlying ideas from a new field, its most relevant results within and also across the fields. To a certain extent, the articles serve as an appetizer and, with this in mind, should in particular answer the question why someone should be interested in the area presented.

In addition to this general level, there is also a certain type of question that is often not addressed in research papers or in the standard literature that informs the teaching of basic university courses: There is a difference between the possibility to prove something and the formulation of this something as a theorem. Or in other words: To formulate something as a theorem is related to some justification, which often remains implicit but is made explicit in the text considered, and would substantially contribute to the logos-block of praxeologies, connecting, for example, praxeologies.

To provide an example, which is related to i. in the consideration of the transition from Basic Approximation Theory to Nonlinear Approximation: In the Appendix there is the following reason stated for highlighting the observation in (A15-A17), which would typically be presented as a theorem in a lecture: If the estimate (A17) holds for every possible partition, then necessarily $f \in \mathrm{Lip}_M \alpha$. In Approximation Theory and sometimes also in Numerical Analysis, such a statement is called an inverse theorem (cf. also A18-A19). For an elementary proof of this statement cf. (p. 62). This inverse theorem justifies the assumption $f \in \mathrm{Lip}_M \alpha$ in the stated result, which means that this assumption does not only allow the application of the arguments in the proof, which is a technically orientated argument, but is inherently related to the estimate. This relates to the general aim of getting rid of those assumptions which are related to the method or approach to prove something but is possibly not necessarily linked with the proved assertion. In basic Analysis courses, there are often results presented without such questioning and related justifications to keep things more simple. Getting to know such arguments could encourage students to search for comparable situations in their knowledge and to try to clarify the questions that arise. By the way, a further justification is given by the saturation theorem (A19-A21).

Finally, the presentation of the result by Kahane and also the presentation of the proof are further explicitly justified by the argument that both express two fundamental characteristics of Nonlinear Approximation:

a) The first characteristic is that the partitions, hence the approximation scheme, providing the claimed approximation property depend on the specific given function *f*. This means that in contrast to linear approximation the scheme is not given in advance independently of the target function f.

b) Secondly, the f-depending partition is obtained in the proof by balancing the variation of *f* over the intervals in the partition, that is, the partition is chosen such that the specific error is somehow equally distributed over the interval (A38-A40). In other situations, $\mathrm{Var}(f)$ is replaced by something else related to the respective error norm and also depending on *f* and the type of partitions. But balancing or equilibration often remains a crucial idea. This is also true for other adaptive schemes like for example adaptive finite element or also finite volume schemes utilizing a posteriori error estimates.

COMPARTMENTALIZATION

Our elaboration of transition situations within the little piece of work from Nonlinear Approximation relates to the second type of compartmentalization, i.e. the compartmentalization of several correct concepts (cf. the explanations in the introduction). Each transition type and the corresponding examples presented above in a specific way address relations between correct concepts, relations between praxeologies, as well as between practical and logos blocks, and point to a possible intervention against the compartmentalization of knowledge. In addition, many explicit references to relations between pure mathematics and applications are presented in further parts of the paper by DeVore on which this contribution is based. These cover examples from Signal Theory, Image Compression and regarding numerical schemes for Partial Differential Equations modelling physical phenomena. Such references in particular address the compartmentalization between symbol systems and real world entities, which was also mentioned in the introduction as a type of compartmentalization.

OUTLOOK

The presented analyses have to be continued in the future and transferred to other mathematical areas. With regard to the issues of transitions, rationales and compartmentalization the contribution could demonstrate that Nonlinear Approximation possesses a strong potential for bridging and extending praxeologies from Analysis and beyond. It seems remarkable that this could be illustrated in so many (with respect to praxeological aspects) structural different ways by a brief example. It is reasonable to ask whether such learning potentials and manifold praxeological patterns of transitions could also be indicated for examples regarding other mathematical domains. Many questions were left open also for the analysed case: For example, it is not clear to what extent the potentials can be realised under the currently dominant teaching and learning conditions and how this could eventually be done effectively. The latter issue stimulates for example to think about whether inquiry orientated education approaches (Artigue & Blomhøj, 2013; Barquero, Serrano & Ruiz-Munzón, 2016) are particularly suitable. With regard to the education of prospective secondary school teachers and against the background of the job analysis of Bass and Ball (2004), tasks for Analysis courses are presented in (Hochmuth, 2015), which already take up some of the observations presented here. In corresponding future empirical studies, emotional-motivational aspects would also have to be considered.

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APPENDIX

The following sketch of the mathematical ideas and notions around linear vs. nonlinear approximation and the famous result by Kahane (1961) closely follows DeVore (1998, 60-69).

For functions f on an interval $I := [0,1]$ two types of approximation are considered in the following, linear and nonlinear approximation. Linear approximation starts with an a priori given sequence of partitions T, $0 =: t_0 < t_1 < \cdots < t_N := 1$, $N \in \mathbb{N}$, related sets $\Pi := \{I_k\}_{k=1}^{N}$ of intervals $I_k := [t_{k-1}, t_k), 1 \leq k \leq N$, and linear spaces of piecewise constant functions relative to the partitions T of dimension N denoted by $S(T)$. For uniformly continuous functions f and with respect to $S(T)$ the (linear) approximation error in the uniform (L_∞) norm is defined by $s(f,T) := \inf_{\chi \in S(T)} \|f - \chi\|_\infty$. Hereby and in the following, norms and semi-norms without indicating the considered interval etc. are to be understood w.r.t. the interval I.

The approximation error is related to the mesh length $\delta_T := \max_{0 \leq k \leq N-1} |t_{k+1} - t_k|$ and the smoothness of *f*: For $\alpha \in (0,1]$ and $M > 0$, $\mathrm{Lip}_M \alpha$ denote the set of all functions f on I such that

$$|f(x) - f(y)| \leq M|x - y|^\alpha$$

and Lip $\alpha := \bigcup_{M>0} \mathrm{Lip}_M \alpha$. In particular, $f \in \mathrm{Lip}\ 1$ if and only if *f* is absolutely continuous and $f' \in L_\infty$, where the derivative could be understood in the distributional sense. Then for $f \in \mathrm{Lip}_M \alpha$ holds

$$s(f,T) \leq M \left(\frac{\delta_T}{2}\right)^\alpha :$$

For the midpoints ξ_J of $J \in \Pi$ hold by definition $|x - \xi_J| \leq \frac{\delta_T}{2}$, $x \in J$, and for the related piecewise constant function $\chi \in S(T)$ defined by $\chi(x) := f(\xi_J)$, $x \in J, J \in \Pi$, since $f \in \mathrm{Lip}_M \alpha$,

$$\|f - \chi\|_\infty \leq M \left(\frac{\delta_T}{2}\right)^\alpha .$$

Furthermore, one can show: If the estimate holds for every possible partition, then necessarily $f \in \mathrm{Lip}_M \alpha$. Such a type of statement is usually called as inverse theorem. Additionally a saturation theorem can be shown: If $s(f,T) = o(\delta_T)$for partitions T then f is a constant, which means, that only trivial functions can be approximated with order better than $O(\delta_T)$.

Replacing the L_∞-norm by L_p-norms, $0 < p < \infty$, similar results hold. It is interesting to note that for $p \geq 1$the adequate constants are given by the average of f over J, that is $\chi(x) \coloneqq \frac{1}{|J|}\int_J \ f(t)dt$, $x \in J, J \in \Pi$, and for $0 < p < 1$ adequate constants are the medians of f on the intervals J, where medians are defined to be any number q for which $|\{x \in J | f(x) \geq q\}| \geq \frac{|J|}{2}$ and $|\{x \in J | f(x) \leq q\}| \geq \frac{|J|}{2}$.

Nonlinear approximation is related to $\Sigma_n \coloneqq \bigcup_{\#T=n+1} S^1(T)$ ($\#T$ denotes the cardinality of the set of breaking points T), which is the set of piecewise constants with at most n pieces. Obviously, Σ_n is not a linear space, since adding two functions from Σ_n results in a piecewise constant function with possibly more than n breaking points. Given a uniformly continuous function f the uniform error of nonlinear piecewise constant approximation is defined by $\sigma_n(f) \coloneqq \inf_{\chi \in \Sigma_n} \|f - \chi\|_\infty$.

Kahane (1961) has proven that for a function $f \in C(I)$ one has

$$\sigma_n(f) \leq \frac{M}{2n}, n = 1,2, \ldots,$$

if and only if $f \in \mathrm{BV}$, that is, f is of bounded variation on I, and $|f|_{\mathrm{BV}} \coloneqq \mathrm{Var}(f)$ is identical with the smallest constant M for which the inequality (33) holds. Hereby, for a function $f: [a,b] \to \mathbb{R}$ the variation $\mathrm{Var}_{[a,b]} f$ is defined by $\sup_P \sum_{i=1}^{m-1} |f(x_{i+1}) - f(x_i)|$, where the supremum is considered with respect to arbitrary partitions $P = \{x_1 < x_2 < \cdots < x_m | m \in \mathbb{N}\}$.

A proof goes as follows: For $f \in \mathrm{BV}$ with $M \coloneqq \mathrm{Var}(f)$ there is a partition $T \coloneqq \{0 \coloneqq t_0 < t_1 < \cdots < t_n \coloneqq 1\}$ such that $\mathrm{Var}_{[t_{k-1},t_k)} f \leq \frac{M}{n}$, $k = 1, \ldots, n$. If $a_k$is the median value of f on $[t_{k-1}, t_k]$, and $\chi_n(x) \coloneqq a_k, x \in [t_{k-1}, t_k), k = 1, \ldots, n$, then $\chi_n \in \Sigma_n$ and $\|f - \chi_n\|_\infty \leq \frac{M}{2n}$. Now to the other direction: If the inequality in (33) holds for some $M > 0$, let $\chi_n \in \Sigma_n$ satisfy $\|f - \chi_n\|_\infty \leq \frac{M+\epsilon}{2n}$ with $\epsilon > 0$. If $x_0 \coloneqq 0 < x_1 < \cdots < x_m \coloneqq 1$is an arbitrary partition for I and ν_k is the number of values that $S\chi_n$ attains on $[x_{k-1}, x_k)$, then

$$|f(x_k) - f(x_{k-1})| \leq 2\nu_k \|f - \chi_n\|_\infty \leq \frac{\nu_k(M + \epsilon)}{n}, k = 1,2, \ldots, m.$$

Since $\sum_{k=1}^m \nu_k \leq m + n$, we get

$$\sum_{k=1}^m |f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^m \frac{\nu_k(M + \epsilon)}{n} \leq (M + \epsilon)\left(1 + \frac{m}{n}\right).$$

Letting $n \to \infty$ and then $\epsilon \to 0$ we find

$$\sum_{k=1}^m |f(x_k) - f(x_{k-1})| \leq M,$$

which shows $\mathrm{Var}(f) \leq M$.

Summarizing the results with respect to linear and nonlinear approximation by piecewise constants, one has the convergence order 1 for linear approximation if $f \in \mathrm{Lip}\ 1$ and for nonlinear approximation if $f \in \mathrm{BV}$. Since $f \in \mathrm{Lip}\ 1$ means $f' \in L_\infty$ and $f \in \mathrm{BV}$ that $f' \in L_1$, this result shows that the second condition is essentially weaker than the first one.

Analyse discursive de l'enseignement des fonctions trigonométriques dans la transition lycée/université

Faten Khalloufi-Mouha

Université de Carthage. Faculté des sciences de Bizerte. Faten.khalloufi@fsb.u-carthage.tn

Abstract: Utilisant le cadre théorique de l'approche commognitive, ce travail explore l'évolution au niveau des attentes et des exigences relatives à l'enseignement de la notion de fonction trigonométrique lors de la transition lycée/université, à travers l'étude des caractéristiques des routines visées. Les résultats des analyses des organisations proposées pour l'enseignement de cette notion, fait apparaitre une fausse continuité au niveau de ces routines visées qui se traduit par une continuité apparente au niveau des tâches et un changement important au niveau des procédures.

Keywords: enseignement apprentissage d'une notion spécifique au niveau universitaire, transition lycée/université, approche commognitive, routine, fonctions trigonométriques et leurs réciproques.

INTRODUCTION ET PROBLÉMATIQUE

Les fonctions trigonométriques sont les premières fonctions périodiques et transcendantes que les élèves rencontrent au niveau de l'enseignement secondaire. Ces fonctions qui se situent au carrefour de plusieurs cadres mathématiques (la géométrie, l'algèbre et l'analyse) sont connectées aux différents concepts mathématiques appartenant à ces cadres (angles, arc, fonctions, équations, …) et mettent en relations différents registres sémiotiques et différents types de représentation (Khalloufi-Mouha, 2014). Dans nos travaux antérieurs (Khalloufi-Mouha, 2018, 2014, Khalloufi-Mouha & Smida, 2012) nous avons identifié l'importance des difficultés relatives à la construction d'une signification mathématique de la notion de fonction trigonométrique qui soit cohérente avec les connaissances antérieures du cadre géométrique de la trigonométrie chez des élèves de 2ème année de l'enseignement secondaire tunisien (16/17 ans). Au niveau de l'enseignement supérieur, les fonctions trigonométriques jouent un rôle très important dans l'introduction et l'apprentissage de l'Analyse, notamment dans leurs relations avec les notions d'intégrales, de séries numériques, de séries de Fourier …. Par conséquence, cette notion est considérée comme très importante dans les études universitaires pour la plupart des disciplines scientifiques utilisant les mathématiques. Les recherches ayant abordé l'enseignement et l'apprentissage de la notion de fonction trigonométrique au niveau de l'enseignement universitaire ont souligné que les étudiants continuent à rencontrer des difficultés importantes (Gueudet & Quéré, 2018, Mesa & Goldstein, 2017, Weber, 2005).

Dans son article, Weber (2005) a analysé l'apprentissage du concept de fonction trigonométrique par deux groupes d'étudiants universitaires. Le premier groupe a été enseigné par un professeur utilisant un cours magistral classique, tandis que le second a été enseigné selon un paradigme d'enseignement expérimental basé sur la notion de procept de Gray et Tall (1994) et les théories cognitives qui conceptualisent l'apprentissage à travers les notions de processus-objets. L'analyse d'interviews et d'un test papier-crayon ont mis en évidence que les étudiants ayant suivi le cours magistral ont développé une compréhension de type procédurale qui n'est pas liée à une signification mathématique. Cependant, les étudiants qui ont reçu un enseignement expérimental ont développé une compréhension plus approfondie. Les étudiants interrogés ont pu opérationnaliser leurs connaissances sur le processus de calcul du sinus d'un réel pour justifier les propriétés de la fonction sinus. De plus, les réponses des étudiants interrogés ont indiqué qu'ils considéraient les expressions trigonométriques comme des procepts. Mesa et Goldstein (2017) ont identifié l'existence de différentes significations relatives aux notions d'angles, de fonction trigonométrique et de fonctions trigonométriques inverses à travers l'étude des organisations proposées dans les manuels scolaires relatives au niveau collégial (post-secondaire) pour l'enseignement de la trigonométrie. Ils supposent que cela pourra engendrer des difficultés chez les étudiants lors de la résolution de situations problèmes surtout que l'enseignement proposé dans ces manuels ne prend pas en charge les connexions nécessaires entre les cadres et les registres mis en jeu dans cet enseignement. La problématique de l'articulation entre les différents cadres et registres lors de l'enseignement des fonctions trigonométrique a été également l'objet de l'étude de Gueudet et Quéré (2018). L'étude a porté sur l'enseignement du thème de la trigonométrie dans les ressources en lignes proposées aux futurs ingénieurs (ingénierie électrique) en France et cela en comparant l'utilisation de la trigonométrie en tant qu'outil, dans un cours d'ingénierie électrique et le contenu des cours en lignes de mathématiques proposé au ingénieurs relatif au chapitre trigonométrie. Dans ce travail, les auteurs ont identifié l'écart entre les besoins relatifs aux notions de trigonométrie dans le cours d'électricité et les cours de mathématiques. Les résultats des travaux précédents, attestent que les difficultés des étudiants relèvent essentiellement d'un manque au niveau de l'habilité à faire des connexions entre les différents cadres, registres et concepts mis en jeu (Gueudet & Quéré, 2018), afin d'identifier les plus pertinents lors de la résolution d'une situation problème (Mesa et Goldstein (2017). Les connaissances des étudiants restent ainsi à un niveau procédural (Webr, 2005) ne permettant pas une appréhension plus profonde articulant les cadres, registres et concepts relatifs à la notion de fonction trigonométrique.

En nous appuyant sur les résultats des travaux précédents, ainsi que sur nos travaux antérieurs sur l'introduction des fonctions trigonométriques au niveau de l'enseignement secondaire (Khalloufi, 2018, 2014, 2009, Khalloufi & Smida, 2012), nous avons choisi d'explorer, dans le contexte Tunisien, en utilisant une approche discursive, l'enseignement des fonctions trigonométriques dans la transition lycée/université. Le but étant d'étudier les changements au niveau des exigences et des attentes auprès des étudiants lors de leur entrée en première année de l'enseignement universitaire entamant des études en sciences de l'informatique. Le travail est guidé par les deux questions de recherches suivantes :

Q1 : Quels sont les changements au niveau des exigences et des attentes auprès des étudiants, autour de la notion de fonction trigonométrique, lors de la transition lycée/université ?

Q2 : Quels sont les changements au niveau des caractéristiques des routines visées par l'enseignement des fonctions trigonométriques au niveau de la première année universitaire ?

CADRE THÉORIQUE

La théorie commognitive (TCM) (Sfard, 2008) est une approche discursive qui définit les mathématiques comme une activité de communication (Sfard, 2012) et l'apprentissage des mathématiques comme un développement du discours. Selon cette approche, le discours mathématique sur la notion de fonction trigonométrique émerge lorsque les apprenants sont engagés dans une communication avec les autres ou avec eux-mêmes à propos de cette notion. Comme tout discours mathématique, le discours sur les fonctions trigonométriques, se distingue par quatre caractéristiques. Le vocabulaire spécifique, qui constitue la première caractéristique, correspond à l'utilisation de la terminologie mathématique et les termes techniques. L'utilisation de ce vocabulaire obéit à des définitions explicites (fonction sinus, cosinus et tangente, les fonctions réciproque des fonctions trigonométriques…). La deuxième caractéristique correspond aux médiateurs visuels. Nous distinguons les médiateurs visuels graphiques (le cercle trigonométrique et les représentations graphiques des fonctions trigonométriques et leurs réciproques) et les médiateurs visuels symboliques tels que les expressions *sin(x), cos(x), tan(x), arccos(x), arcsin(x)* ou *arctan(x)*. La troisième caractéristique correspond aux récits approuvés qui comportent les textes écrits ou oraux décrivant les objets, les processus et les relations entre eux et qui sont soumis à la validation, à la modification ou au rejet selon des règles définies par la communauté (définitions, théorèmes et preuves). La dernière caractéristique est la notion de routine. Les routines comprennent les pratiques régulièrement utilisées et bien définies par la communauté (telles que la définition, la conjecture, la preuve, l'estimation, la généralisation et l'abstraction). Dans les travaux récents (Lavie, Steiner & Sfard, 2019 et Lavie & Sfard, 2019) la notion de routine gagne en opérationnalité dans sa nouvelle définition et devient une unité d'analyse pour étudier l'apprentissage. Pour définir la notion de routine, Lavie et al. (2019) introduisent le concept de *task situation* (la situation dans laquelle une personne ressent le besoin d'agir) et le concept d'*espace de recherche précédent (precedent-search-space)* qui comprend les événements passés considérés comme suffisamment similaires à la tâche actuelle pour justifier la répétition de la même procédure. En utilisant ces notions, Lavie et al (2019) définissent la routine comme étant le couple tâche-procédure. Dans cette perspective, Lavie et ses collègues considèrent l'apprentissage comme un processus de routinisation progressive et l'étude de l'apprentissage revient à l'étude du processus d'émergence et de développement des routines. Dans cette partie de notre travail, relative à l'étude des changements au niveau des exigences et des attentes auprès des étudiants, autour de la notion de fonction trigonométrique, lors de la transition lycée/université, nous avons désigné par « routines visées » l'ensemble des routines proposées dans un objectif d'enseignement d'une notion mathématique spécifique. Ces routines sont évoquées à partir de leurs tâches sans référence à une task situation particulière ou à une personne spécifique. Ce sont les routines telles qu'elles sont susceptibles d'être interprétées et réalisées par un expert en utilisant les connaissances relatives au niveau scolaire en question. Les procédures sont décrites soit à l'aide d'algorithmes qui déterminent la façon d'agir, soit à partir des règles qui guident l'action de l'apprenant comme exemple les règles utilisées pour prouver des théorèmes ou définir de nouveaux termes mathématiques (Morgan & Sfard, 2016, p.101). Nous admettons dans notre travail l'hypothèse que les exigences de l'enseignement se traduisent dans les organisations mathématiques proposées pour l'apprentissage des notions en jeu, à travers la mise en place de routines visées, auxquelles les apprenants sont amenés à s'engager lors de la résolution de problèmes. Ainsi, approcher nos questions de recherche en termes de routines, nos amène à identifier les changements au niveau des routines relatives à l'enseignement des fonctions trigonométriques que les auteurs des manuels, au niveau du lycée, et que l'enseignant du cours de mathématique, au niveau de la première année universitaire, cherchent à installer.

ANALYSE DE L'ÉVOLUTION DES ROUTINES

Méthodologie

Dans ce travail nous analysons le manuel de 3ème (17/18 ans) l'année où les fonctions trigonométriques commencent à être un objet d'enseignement ainsi que le manuel de 4ème année (18/19 ans). Nous rappelons que dans le contexte Tunisien, à chaque niveau scolaire correspond un manuel officiel unique, utilisé comme une ressource pour les enseignants et comme un outil de travail pour les élèves. L'analyse du manuel de 3ème porte sur les parties « cours » et « exercices et problèmes » du 8ème chapitre « Fonctions trigonométriques ». Pour le manuel de 4ème année (Tome 1), nous avons examiné les chapitres où les fonctions trigonométriques sont utilisées dans l'apprentissage de nouvelles notions mathématiques. Au niveau de l'enseignement supérieur, nous avons analysé dans les notes de cours de l'enseignant responsable du module mathématique du premier semestre, la partie du chapitre « Fonctions numériques » relative aux fonctions trigonométriques et leurs réciproques ainsi que la série d'exercices associées à cette partie.

Pour l'identification et l'analyse des routines et en adaptant les indicateurs des routines explicités dans le schème analytique Morgan et Sfard (2016), nous identifions les routines visées pour l'apprentissage des fonctions trigonométriques à travers l'identification des tâches proposées et des procédures susceptibles d'être utilisées pour accomplir ces tâches. L'analyse des routines est guidée par les questions suivantes :

Composantes des routines visées	**Questions guidant l'analyse**
Catégorisation des tâches relatives aux fonctions trigonométriques, auxquelles	• Quels sont les cadres mathématiques mis en jeu ?

les élèves sont amenés à s'engager et qui sont proposées dans les manuels et dans les notes du cours et la série d'exercices proposées aux étudiants de 1ère année sciences de l'informatique.	• Quels sont les différentes réalisations associées à la notion de fonctions trigonométriques (les différentes façons de définir et de représenter ces fonctions) ? • Quels sont les médiateurs visuels évoqués ? • Quels sont les objets et les concepts mathématiques mis en relation avec les fonctions trigonométriques ?
Les caractéristiques des procédures routinières que les élèves doivent être capables de mobiliser pour accomplir ces types de tâches.	• Les procédures sont-elles algorithmiques ou heuristiques ? • Quel est le degré de complexité associé à ces procédures ? • Les procédures sont-elles implicites (à identifier par les élèves) ou explicites dans l'énoncé de la tâche ?

Table 1 : méthodologie de l'identification et l'analyse des routines visées.

Analyse des manuels scolaires

L'analyse de l'organisation proposée dans les manuels pour l'introduction et l'étude de la notion de fonction trigonométrique, fait appel à différentes réalisations[1] de cet objet, qui font appel à différents types de médiateurs visuels (symbolique, tableau, graphique, …). Les activités proposées utilisent ces différentes réalisations afin d'étudier les propriétés des fonctions trigonométriques et leurs relations avec les différents objets mathématiques. L'analyse en termes de routines visées, a permis d'identifier trois catégories de routines selon la nature de la propriété mobilisée des fonctions trigonométriques : la première catégorie est celle mobilisant l'une des propriétés globale, locale ou ponctuelle des fonctions trigonométriques (Vandebrouck, 2011). La seconde catégorie est relative aux routines articulant entre deux de ces propriétés et la troisième catégorie est relative à celles articulant les trois aspects, global, local et ponctuel.

Catégorie 1 : dans cette catégorie nous distinguons entre les routines ayant un aspect ponctuel, local ou global. *Les routines ponctuelles* sont les routines essentiellement associées à des tâches de calcul des images de certains réels par une fonction trigonométrique et les routines de résolution algébrique ou graphique des équations trigonométriques où le travail est localisé en des points d'intersection. Le second type, *les routines locales*, ce sont les routines de détermination de la continuité et de la

[1] Sfard (2008) définit les réalisations comme « perceptually accessible objects that may be operated upon in the attempt to produce or substantiate narratives" about a signifier (p. 154)

dérivabilité, en un point donné, des fonctions trigonométriques simples ou composées avec des fonctions algébriques. Les procédures associées sont essentiellement de type algébrique. Elles mettent en jeu la notion de voisinages et sont mobilisables lors du calcul de limites. Selon les procédures de calcul des limites on distingue deux sous-routines. L'une émerge lorsqu'il s'agit d'une application directe d'une limite usuelle et la seconde, lorsqu'il s'agit de procéder à une modification de l'expression de la fonction afin de faire apparaitre une limite usuelle à travers des minorations, majorations et encadrement avec des fonctions usuelles. Le troisième type, désigné par *routines globales*, comporte les routines de détermination de l'ensemble de définition, l'étude de la continuité et la dérivabilité sur un intervalle ainsi que l'étude des variations. Ces études, vu la propriété de périodicité des fonctions trigonométriques, sont généralement réduites à un intervalle, déterminé à partir de l'étude de la parité et la périodicité de ces fonctions. L'analyse des différents problèmes proposés, fait apparaitre qu'en général, l'intervalle d'étude est proposé dans l'énoncé et sa détermination n'est pas à la charge des élèves notamment au niveau des tâches de l'étude de l'existence d'une fonction réciproque où la procédure proposée consiste à appliquer le théorème de la fonction réciproque dans un intervalle imposé par l'énoncé (fig1).

Catégorie 2 : la seconde catégorie est relative aux routines articulant deux types des routines précédentes. Elles comportent les routines de construction des représentations graphiques qui articulent entre le ponctuel (construction des points) et le global (la forme de la courbe). Il y a également les routines de détermination et de représentation des tangentes ou des asymptotes à la représentation graphique des fonctions trigonométriques. Ce type de routines articule entre le ponctuel et le local.

Catégorie 3 : c'est la catégorie des routines articulant les trois aspects, ponctuel, local et global. Elle comporte à titre d'exemple, les routines de réalisation de tableau de variation des fonctions trigonométriques simples ou composées avec des fonctions algébriques. Les procédures relatives à ces routines comportent la détermination des valeurs de la fonction en des points, l'étude des variations sur le domaine d'étude ainsi que le calcul des limites.

À travers nos analyses des caractéristiques des routines identifiées, nous avons repéré que dans la plupart de ces routines, les procédures sont imposées dans les énoncés à travers des questions intermédiaires qui guident entièrement le travail des élèves en indiquant la procédure visée. Cela permet de fournir des modèles unifiés des routines visées. Nous avons également noté que les routines relatives à la parité et la périodicité ne font pas l'objet de beaucoup de travail au niveau des exercices proposés. Dans ces exercices l'énoncé impose l'intervalle d'étude de la fonction trigonométrique donnée. Nous illustrons par l'exercice suivant, extrait du manuel de 4ème année (Tome 1, exercice 26, p93)

26 A/ Soit la fonction f définie sur $[0, 1]$ par $f(x) = \sin\left(\frac{\pi}{2}x\right)$.

1. a. Montrer que f réalise une bijection de $[0, 1]$ sur $[0, 1]$.

On notera f^{-1}, la fonction réciproque de f.

b. Calculer $f^{-1}\left(\frac{1}{2}\right)$ et $f^{-1}\left(\frac{\sqrt{2}}{2}\right)$.

2. a. Montrer que f^{-1} est dérivable sur $[0, 1[$.

b. Calculer $(f^{-1})'\left(\frac{1}{2}\right)$.

c. Montrer que pour tout x de $[0, 1[$, $(f^{-1})'(x) = \frac{2}{\pi\sqrt{1-x^2}}$.

3. Tracer dans un repère orthonormé C_f et $C_{f^{-1}}$.

L’exercice est extrait de la partie « Exercices et problèmes » du chapitre « Fonctions réciproques ». L’objectif de l’exercice est la définition, l’étude de la dérivabilité et la représentation graphique de la fonction réciproque de *f*. Les questions 1.a, 2.a et 3 renvoient à des routines globales de détermination du domaine de continuité et de dérivabilité. L’énoncé fait également apparaitre une importante utilisation des routines ponctuelles de détermination de l’image d’un réel par la fonction réciproque et par sa dérivée. La procédure de cette routine visée, consiste à appliquer la relation entre une fonction et sa réciproque ainsi que l’utilisation des angles remarquables pour la résolution d’équations trigonométriques simples. Nous supposons que l’utilisation de ces routines ponctuelles vise une justification de l’existence de la fonction réciproque de *f* qui ne peut pas être explicitée à travers une expression spécifique comme le cas des fonctions précédemment étudiées. De même, la représentation graphique de cette fonction qui est une autre réalisation de la fonction réciproque vise également la justification de son existence. Nous considérons que à ce niveau, les fonctions réciproques des fonctions trigonométriques restent au niveau procédural et son approchées à travers un algorithme qui se caractérise par l’application directe du théorème de la fonction réciproque ainsi que la relation entre une fonction et sa réciproque. Dans l’énoncé de cet exercice, nous avons également noté que le domaine de définition est imposé et que l’intervalle image par *f* et le domaine de dérivabilité de sa réciproque son donnés. La tâche de l’élève revient à justifier ces intervalles à travers l’application du théorème du cours. Les procédures des routines visées par cet exercice son imposées par l’énoncé et l’élève est guidé à travers les questions intermédiaires qui visent l’exécution de routines précises.

Analyse des notes de cours et de la série de TD proposés aux étudiants

Le module de mathématiques du premier semestre des sciences de l’informatique a pour objectif d’introduire et de reprendre des notions de base. La notion de fonction trigonométrique est revisitée dans ce module au niveau du 2ème chapitre qui s’intitule « Fonctions numériques ». Ce chapitre consiste d'abord à reprendre succinctement des résultats rencontrés durant les deux dernières années du lycée relatives à l’étude de fonctions numériques et leurs représentations graphiques. Il reprend également des éléments de base sur le calcul des limites, l'étude de la continuité, la dérivabilité, la parité et les éléments de symétrie, en plus des théorèmes fondamentaux comme le théorème des valeurs intermédiaires et celui des accroissements finis. Par la suite il y a un passage à l'étude des propriétés de la réciprocité d'une fonction, et les conséquences qui en découlent sur les représentations graphiques, l'explicitation de la fonction réciproque et de sa dérivée. En fin de chapitre, sont introduites les fonctions réciproques des fonctions trigonométriques et des fonctions hyperboliques, comme un nouveau champ d'application de tout ce qui précède. L'étude des fonctions trigonométriques et hyperboliques réciproques s'étale sur deux séances de cours (3heures) et deux séances de travaux dirigés (3heures).

L'analyse des notes de cours de l'enseignant et de la série d'exercices associée à ce chapitre a permis d'identifier l'émergence de nouveaux types de routines associées aux fonctions trigonométriques réciproques. Ces routines coexistent avec les routines déjà rencontrées au niveau du secondaire. Cependant, nous avons noté une augmentation remarquable au niveau du degrés de complexité des fonctions proposées ainsi que l'émergence de nouvelles fonctions transcendantes (la composée de fonctions trigonométriques avec des fonctions algébriques ou transcendantes, la composée de fonctions trigonométriques avec des fonctions trigonométriques réciproques). Nous illustrons par l'exercice suivant extrait de la série de TD

Exercice 7

Soit $f(x) = \arcsin(\frac{2x}{x^2+1})$.
1) Montrer que f est définie sur $\mathbb{R}$.
2) Vérifier que f est impaire et que pour $x \neq 0$, $f(\frac{1}{x}) = f(x)$.
3) Déterminer le domaine de dérivabilité de f.
4) Calculer $f'(x)$.
5)Simplifier f.

L'objet de l'exercice est l'utilisation des propriétés de la fonction arcsinus pour l'étude de la composée de la fonction arcsinus avec une fonction rationnelle. L'énoncé de l'exercice fait apparaitre l'importance du degré de complexité au niveau de la fonction étudiée et des procédures algébriques à utiliser pour déterminer le domaine de définition, le domaine de dérivabilité ainsi que pour le calcul de la dérivée.

L'analyse des notes de cours et des exercices proposés dans la série a permis de relever une importance du nombre de routines relatives à la troisième catégorie puisque les tâches proposées articulent généralement entre les différents aspects ponctuel, local et global des fonctions. Les routines de représentation graphique de fonction trigonométrique et/ou de sa réciproque apparaissent uniquement dans la partie cours lors de l'introduction et la caractérisation des fonctions *arcsinus, arccosinus* et *arctangente* ce qui atteste une régression au niveau de l'importance accordée aux médiateurs visuels graphiques par rapport à l'enseignement secondaire. L'analyse a également mis en évidence l'émergence de routines reliant une fonction trigonométrique et sa réciproque ou entre une fonction trigonométrique et la réciproque d'une autre fonction trigonométrique. Ces routines permettent la mobilisation des propriétés de parité et de périodicité des fonctions trigonométriques. Ces routines sont de type global dans le cas de la détermination du domaine de définition des fonctions étudiés. Cependant nous avons également identifié la mobilisation de ces propriétés dans le cas de nouvelles routines ponctuelles visant la détermination des images de certaines valeurs par ces fonctions composées. Nous donnons à titre d'exemple la détermination de $Arcsin[sin(\frac{15\pi}{7})]$.

Nos analyses font apparaitre une fausse continuité au niveau des routines relatives aux fonctions trigonométriques qui sont susceptibles de cohabiter avec les nouvelles routines relatives aux fonctions trigonométriques réciproques introduites en première année universitaire. En fait, le changement des contextes des tâches routinières au niveau de l'université, la complexité des fonctions étudiées et l'introduction des fonctions trigonométriques réciproques nécessite une modification au niveau du domaine d'applicabilité des routines. Cela est susceptible d'engendrer certaines difficultés à individualiser les routines visées par l'enseignant vu la nécessité de développer de nouvelles procédures plus complexes que celles au niveau du secondaire. L'analyse fait apparaitre également qu'au niveau de l'université, les étudiants ont plus d'autonomie pour l'élaboration des tâches proposées ce qui engendre une diversité des routines susceptibles d'être effectués et par conséquence, l'existence de plusieurs routines associées à une même tâche.

CONCLUSION

Dans ce travail nous avons exploré les exigences et les attentes relatives à l'enseignement de la notion de fonction trigonométrique, à travers l'identification de l'évolution des routines visées et l'analyse de leurs caractéristiques en utilisant le concept de routine de l'approche commognitive. Les résultats nous ont permis d'identifier trois catégories de routines selon la nature de la propriété de fonction trigonométrique visée par cette routine. L'analyse de l'évolution des caractéristiques de ces routines a fait apparaitre une fausse continuité qui se traduit par une continuité apparente au niveau des tâches et un changement important au niveau des procédures. Ces procédures relèvent d'une application directe des théorèmes et définitions du cours au niveau de l'enseignement secondaire et sont entièrement imposées aux élèves qui sont guidés par des questions intermédiaires. Au niveau universitaire, la complexité des fonctions étudiées et l'introduction des fonctions trigonométriques réciproques nécessite le développement de nouvelles procédures plus complexes.

Dans ce travail, nous avons focalisé sur les routines spécifiques à la notion de fonction trigonométriques. Cependant, plusieurs autres routines peuvent entrer en jeu et influencer l'apprentissage de cette notion. Parmi ces routines il y a les routines d'instanciation. En fait, l'importance du niveau de rigueur associé à l'utilisation d'un ensemble de règles formelles bien définies au niveau de l'enseignement supérieur rend le discours mathématique universitaire loin de ce qu'un nouveau bachelier connait de l'enseignement secondaire.

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“I only know the absolute value function” – About students’ concept images and example spaces concerning continuity and differentiability

Elisa Lankeit[1] and Rolf Biehler[2]

[1]University of Paderborn, Germany, elankeit@math.upb.de;
[2]University of Paderborn, Germany

We analyse students’ work on a task concerning relations of various concepts of differentiability in $\mathbb{R}^n$ *to find out about their concept images about continuous but non-differentiable functions other than the prototypical examples of functions with a “cusp” like the absolute value function. We identify different types of continuous, non-differentiable functions and show which types seem to be more accessible for the students than others. We use a study with sixteen students in an Analysis-II-course at a German university.*

Keywords: Teaching and learning of Analysis and calculus, epistemological studies of mathematical topics.

INTRODUCTION

Differentiability and derivatives are essential topics in school and university mathematics and have been studied extensively in different contexts (e. g. Orton, 1983; Zandieh, 2000). In another article (Lankeit & Biehler, 2019), we described a task where Analysis-II-students were asked to explore the relations between different concepts of differentiability in $\mathbb{R}^n$ such as total differentiability, partial differentiability, one-sided directional differentiability and continuity. We found out that one of the implications students had the most difficulties with was the question of whether continuity implied the existence of all one-sided directional derivatives. Only one out of 31 students who handed in their written work, produced in a tutorial group meeting, stated a correct example for a function that is continuous but for which not all one-sided directional derivatives exist in $x = 0$: the function $\sqrt{|x|}$. Five of the students gave the absolute value function as an example, which is not a legitimate counterexample since all one-sided directional derivatives exist. To find out why so many students could not come up with a valid example, we will have a look at the transcripts of a subgroup of sixteen students whom we videotaped while working on this task. We will examine how they argue and what functions they consider. This analysis will provide exciting insights into these students’ concept images concerning continuous and differentiable or non-differentiable functions.

THEORETICAL FRAMEWORK AND LITERATURE REVIEW

We based our task design and analysis on Brousseau’s Theory of Didactical Situations (TDS) (Brousseau, 2006). A *situation* describes the circumstances in which students find themselves concerning their *milieu* (the set of objects on hand, available knowledge and interaction with others). In this theory, we distinguish between didactical and adidactical situations. A situation is of adidactic nature if the teacher does not instruct, but students work autonomously and learn by adapting to the milieu whereas, in a didactical situation, acculturation happens through institutionalisation and devolution. For a more detailed description and a well-presented introduction of TDS, see for example, Artigue, Haspekian, and Corblin-Lenfant (2014). For the analysis of students' work on the task, we use the notions of concept image and example space. Tall and Vinner (1981, p. 152) describe the concept image as the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes". It is important to note that a concept image does not have to be coherent. It is also notable that not all parts of the concept image are evoked at the same time. An essential element of the concept image is the related example space (Goldenberg & Mason, 2008) which contains examples, non-examples and counterexamples for the concept. We consider the concept image as a part of students' milieu when working in a specific situation.

Not much is known about university students' concept image of differentiability or non-differentiable functions after they have been taught a formal approach as compared to the situation in school where arguing with interpretations like "tangent slope" is more common. Viholainen (2008) presents the case of a student who claimed that several piecewise-defined functions with jumps were differentiable because it was "constant where the jump occurred" so that the derivative in that point "was zero". However, that functions whose graphs depict "corners" could not be differentiable was clear to him. This example illustrates that this student's concept image concerning non-differentiable functions is not complete, and especially it was not clear to him that differentiability implies continuity. A problem for students correctly linking differentiability and continuity is also reported by Juter (2012) and Duru, Köklü, and Jakubowski (2010) who found that many students believed continuity implied differentiability in the one-dimensional case. Klymchuk (2005) showed (with a small sample) that in a group of students where counterexamples were not used regularly and explicitly in the lecture, less than half of the students were able to sketch a graph that was continuous, looked smooth and was at one point not differentiable.

RESEARCH QUESTIONS

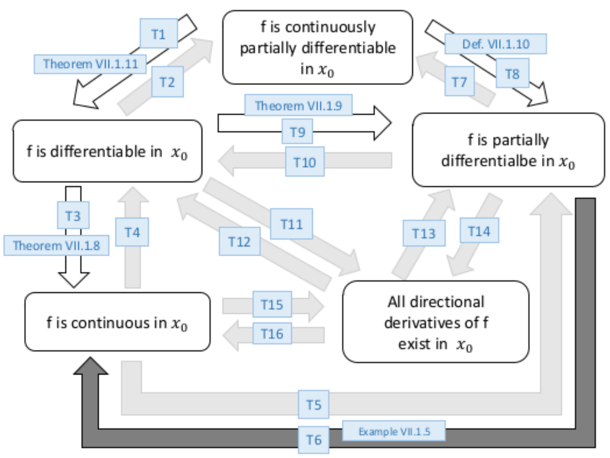
The broader aim of the whole study is to improve our understanding of students' difficulties concerning the different concepts of multivariable differentiability and their connections to one-dimensional differentiability. This understanding can inform the teaching of these topics. The understanding of multivariable differentiability cannot be separated entirely from that of one-dimensional differentiability because it builds on it.

In this article, we are interested in the following research questions: What makes the task of deciding whether continuity implies one-sided directional differentiability difficult for the students: Do difficulties occur in translating the item into the one-dimensional case or do the problems lie in insufficient concept images for non-differentiable functions in the one-dimensional case? What kinds of functions do the students consider when trying to find an example and what can we learn about students' concept images and example space concerning differentiability and continuity in the one-dimensional case from their work on this task? What can be done to improve students' performance on this task?

METHODOLOGY AND STUDY DESIGN

Our study took place in an Analysis-II-course (which is, from an international perspective, more on the level of upper-division proof-oriented Real Analysis courses in the US than typical lower-division Calculus courses). We did not influence the lecture the students participated in but designed two tasks concerning differentiability in $\mathbb{R}^n$ in cooperation with the lecturer and his teaching assistant. The students (second or higher semester, depending on their study program) worked on these tasks in two of their weekly tutorial group meetings. The task that we are concerned with here is part f) of the task shown in figure 1, for a more detailed description of the task and the design principles (guided by different "task potentials" that Gravesen, Grønbæk, and Winsløw (2016) formulated building on TDS) see Lankeit & Biehler (2019).

In this diagram it is marked which logical relations should be known from the lecture. (White means the implication is valid, dark grey means the implication is invalid.)



b) The function

$$f:\mathbb{R}^2\to\mathbb{R},\quad f(x):=\begin{cases}\frac{x_1x_2}{x_1^2+x_2^2}, & \text{if } x=(x_1,x_2)\in\mathbb{R}^2\setminus\{(0,0)\}\\ 0, & \text{if } x=(0,0)\end{cases}$$

was used to reason that implication T6 is invalid. Why does this example mean the implication is not true?

c) Next, consider the converse directions of the by lecture valid implications, thus the implications T10, T7, T2 and T4. Tick whether they are true or false and prove your claims.

d) How do differentiability and the existence of all one-sided directional derivatives correlate? Investigate whether the implications T11 and T12 are valid.

e) Examine the logical relations between partial differentiability and the existence of all one-sided directional derivatives by checking whether T13 and T14 are true.

f) Is it possible to deduce from continuity in a point x_0 that a function is partially differentiable or all one-sided directional derivatives exist in this point? (I.e., investigate the implications T5 and T15.)

Figure 1: The discussed task (translated by the first author).

We chose eight pairs of students to work on this task not in their usual tutorial group but separately in a situation where the first author acted as a tutor. The selected students were in their second or higher semester and studied Mathematics (4 students), Computer Science (1 student) or were pre-service teachers (11 students). Each group was filmed while working on this task. The written work they produced while working on the task was collected as well. The videos were transcribed after collecting the data. The transcripts were then analysed concerning our research questions. Transcripts shown in this article are translated from German by the first author.

The situation could be (and was in all of the cases) transformed from an adidactical to a didactical situation when the tutor stepped in, asked questions or gave hints. Since the time we had for the interviews was limited and this was the last task, we did not in all cases let the students think on their own or allow them "walk in the wrong direction" as long as we would have done for other tasks. Therefore, we can only conclude that some functions might not be as readily available in the students' example spaces as we would like them to be, and not that they are not at all contained in the example spaces.

EPISTEMOLOGICAL ANALYSIS AND A PRIORI ANALYSIS

The question of whether or not continuity implies the existence of all one-sided directional derivatives translates in the one-dimensional case to the question of whether continuity implies right- and left-sided differentiability. There are different reasons for functions not to be differentiable in the one-dimensional situation. As known, a function $f: \mathbb{R} \to \mathbb{R}$ is differentiable in $x_0 \in \mathbb{R}$ if the limit $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists. This limit does not exist if (a) the right- and left-sided limits exist but are not the same, (b) the term tends to infinity (for $h \searrow 0$, $h \nearrow 0$ or both) or (c) oscillatory behaviour (from at least one side) occurs in a way that makes the limit not exist. Case (a) happens for example in the absolute value function and means that the graph of the function has some sort of "corner" or "cusp", i.e. an abrupt change of the slope. Case (b) means that a tangent line to f at the point x_0 is vertical. This kind of behaviour can be found for example at $x_0 = 0$ in the cubic root function $f: \mathbb{R} \to \mathbb{R}, f(x) = \sqrt[3]{x}$ or in a suitably continued square root function, e. g. the functions $f: \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{|x|}$ or $f(x) = -\sqrt{-x}$ for $x \in \mathbb{R}^{<0}$ and $f(x) = \sqrt{x}$ for $x \in \mathbb{R}^{\geq 0}$. Case (c) occurs for example in the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x \cdot \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$, at the point $x_0 = 0$. A function that is not differentiable because of (a) still has both one-sided directional derivatives. In cases (b) or (c), at least one of the one-sided directional derivatives does not exist. This means that the standard example for a continuous but not differentiable function, i. e. the absolute value function, cannot be used to contradict the implication "continuity implies one-sided directional differentiability" (as well as any other function that is not differentiable because of a cusp). However, the example functions for cases (b) and (c) provide valid counterexamples since all of them are continuous.

In our a priori analysis, we expected that students would first think of the absolute value function and then quickly come to the conclusion that this is not a counterexample because the one-sided derivatives in 0 exist (although they are not the same). We expected them to try to come up with other functions $\mathbb{R} \to \mathbb{R}$ that are continuous but not differentiable and assumed that most students would – maybe with some help – think of some sort of a root function. A more detailed a priori analysis was done in Lankeit and Biehler (2019). The example space concerning non-differentiable functions was not explicitly cared for in the lectures. As usual, the absolute value function was given as an example for a non-differentiable, continuous function. Examples from the case (b) or (c) were not addressed in particular in the analysis I course preceding the discussed Analysis II course. The function $\sqrt[n]{x}$ on $(0, \infty)$ for $n \in \mathbb{N}$ was used as an example for a differentiable function as well as the function $f(x) = x^3 \cdot \sin\left(\frac{1}{x}\right)$ for $x \in \mathbb{R} \setminus \{0\}$, $f(0) = 0$.

RESULTS

We will now give an overview over the groups' work on the question whether or not continuity implied the existence of (one-sided) directional derivatives with an emphasis on our research questions which different functions they debated and what we can learn about their concept images. For space reasons, we will not show detailed case studies for the groups independently but rather give summaries over all of the groups concerning example functions they mentioned and students' reactions to the idea of a vertical tangent.

When they started looking for an example function, all of the pairs considered functions $\mathbb{R} \to \mathbb{R}$ and had no trouble translating the question into the question whether continuity implied the existence of the limits $\lim\limits_{h\searrow 0}\frac{f(x_0+h)-f(x_0)}{h}$ and $\lim\limits_{h\searrow 0}\frac{f(x_0-h)-f(x_0)}{h}$. Some of them started by actually trying to find a proof for the implication but recognised errors in their "proofs" themselves or with the help of the tutor. In this article, we will concentrate on the attempts to falsify the statement. Some of the groups also tried to use the logical structure of the diagram given in the task (see figure 1) which is something we found students doing for many of the implications, see Lankeit and Biehler (2019). It is not possible to use this diagram-based strategy successfully for this task if all earlier implications have been assigned the correct truth values.

Example functions the groups used

We will now have a look at example functions the groups came up with themselves, without the tutor hinting at a specific function (e. g. by saying the name, sketching the graph or asking for a function's inverse function). We will group the examples the different pairs came up with by the different cases ((a)-(c)) we described above. We additionally add the group (d) of discontinuous functions that were wrongly discussed as counterexamples, even though discontinuous functions could also be grouped into the cases (a)-(c). Still, it seems helpful to differentiate between continuous and discontinuous examples because their non-continuity makes them unsuitable as counterexamples in this task. We also added a category of differentiable functions (e) that students wrongly mentioned as candidates for non-examples. The groups came up with the following examples [1] in the five categories (a)-(e) on their own:

(a) $|x|^*$ (7 groups), $f(x) = \begin{cases} x^2, x < 0, \\ \sin(x), x \geq 0, \end{cases}$ (1 group), $f(x) = \begin{cases} 0, x \leq 0, \\ \mathrm{x}, x > 0, \end{cases}$ (1 group), other (2 groups)

(b) (vertical tangent): None

(c) $x \sin\left(\frac{1}{x}\right)$ (2 groups)

(d) Step function* (1 group), $\frac{xy}{x^2+y^2}$* (1 group), other (1 group)

(e) $e^{x^{**}}$ (2 groups), x^2* (2 groups), $x^2 \sin\left(\frac{1}{x}\right)$* (1 group), $\frac{1}{x}$ (1 group), $\frac{1}{x}^{**}$ (1 group), $\tan\left(\pi x - \frac{\pi}{2}\right)$** (1 group), other (1 group)

When a group immediately (i.e. in the same or one of the two following turns) after stating the example recognised that it is not a suitable counterexample, it is marked with "*". The tag "**" is used to mark examples the students came up with after the tutor introduced the idea of a vertical tangent. "Other" means the group talked about some other functions from the respective category without explicitly specifying it. It should be noted that none of the groups came up with an example of case (b) ("vertical tangent"), but two groups found an example of case (c). All groups who considered the absolute value function as a counterexample immediately realised it was not a suitable one. Only one group did not discuss this function (or any other function with a cusp).

Most of the groups at first only came up with functions with cusps or only the absolute value function. Some of them commented on this like the following quotes:

Peter: Can you come up with anything? Because mine [my example for a continuous, non-differentiable function] is the absolute value function by default. Because there it is nice that one has the visual evidence why it doesn't work.

The tutor asked five of the groups for reasons why a function could be continuous but non-differentiable. All of them only mentioned functions with cusps and in some cases, discontinuous functions, similar to group 1:

Tim: What other functions that are not differentiable do I know? […][2] It would have to be functions that have some kind of cusp, right?

Michael: Yes.

Tutor: Have a cusp, or what else would be possible?

Tim: Have a gap. But then it would not be continuous.

Group 8 reacted similarly:

Carl: Then I don't know any other class of functions that is non-differentiable and continuous, other than cusps.

Only the groups 3 and 7 came up with an example that could be successfully used to falsify the statement "continuity implies the existence of all one-sided directional derivatives". Both groups used the same example, the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x \cdot \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$, at the point $x_0 = 0$. How they came up with it was different in those cases: The tutor had shown the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 \cdot \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$, to group 3 in part c) of the considered task as an example for a differentiable function for which not all (partial) derivatives are continuous. They remembered having seen this example before but said they wouldn't have hit on it by themselves. After having seen this in the earlier task, they had a look at it again in part f) when they were looking for a function that is continuous but does not have all one-sided directional derivatives. They noted the function was continuous and then wanted to check out whether the one-sided directional derivatives existed. When asked by the tutor what this function had been an example for, they realised it was differentiable which implied the existence of all directional derivatives. They then decided to modify the function:

Sophie: What happens if we only take x here? Because she [the tutor] said before it's not differentiable if we only take x.

In group 7, the tutor had not introduced this or a related example before. Marc mentioned $\frac{\sin(x)}{x}$ wanting to find "that function that oscillates around zero the closer we get to zero" and finally developed (guided by the tutor's questions) the above function. It is clear from what he said that he had seen this or a related function before.

In the other six groups, the tutor introduced the idea of using a variant of the square root function (see above), either by saying the name of the function, sketching the graph or hinting at it by asking for the inverse function of the quadratic function. Depending on the time, the groups then checked themselves or together with the tutor that the one-sided directional derivatives in 0 do not exist.

The idea of a "vertical tangent."

The tutor gave different hints for each group, depending on what they needed. In groups 1 and 4, the tutor introduced the idea of a function with a "vertical tangent", or that is very steep at some point. In the first group, Michael answered with the exponential function but immediately stated that this function is, in fact, differentiable. In the fourth group, Laura mentioned a function with an asymptote, a function that approaches the y-axis like $\frac{1}{x}$, and later the exponential function. In group 8, David explained that another function they had discussed in a previous task (which was not continuous) did not have all one-sided directional derivatives in the following way, thus introducing the idea of an infinite slope himself:

David: The problem was that there wasn't really a slope but rather a steep ascent tending to infinity. But that function was not continuous.

When asked what this would mean visually for the graph of a function $\mathbb{R} \to \mathbb{R}$, David and Carl talked about functions with bounded domain and unbounded codomain and mentioned a bijective tangent function (we believe they meant $f: (-1,1) \to (-\infty, \infty), x \mapsto \tan\left(\pi x - \frac{\pi}{2}\right)$) but realised there is not one specific point where the slope is infinite. When asked for a function with a particular point where the slope is infinite, the following dialogue happened:

Carl: Some kind of… a vertical line, somehow.

Tutor: Do you know any function that behaves like that?

David: Yes, a step function, but that is not continuous. (laughs)

Carl: Exactly. It is either not continuous or not well-defined in that sense.

The second group worked with the definition of one-sided directional derivatives. It came up with the idea that the limit should be $\pm\infty$ so that the one-sided directional derivative does not exist. The first example they tried was $\frac{1}{x}$. They did not translate this into the idea of a vertical tangent and did not advance this idea but tried other strategies next.

On the other hand, after discussing $\sqrt{|x|}$, when the tutor asked the fifth group whether they could have seen the function is not differentiable in 0 before calculating that the limit of the difference quotient does not exist, Peter answered in the following way:

> Peter: […] We don't have an unambiguous way to find the tangent is zero. […] Respectively, the tangent gets steeper and steeper and steeper […] until it would be vertical, which is impossible.

At first, he uses the argumentation he used to explain why the absolute value function is not differentiable in 0. Still, he then realises that the square root function is a different case and gives the idea of a vertical tangent himself. It can also be seen from this excerpt that he does not accept a vertical tangent as a tangent to the function graph.

DISCUSSION

Students correctly had the idea to look for counterexamples in the one-dimensional case. However, finding examples for functions $\mathbb{R} \to \mathbb{R}$ that are continuous but not differentiable from the left and right side was problematic. The students' remarks show that for a non-negligible part, the accessible example space concerning continuous but non-differentiable functions contains only the absolute value function. Most of the groups seemed to be limited to functions with a cusp when thinking about continuous, non-differentiable functions. This finding is not very surprising since the absolute value function is the prototypical function for a non-differentiable, continuous function that was shown to them in the Analysis I lecture preceding the discussed Analysis II course when differentiability was introduced. Differing from findings in the analysis of students' written solutions (Lankeit & Biehler, 2019), the problem of using the absolute value function as a counterexample did not occur. Most groups discussed this function but quickly realised it was not a suitable counterexample. Additionally, a broader range of functions was considered as possible examples. Both differences might be due to the different setting: The didactical contract is slightly different when under individual observation than in the usual group work. It can be assumed that the students had a greater need for a solution. Additionally, while the situation was purely adidactical for the students in their usual tutor group meeting since the tutors were advised not to help them, the groups in this video study had the help of the tutor who interacted with them.

None of the groups came up with an example from the group (b) by themselves, a function that has a "vertical tangent" at a point. Two groups mentioned that the limit might not exist because it is $\pm\infty$ but could not find an example. The idea of a "vertical tangent" evoked images of functions that become steeper when approaching infinity or a pole but not of functions with "infinite slope" at one point. This idea seemed to be new for the students, which shows that this type of behaviour of functions needs to be addressed more explicitly. It is, however, not unexpected since the tangent to a graph at a point where the function is not differentiable is often not defined distinctly. In Biza, Christou, and Zachariades (2008), the task where a vertical tangent to a point of a curve needed to be drawn was solved successfully by only 33 % of the students. Problems with drawing the tangent to the graph of the function $\sqrt{|x|}$ in 0 are documented by Vinner (2002) in the school context as well, showing that case (b) is not easy for learners. Peter's explanation “until it would be vertical which is impossible” additionally shows that vertical tangents are perceived as “not allowed”. The spontaneous extension of the notion of tangent should not be expected. Therefore, the hint to think about “vertical tangents” proved to be not helpful to the students. When trying to bring students to think about functions from group (b), it might be more suitable to work in a symbolic rather than a graphical way and lead the students to think about why the limit of the difference quotient might not exist.

The fact that the rather “strange” oscillating sine-function was more accessible for the students than a root function was surprising to us. The students had seen this example in an additional, voluntary task at the end of their Analysis I course (not in the lecture), and not much time was spent on it. A possible explanation is the following: A problem with the square root function is that it is defined only on $[0,\infty)$. Therefore, differentiability is often, as in the Analysis I course preceding the discussed class, only examined on $(0,\infty)$, leading students to remember this as an example for a differentiable function and not thinking about the vertical tangent in 0. The variants of the square root function mentioned above were not discussed in the lecture, and neither was $\sqrt[3]{x}$. Additionally, while the oscillating sine-function is – if introduced – always framed as a strange function serving as a counterexample to something, the square root function might be considered a too “normal” function to even consider it as a counterexample. Therefore, one should probably not expect students to invent functions from group (c) themselves without help but, if they have already seen a variant of $\sin\left(\frac{1}{x}\right)$, it might be easier to find these example functions than suitable modifications of the square root function.

These findings suggest that the students’ example space as part of their milieu when in the situation of solving this task is not rich enough. These difficulties can be met in different ways. One way is addressing the cases (b) and (c) and not only the absolute value function when discussing continuous, non-differentiable functions in Analysis I. Another is enriching the milieu for this situation by hints and preceding tasks helping the students explore different cases why functions might not be differentiable and find example functions. It might be more suitable to lead them towards the group (c) than the group (b). Hinting at specific functions enables the students to solve the task by calculating that the one-sided derivatives do not exist, but this does not improve their understanding of their concept image and should therefore not be preferred.

NOTES

1. The functions are shortened for space reasons, of course, "xsin $\left(\frac{1}{x}\right)$" actually means the function $f\colon \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x \cdot \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$ etc.

2. If transcripts contain "[…]", this means that we omitted a (not relevant) part from the discussion in the transcript here to shorten the paragraph. In contrast, "(…)" means there was a pause.

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Review of our work on the teaching and learning of functions of two variables

Rafael Martínez-Planell and María Trigueros Gaisman

[1]University of Puerto Rico at Mayagüez, Mathematical Sciences, Puerto Rico, rmplanell@gmail.com; [2]Instituto Tecnológico Autónomo de México, México

In this article we present a short review of our research on student understanding of function of two variables. We describe results dealing with basic aspects such as geometrical understanding and understanding of the definition, then we consider results on student understanding of some notions of the differential calculus: plane, tangent plane, partial derivatives, directional derivatives, and the total differential.

Keywords: Teaching and learning of analysis and calculus, teaching and learning of specific topics in university mathematics, functions of two variables, APOS.

INTRODUCTION

It can be expected that natural or human induced phenomena in science, technology, and engineering will commonly involve quantities which are on an invariant (or nearly invariant) relationship with two or more other different quantities. This makes the multivariable calculus an important tool for the modelling and exploration of these phenomena. It is only in the last 10-15 years that research on the didactics of multivariable functions has gathered some momentum. The reason for this is that perhaps for a long time there was the expectation that understanding the didactics of single-variable calculus was enough since generalization to the multivariable context seemed to be non-problematic. However, this was quickly found not to be the case. A common undercurrent to research in multivariable functions is that these functions have their own particularities which make generalizing from the one-variable case challenging for students. In this article we describe research on the teaching and learning of two-variable functions. Because of lack of space, we concentrate on discussing mainly our own research results giving only a brief hint of some research by others. We also try to minimalize theoretical considerations and technical nomenclature so that the article may be of value to practitioners, as well as informative to researchers from different perspectives.

Our research question is: What do our research results about the learning of two-variable functions Calculus taken together tell and how can they be used to help promote students' learning in the classroom?

THEORETICAL FRAMEWORK

Since most of the research we have undertaken is APOS-based, we start with a brief summary. In APOS (for more detail see Arnon et al., 2014), an Action is a transformation of a previously constructed Object that the individual perceives as external. It is external in the sense that it is relatively isolated from other mathematical knowledge and so the individual will not be able to justify the Action. An Action could be the rigid application of an explicitly available or memorized procedure or fact.

When an Action is repeated and the individual reflects on the Action, it may be interiorized into a Process. The Process is perceived as internal in the sense that it has meaningful connections to other mathematical knowledge. This enables the individual to imagine the Process, omit steps, and anticipate results without having to explicitly perform the Process. A Process is thus perceived as dynamic. The meaningful connections also allow the individual to justify the Process and interrelate its different representations. Different Processes may be coordinated to form new Processes.

As new problem situations arise the individual may feel the need to apply Actions on a Process. For this to occur the Process has to be conceived as a totality, an entity in itself. When an individual is able to do Actions on a Process or imagine doing so, it is said that the Process has been encapsulated into an Object.

A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schema dealing with a specific mathematical notion. The Schema is coherent in the sense that its different components are interrelated in a way that allows the individual to determine if a problem situation falls within the scope of the Schema.

In their research, Martínez-Planell and Trigueros used semi-structured interviews with students who had finished a multivariable calculus course and, in some studies, had not used APOS-based didactical strategies. Students are asked several questions and the interviewer interacts with them in order to give them opportunities to show evidence of their constructed structures as defined in the theoretical framework. They analyse students' responses to find evidence showing if they were using Actions, evidenced by students' relying mostly on memorized facts, or in their need to use specific information to do calculations, or if they showed to have interiorized Actions into Processes demonstrated by their capability to generalize their strategies or to skip steps in procedures, for example, or encapsulated them into Objects shown by being able to perform Actions on notions studied to find their properties, for example.

When students work with different problems related to a mathematical notion. they may evidence that they use mainly Actions in their responses. It is then said that those students have constructed an Action conception of that notion. The same can be said about Process and Object conception.

A genetic decomposition (GD) is a conjecture of mental constructions students may do in order to understand a specific mathematical notion. In this article "to understand" a mathematical notion means to attain at least a Process conception of the notion. After a research study where a dialogue between APOS Theory and the Anthropological Theory of Didactics was carried out (Bosch, Gascón & Trigueros, 2017), interesting theoretical constructs emerged. In some researches we have used some of them to include both the cognitive and the institutional aspects that play a role in the teaching and learning of mathematics. Those mentioned in this paper will be discussed when they are needed.

BASIC ASPECTS OF FUNCTION OF TWO VARIABLES

Geometric understanding of functions of two variables

The first article we found on the literature of mathematics education that deals with generalization of function of one variable and student understanding of the basic notions of functions of two variables was by Yerushalmy (1997). She studied a group of seven-graders as they generalized notions from single to two-variable functions in a modelling context. Her results, among other things, stress the importance of the interplay between different representations in this generalization. Trigueros and Martínez-Planell (2010) followed with a study of students' geometrical understanding of functions of two variables. It may be argued that the most basic technique for solving problems about functions of two variables is reducing them to problems about functions of one variable; this is done by holding a variable fixed, equivalently, by intersecting a surface with a plane of the form x=c, y=c, z=c (fundamental plane). However, students tend not to do this on their own. Trigueros and Martínez-Planell (2010) found that intersecting a surface with a fundamental plane and placing the resulting curve in its appropriate place in space needs to be explicitly discussed, since students frequently confuse transversal sections with projections onto a coordinate plane, and need help to construct those Processes needed to relate contours with the graph of a function. Students were found not to easily relate their intuitive knowledge of space with the mathematical notion of R^3. This suggests the potential of using concrete manipulatives for students to perform Actions during instruction. Like Yerushalmy (1997), students' constructions were found to be dependent on the specific representation used so that it is important to help students construct Processes to interrelate different representations, including verbal representations, where phrases like "cut the surface", "lift a curve" and so on, are sometimes misunderstood by students. Given the importance of fundamental planes for the study of two-variable functions it may be expected that they would be institutionalized as a technique by explicitly referring to them and exploring their use throughout the course in different problem situations. Follow up studies (Trigueros and Martínez-Planell, 2011, 2015) of institutional conditions (textbook and instruction) using tools from the dialogue between the Anthropological Theory of the Didactic (ATD) and APOS (Bosch, Gascón, and Trigueros, 2017) show that this is not the case. Specifically, the moments of study of the ATD were used to examine the mathematical organization of the Stewart (2006) textbook as it regards the treatment of geometrical aspects of functions of two variables. Results obtained supported APOS based observations.

Understanding of the definition of function of two variables

After studying geometrical representations of functions of two variables, Martínez-Planell and Trigueros (2012) turned their attention to student understanding of the definition of functions of two variables: domain, range, uniqueness of functional values, and the possible arbitrary nature of a functional assignment. They found that generalizing from function of one variable to function of two variables is fraught with difficulties. Students frequently think of the domain of a function of two variables as a collection of real numbers, and sometimes as the *x* values with the *y* values being the range (see also Dorko and Weber, 2014) which implies that students were not offered enough opportunities to construct the notion of domain of two-variable functions through Actions involving ordered pairs and their representation in 3D-space, particularly in cases where the domain is given symbolically and includes a restriction. However, in a later study, students in Şefik and Dost (2019) did not exhibit this difficulty with restricted domains, so institutional issues may play a role here. Other aspects which need specific attention in classrooms include the vertical line test and its use for ascertaining uniqueness of functional value. It was observed that even good students occasionally have difficulties to recognize the uniqueness of functional value and the possible arbitrary nature of a functional assignment. So, to summarize, students tend not to generalize to a notion of function of two variables that allows them to recognize or show behaviour consistent with the modern set theoretic definition. Students need opportunities to reflect on the construction of domain, range, and uniqueness of functional value so they can distinguish functions of two variables from one-variable functions through activities designed to foment student reflection. As before, it was found that student constructions tend to be dependent on specific representations so that treatment Actions and conversion Processes (Duval, 2006) need to be explicitly fostered throughout instruction.

Use of designed activities in the classroom

The Trigueros and Martínez-Planell (2010) and Martínez-Planell and Trigueros (2012) studies were part of a first cycle of APOS research. Study results led to revising the GD where constructions were introduced taking into account results obtained in the first cycle. The new GD was used to design specific activities to help students do the constructions they needed to better understand two-variable functions. After classroom testing the activities for several semesters a second research cycle was undertaken (Martínez-Planell and Trigueros, 2013). It was found that students frequently did not work well with free variables, for example, as in $f(x,y)=x^2$ (missing variable) or as in $f(x,y)=x\sin(y)$ when you set $x=0$ (all variables present). Difficulties associated to both of these types of functions are independent of each other so it was concluded that they need to be explicitly addressed in class. In particular, the use of fundamental planes to graph cylinders (like $z=x^2$), needs to be explicitly considered through reflection on Actions during instruction. It was also found that activities that review function transformations help students construct the Process needed in graphing activities; that students overgeneralize familiar algebraic expressions so they can act as an obstacle; that in order to do the Action of intercepting a surface with a given fundamental plane, some students attempt to first graph the entire surface (consistent with an Action conception), and that some students needed help to interiorize point by point evaluations Actions into a Process (especially in the case of a missing variable). Results of the use of these activities were encouraging since students in the experimental section, who used the designed activities, performed better than those in a lecture-based section. Since it was considered that more reflection opportunities were needed to encourage the construction of a Process conception, it was decided that a third cycle of APOS research would be needed. So, the GD was further refined and the activity sets were improved to reflect the results of the second cycle. After classroom implementation of the activity sets, a third cycle of student interviews was undertaken.

Comparing understanding in two groups

Results of the third research cycle and their comparison with the main results of the first two cycles and the evolution of GD and activity sets through the three cycles showed that the activity sets were adequate in terms of helping students to interiorize their Actions into Processes (Martínez-Planell and Trigueros, 2019). Students from the sections using the activity sets were found to be more likely to show they had constructed a Process conception of function of two variables (8 of 12) than students not using the activity sets (0 of 7). Students who did not construct Processes did not show to have reflected on their Actions while working on the activity sets.

DIFFERENTIAL CALCULUS OF FUNCTIONS OF TWO VARIABLES

Vertical change on a plane and the genetic decomposition

Tall (1992) proposed a way to interpret the total differential $dz=(\partial z/\partial x)dx + (\partial z/\partial y)dy$ as $dz=(dz_x/dx)dx + (dz_y/dy)dy$ where dz_x, dz_y, dx, and dy are lengths of segments in the tangent plane (see Figure 1) and so may be cancelled. This makes more sense than the two ∂z's occurring in the original equation, where they mean different things in each of the two terms. We adapted this idea using the notion of vertical change on a plane to model the point-slopes equation of a plane, the tangent plane, the directional derivative, and the total derivative as suggested in Figures 2 and 3, thus aiming to bring coherence to students' differential calculus Schema (see Arnon et al., 2014, for a discussion of Schemas and Schema coherence).

A genetic decomposition was proposed in terms of vertical change on a plane (Martínez-Planell, Trigueros, and McGee, 2015). The vertical change on a plane dz from an initial base point (a, b, c) to a final generic point (x, y, z) is the sum of the vertical changes in the x direction (dz_x) and the y direction (dz_y) as one moves from the initial to the final point. But each one of these vertical changes can be expressed in terms of slope: dz_x is the slope in the x direction times the horizontal change in the x direction, $(dz_x/dx)dx$ and similarly for dz_y. From here one may obtain the point-slopes equation of a plane (Figure 2) $z - f(a,b) = m_x(x-a) + m_y(y - b)$, the tangent plane $z - f(a,b) = f_x(a, b)(x-a) + f_y(a, b)(y - b)$, and the total differential $df(a,b) = f_x(a,b)dx + f_y(a,b)dy$. One may also obtain the directional derivative in direction of a given vector (Figure 3) $D_{<\Delta x,\Delta y>}f(a,b) = (f_x(a,b)\Delta x + f_y(a,b)\Delta y)/ \| < \Delta x, \Delta y > \|$.

The original genetic decomposition is detailed in Martínez-Planell, Trigueros, and McGee (2015), a revised version in Martínez-Planell, Trigueros, and McGee (2017), and further refinements for the total differential in Trigueros, Martínez-Planell, and McGee (2018). These three papers will be discussed in the following subsections.

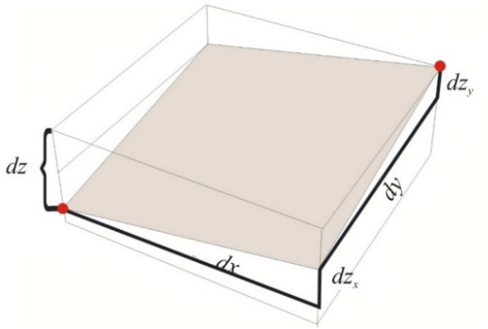

Figure 1: Tall's suggestion

$dz = dz_x + dz_y = (dz_x / dx)dx + (dz_y / dy)dy$

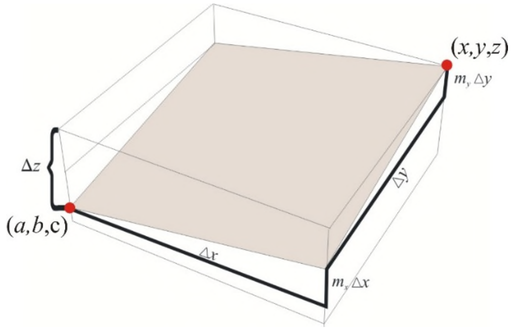

Figure 2: Vertical change on a plane $\Delta z = m_x\Delta x + m_y\Delta y$ may be used to model point-slopes $z - c = m_x(x - a) + m_y(y - b)$, tangent plane $z - f(a,b) = f_x(a,b)(x - a) + f_y(a,b)(y - b)$, and total differential $df(a,b) = f_x(a,b)dx + f_y(a,b)dy$

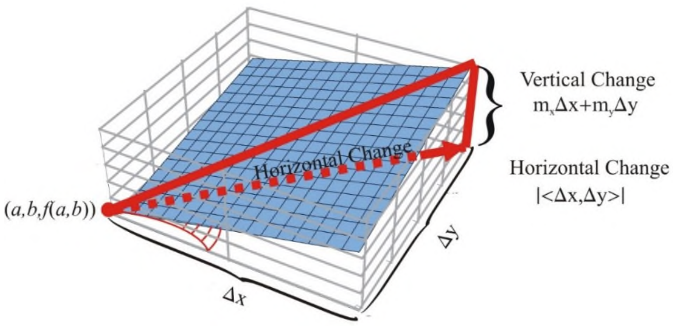

Figure 3: Directional derivative at (a,b) in the direction $<\Delta x, \Delta y>$, $D_{<\Delta x,\Delta y>}f(a,b) = (f_x(a,b)\Delta x + f_y(a,b)\Delta y) / \|<\Delta x, \Delta y>\|$

Slope and other pre-requisites to the differential calculus

One of the most important pre-requisite ideas necessary for the differential calculus is the notion of slope. It was found that even students that showed behaviour consistent with a Process conception of slope in two dimensions sometimes were restrained to working with slope as an Action when dealing with lines in three dimensions. Moore-Russo, Conner, and Rugg (2011) had previously observed student difficulty in generalizing slope to three dimensions. They observed that students would have difficulty realizing that in 3D slope must be directed slope. Indeed, in Martínez-Planell et al. (2017) these results were confirmed when discussing directional derivatives even in the case of some of the best interviewed students. Results showed that students need to have constructed an Object conception of slope in 3D to be able to think of a Process of slope as a totality upon which Actions can be performed. The notion of slope is central in the construction of a Process of vertical change which is the main concept giving coherence to a differential calculus Schema that includes the notions of point-slopes equation of a plane, tangent plane, directional derivative, and total differential. Indeed, McGee and Moore-Russo (2015) and McGee, Moore-Russo, and Martínez-Planell (2015) showed that students who explicitly consider 3D slopes during instruction seem to develop a better understanding of the differential calculus than students who do not.

Other basic pre-requisite ideas also play a role in students' possible construction of the differential calculus (Martínez-Planell et al., 2015): Some students do not realize that slope in the x and y directions are invariants of planes that do not depend on a base point, or that the total vertical change Δz is the sum of the vertical changes in the x and y directions, $\Delta z_x + \Delta z_y$. This behaviour is reminiscent of the "two-change problem" as discussed in Weber (2015). Some students need to develop flexibility in their notion of variable and still consider slope to be $\Delta y / \Delta x$, similar to the before-mentioned observation of Dorko and Weber (2014).

Partial derivatives

The construction of partial derivatives is conjectured as the coordination of Processes of fundamental planes and derivative of a one-variable function (Martínez-Planell et al., 2015), thus being restricted to an Action conception of either notion will not allow the construction of a Process of partial derivative or any of the other notions that depend on this. Given observed student difficulty with derivative of function of one variable, this suggests that the basic ideas in the construction of derivative of a one-variable function should be repeated in the discussion of partial derivatives, but now inserted in corresponding fundamental planes. As observed in Asiala et al. (1997), the notion of derivative needs the construction of graphical and symbolic trajectories that then must be coordinated. The same care should be taken by explicitly considering conversions between different representations in the construction of the ideas of the differential calculus (Martínez-Planell et al., 2015, 2017).

Directional derivative

The directional derivative is considered in detail in Martínez-Planell, Trigueros, and McGee (2017). One unexpected finding was that even some of the best performing students may have difficulty representing or imagining the direction vector $\langle a, b \rangle$ in space as $\langle a, b, 0 \rangle$. This has implications in their geometrical interpretation of directional derivative. These students can only perform Actions when doing symbolic computations and are not always sure of what they are doing when looking for directional derivatives. Students who have constructed an Action conception of vertical change may be expected to have difficulty to interpret the usual formula found in textbooks (Stewart, 2006) $D_{\langle u1,u2 \rangle} f(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2$ where $\langle u_1, u_2 \rangle$ is a unit vector, or the formula proposed in the genetic decomposition $D_{\langle \Delta x, \Delta y \rangle} f(a,b) = (f_x(a,b)\Delta x + f_y(a,b)\Delta y) / \|\langle \Delta x, \Delta y \rangle\|$. So again, vertical change plays a key role. Also, as mentioned before, some students did not realize that slopes in 3D must be directed slopes. This was a difficult construction for students; only one of 26 interviewed students managed to construct a Process conception of directional derivative. Our findings suggest that the construction of geometric meaning for the directional derivative should be carefully worked with students during instruction, thus helping them find a solution to the “two change problem” discussed by Weber (2015).

The total differential

Trigueros, Martínez-Planell and McGee (2018) considered student understanding of the total differential. They found this to be a very hard construction for students as none of 26 interviewed students managed to construct a Process conception of the total differential, only one of the 26 was judged to be in transition to such a conception, six were found to have an Action conception, and 19 of the 26 students showed to have no knowledge or recollection of the total differential. As with the previously described concepts, it is fundamental to develop a Process conception of vertical change on a plane in order for students to construct the geometric interpretation of the total differential. Most students do not think of the total differential as a function of independent variables *dx* and *dy*. Those students showing an Action conception were able to do mechanical computations where *dx* and *dy* were thought of as “very small numbers” which was not enough to develop a geometric insight that would help justify their solution procedures. Some students showed to have constructed the notions of function and tangent plane as isolated concepts thus had difficulties to explain the analytical and graphical consequence of the relation between them that is formalized in the total differential. These results led to revising the genetic decomposition and the activity sets so as to help students be able to coordinate Processes of tangent plane to a surface and of two-variable function into a Process where they can recognize that the tangent plane to a point of the function can be considered as the local linear approximation of the function, both in analytical and graphical representations.

The Trigueros et al. (2018) study included an analysis of some institutional issues (textbook, activities, and instruction) that constraint student learning of the total differential and provide supporting evidence for the APOS-based observations in the paper. The model of the moments of study of the ATD was used together with another tool from the ATD-APOS dialogue (Bosch et al., 2017): the institutional classification of techniques as action-technique (memorized, rigid, or applied to isolated tasks), process-technique (supported by a technological-theoretical discourse, presenting variations, connected to other techniques), and object technique (taken as an object of study). It was found that the moments of study were unbalanced (i.e. some moments and or ideas were not well represented) and may not foster a deep understanding of the total differential and its relation to the tangent plane. The analysis of the different types of techniques indicates that techniques introduced in the textbook are constrained to action-techniques which may limit the understanding of the concepts introduced and their relations while process-techniques are needed to foster students’ understanding.

SUMMARY AND CONCLUSION

In this short paper we have briefly summarized our research on functions of two variables, the basic aspects, and the differential calculus. The results of this research give ideas that may influence the design of better teaching strategies. All results show the role that construction of three-dimensional space and different representations play in students' understanding of the differential calculus of two-variable functions and how students' reflection on detailed activities and class discussion can help them to construct Processes and Objects needed in deepening their understanding of the main ideas of this discipline. Results obtained also open the door for further research on students' understanding of the differential calculus of functions of two variables.

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Un problème de CAPES comme premier pas vers une implémentation du plan B de Klein pour l'intégrale

Gaëtan Planchon[1] et Thomas Hausberger[1]

[1]IMAG, Univ. Montpellier, CNRS, Montpellier, France,
gaetan.planchon@umontpellier.fr

Dans le contexte institutionnel de la formation des enseignants du secondaire en France, nous avons élaboré un problème qui vise à créer des liens entre l'intégrale du Lycée (upper secondary) et les théories de Riemann et de Lebesgue enseignées à l'Université. En d'autres termes, il s'agit d'une réponse à la question de l'implémentation du plan B de Klein, dans le cas de l'intégrale, comme stratégie pour pallier à la seconde discontinuité de Klein. La méthodologie se fonde sur l'exploitation d'un modèle praxéologique de référence pour les différents objets de savoir. Les premiers résultats suggèrent que la modalité sous forme de problème écrit est insuffisante pour produire le transfert des connaissances mathématiques académiques en des connaissances utiles pour un enseignant du secondaire.

Mots clefs: teaching and learning of analysis and calculus, transition to and across university mathematics, teacher training, integral, Klein's plan B

INTRODUCTION

Dans une préface souvent citée, Klein (1908) a mis en avant une double discontinuité dans le parcours des étudiants de mathématiques se destinant à la carrière d'enseignant. La première discontinuité a lieu à l'entrée de l'université (on parle de nos jours de "transition secondaire-supérieur") tandis que la seconde, moins étudiée en éducation mathématique, s'opère à la sortie, lorsque les étudiants quittent l'université pour prendre un poste d'enseignant de mathématiques dans le secondaire. En effet, ces derniers perçoivent en général mal, d'eux-mêmes, les liens entre les savoirs universitaires et ceux, plus élémentaires, qui font l'objet des programmes de l'enseignement secondaire. La seconde discontinuité pose donc la question du transfert des connaissances mathématiques académiques en des connaissances utiles pour un enseignant du secondaire.

En France, la formation initiale des futurs enseignants est dispensée à l'université dans le cadre des masters MEEF[1]. Ces derniers préparent à la fois au concours de recrutement (le CAPES[2], lequel évalue la maîtrise de connaissances disciplinaires couvrant essentiellement les deux premières années d'université, ainsi des capacités liées aux dimensions professionnelles) et à l'entrée dans le métier d'enseignant de mathématiques (pendant la seconde année, les étudiants enseignent à mi-temps). Les étudiants du master MEEF ont, pour la plupart, suivi un cursus de licence de mathématiques. La première année de master est alors un moment propice au travail

1 Métier de l'Enseignement, de l'Education et de la Formation

2 Certificat d'Aptitude au Professorat de l'Enseignement du Second degré

des liens entre les notions mathématiques (dans le monde anglo-saxon, on parle de “capstone course”, point culminant de l’apprentissage qui vise l’intégration des connaissances en un tout cohérent) et en particulier, dans la perspective du concours, des liens entre savoirs mathématiques du supérieur et ceux du secondaire.

Klein (1908) propose en fait un programme de cours de type capstone (deux autres ouvrages suivront), basé sur une stratégie d’apprentissage qu’il appelle plan B (loc. cit. p.77-85). Par opposition au plan A qui opère un morcellement des savoirs en des pans plus ou moins autonomes (c’est la stratégie dominante à l’université, mise en oeuvre dans la division en modules), le plan B promeut une vision davantage holistique des mathématiques dans sa démarche de mise en évidence et d’exploitation des liens entre domaines mathématiques.

Les idées de Klein n’ont pas manqué d’inspirer mathématiciens et didacticiens. Récemment, Winsløw et Grønbæk (2014) se sont appuyés sur des outils de la théorie anthropologique du didactique (TAD ; Chevallard, 1991, 1999) pour modéliser la double discontinuité et proposer une première étude diagnostique des difficultés des étudiants qui se destinent au métier d’enseignant, dans le cadre d’un enseignement de type capstone au Danemark. Dans une seconde étude et avec des outils similaires, Winsløw et Kondratieva (2018) ont construit deux problèmes qui relient la théorie de l’Analyse aux savoirs pratiques du Calculus, dans le but de préciser, exemplifier et raviver l’idée de plan B de Klein. Des données issues d’expérimentations en classe sont analysées afin de discuter la faisabilité et les effets d’une telle approche.

L'objectif de cet article est d'exposer, dans la continuité des travaux de Winsløw et ses collaborateurs, une implémentation du plan B, cette fois dans le contexte des masters MEEF en France et sous la forme d’un problème de CAPES portant sur l’intégrale. Le cadre de la seconde épreuve écrite du CAPES semble être favorable à ce type d’implémentation puisque le programme de cette épreuve “est constitué des programmes de mathématiques du collège et des différentes séries du lycée général et technologique. Les notions traitées dans ces programmes doivent pouvoir être abordées avec un recul correspondant au niveau M1 du cycle master” (Journal Officiel du 8 décembre 2015, texte 8). L’intégrale est rencontrée par les élèves en classe de Terminale (dernière année du Lycée), puis par les étudiants à l’université à différents niveaux du cursus, avec souvent plusieurs théories (intégrale de Riemann, de Lebesgue, intégrale par rapport à une mesure quelconque, etc), ce qui en fait un objet riche et porteur pour l’étude de la deuxième discontinuité de Klein.

Dans un premier temps, nous exposons les outils de TAD qui ont guidé la construction du problème de CAPES et servent également à l’évaluation des effets sur les apprentissages. Puis nous présentons globalement le problème en soulignant les liens entre connaissances mathématiques qu’il vise à tisser. Enfin, nous ciblons deux parties du problème afin d’illustrer, par une analyse de copies d’élèves, les premiers résultats que cette étude a permis de produire.

CADRE THÉORIQUE ET QUESTIONS DE RECHERCHE

Winsløw et Grønbæk (2014) formalisent tout d'abord la double discontinuité à l'aide de la notion de rapport d'un individu à un objet de savoir au sein d'une institution, introduite par Chevallard (1991). Il s'agit de considérer : l'institution I, qui ici sera soit le lycée (L), soit l'université (U) ; un individu x qui occupe différente position dans l'institution I : d'abord élève au lycée (s), puis étudiant à l'université (σ) et enfin enseignant au lycée (t) ; un objet de savoir (ici l'intégrale), qui vit à travers différentes institutions et sera noté o au lycée et ω lorsqu'il s'agit d'une théorie de l'intégration enseignée à l'université. Le problème de Klein se modélise alors ainsi :

$$\mathrm{R_L}(s,o) \rightarrow \mathrm{R_U}(\sigma,\omega) \rightarrow \mathbf{R_U^*(\boldsymbol{\sigma},\boldsymbol{\omega})} \rightarrow \mathbf{R_U(\boldsymbol{\sigma},\boldsymbol{o})} \rightarrow \mathrm{R_L}(t,o)$$

où l'enjeu d'un cours de type capstone dans une formation des enseignants se situe au niveau des éléments en gras : quels types de connaissances complémentaires instaurant un rapport $\mathbf{R_U^*(\boldsymbol{\sigma},\boldsymbol{\omega})}$ sont à apporter sur les théories universitaires de l'intégration pour établir un rapport $\mathbf{R_U(\boldsymbol{\sigma},\boldsymbol{o})}$ qui soit pertinent pour un futur enseignant ?

En TAD, les rapports institutionnels aux savoirs sont décrits en termes de praxéologies (Chevallard, 1999), c'est-à-dire de couples (Π, Λ) où Π dénote le *bloc praxique* et Λ le *bloc théorique*. Le premier bloc est composé d'un *type de tâches T* et d'une technique τ permettant de réaliser ces tâches. Le second, qui est souvent rattaché à une famille de blocs praxiques, comprend la technologie θ, un discours qui justifie la technique, ainsi qu'un niveau supérieur de justification, la théorie Θ, qui unifie en général plusieurs technologies. Dans leur travail, Winsløw et Kondratieva (2018) font l'hypothèse que les cours de Calculus à l'université développent en général des praxéologies notées (Π_i, L_i) aux blocs du logos "faible" (par rapport à la pratique mathématique experte), tandis que les cours plus avancés d'Analyse développent des praxéologies (P_i, Λ_i) aux blocs praxiques artificiels. Une interprétation du plan B est alors de créer des liens du type $\Lambda_i \leftrightarrow \Pi_j$ visant à montrer aux étudiants comment des blocs du logos de l'analyse Λ_i sont "construits" (dans un sens épistémologique à clarifier en fonction des objets d'étude) sur des blocs praxiques Π_j qu'ils ont travaillés au préalable en Calculus.

Le dernier outil de TAD que nous allons mobiliser est la notion de *modèle épistémologique de référence* (MER ; Florensa et al., 2015). Ce modèle consiste en une reconstruction du savoir enseigné, obtenue en étudiant l'objet de savoir à différents niveaux de la transposition didactique (via l'épistémologie historique, les programmes officiels, les études de manuels et documents de cours). Il constitue la référence pour le chercheur et est construit en lien avec les questions de recherche. Dans la section suivante, nous ferons une présentation simplifiée du MER qui a été élaboré pour l'intégrale du secondaire o et ses avatars universitaires ω_{R} (intégrale de Riemann) et ω_{L} (intégrale de Lebesgue dans le cadre de la théorie de la mesure). Les questions de recherche qui guident sa construction sont les suivantes : quels liens peut-on établir entre les praxéologies mobilisant o dans le secondaire et les praxéologies universitaires développées relativement à ω_R et ω_L? Quelle implémentation d'un plan B de Klein proposer dans le contexte des étudiants de master MEEF, pour l'intégrale ? Enfin, notre dernière question de recherche concerne alors les effets sur les apprentissages : la modalité sous forme d'un problème de Capes permet-elle aux étudiants de construire les liens $\mathbf{R}_U^*(\sigma,\omega) \rightarrow \mathbf{R}_U(\sigma,o)$ qui ont guidé sa construction et constituent les objectifs d'apprentissage ?

MÉTHODOLOGIE, PRÉSENTATION DES MER ET DU PROBLÈME

Pour construire le MER pour l'intégrale au lycée, nous avons mené une analyse praxéologique de manuels et nous sommes appuyés sur le programme officiel de 2011. Pour l'intégrale de Riemann à l'université, nous avons complété le MER proposé par Cong Khan (2006) par une analyse praxéologique de documents de cours et travaux dirigés utilisés à l'université de Montpellier en 2016 ainsi que d'un ouvrage collectif en ligne (http://exo7.emath.fr). Enfin, le MER relatif à l'intégrale de Lebesgue, encore en cours d'élaboration, s'appuie également sur des ressources issues des enseignements donnés à Montpellier.

Au lycée comme en début d'université, un bloc praxique prépondérant pour l'intégrale est le bloc Π_1 composé du type de tâches T_1 (calculer l'intégrale définie d'une fonction continue f sur un segment $[a,b]$) et de la technique τ_1 (calculer une primitive F de f puis $F(b)$-$F(a)$). La technique est justifiée par le théorème fondamental de l'analyse (θ_1) qui, à son tour, est justifié au lycée à partir de la notion d'aire et ses propriétés (l'intégrale est définie géométriquement comme l'aire sous la courbe). Ces dernières ne sont pas formalisées, voire demeurent implicites, et restent donc largement fondées sur l'intuition. Le bloc du logos du lycée, noté L_o est donc un bloc faible au sens de la rigueur universitaire, voire incomplet. Un second bloc praxique P_2 rattaché à ce logos est celui du calcul approché des intégrales, avec pour technique la subdivision de l'intervalle et l'encadrement par des rectangles, de façon à définir ce qui sera appelé à l'université les "sommes de Riemann".

À l'université, une plus grande diversité de techniques (intégration par partie, changement de variable) permettent de réaliser T_1. La technique τ_1 n'est cependant pas obsolète et toujours justifiée par θ_1. Par contre, la théorie de Riemann vient justifier la technologie ; elle comprend la définition de l'intégrale de Riemann ainsi que des notions qui fondent l'analyse comme la borne supérieure. Nous notons Λ_R ce nouveau bloc du logos qui accompagne le passage de o à ω_R, ou encore la fondation de l'intégrale sur le nombre plutôt que sur la géométrie. De nouvelle praxéologies apparaissent également à cette occasion, notamment des types de tâches plus théoriques, par exemple "Montrer qu'une fonction donnée est Riemann-intégrable", dont les techniques correspondantes sont justifiées par la théorie Λ_R. Plus tard, en général en troisième année de licence, les étudiants peuvent trouver dans la théorie de la mesure les idées générales d'une formalisation possible du bloc du logos L_o. Des types de tâches relatifs à ω_L tels que "Montrer qu'une application donnée définit une mesure sur un ensemble donné” ou encore “Montrer qu’une mesure (générale) est croissante” facilitent en effet une telle formalisation. Cependant, cette théorie abstraite demeure probablement hors de portée de nombreux étudiants qui abordent le cycle de formation des enseignants et elle ne figure pas au programme officiel du CAPES.

Le problème de CAPES que nous proposons comme implémentation du plan B de Klein a pour objectif la construction d’un nouveau bloc du logos Λ_o de l’intégrale du lycée, pour l'enseignant. La méthodologie de sa construction se base sur l’exploitation des liens que les différents MER ont permis de mettre en évidence, lesquels sont pensés en termes de relations entre blocs de la praxis et du logos, comme expliqué dans le cadre théorique. Les éléments ci-dessus suggèrent la possibilité de relier Λ_o à différents blocs du logos et de la praxis relatifs à ω_R et ω_L. Nous allons voir que l’implémentation de ces liens va se faire en exhumant dans l’histoire des éléments d’une théorie de la mesure des aires due à Jordan et Lebesgue (mais différente de ω_L). En d’autres termes, c’est l’épistémologie historique qui a permis d’enrichir les MER.

Notre problème comporte quatre parties. La première partie commence par la définition axiomatique d’une mesure des aires, inspirée de celle proposée par Perrin (2005). Cette axiomatique repose sur ce que Lebesgue considère comme les éléments essentiels que doit vérifier une mesure des aires (Lebesgue, 1935), renforçant le rôle des transformations géométriques du plan, dans l’esprit de l’algèbre moderne. Cette mesure consiste en la donnée d'une application μ définie sur un ensemble Q appelé ensemble des parties quarrables du plan. Contrairement à l’axiomatique contemporaine de théorie de la mesure, μ n'est pas supposée σ-additive mais simplement additive (la mesure d’une union disjointe est la somme des mesures), d’où sa propriété de croissance (pour l’inclusion). On suppose de plus qu’elle vérifie les propriétés d’invariance par isométrie et d’homogénéité par rapport aux homothéties ($\mu(h(X))=k^2\ \mu(h(X))$ pour une homothétie h de rapport k). Ces propriétés sont démontrées par Lebesgue mais nous les considérons comme faisant partie des axiomes. La mesure du carré unité est 1. Cette axiomatique constitue le fondement de notre logos Λ_o.

Les tâches demandées dans la partie 1 consistent à démontrer un certain nombre de formules élémentaires de calculs d'aires (aire d'un rectangle, découpages dans un triangle, aire d'un parallélogramme) en s’appuyant sur l’axiomatique précédente. Le travail proposé consiste donc à appliquer la méthode axiomatique travaillée à l’université (autant en algèbre qu’en topologie ou théorie de la mesure) pour fonder des praxis enseignées à l’école et au collège. En vertu de la sémantique des objets et du type de formalisme, ce travail est à rapprocher de praxéologies en théorie de la mesure (voir plus haut) ou encore de raisonnements ensemblistes en théorie des probabilités menés dans le secondaire. Mais il s’agit également d’intégrer des éléments de géométrie (usage des transformations), ce qui est susceptible de déstabiliser certains étudiants. Nous détaillerons des exemples dans la partie empirique.

La deuxième partie de ce problème consiste à montrer que, sous l'hypothèse d'existence de la mesure μ et de "quarrabilité" de l'aire sous la courbe $\Omega_x = \{(t,y)\in\mathbf{R^2}\ ,\ 0\leq t \leq x,\ 0\leq y \leq f(t)\}$, la fonction d'aire $x\mapsto\mu(\Omega_x)$ est dérivable de dérivée f (théorème fondamental de l'analyse, TFA). Deux exemples sont proposés pour débuter : le cas des fonctions x^2 et e^x. Les tâches suivantes correspondent aux étapes de la preuve du TFA rencontrée au lycée (donc sous l'hypothèse de monotonie de f), qui utilise la figure et l'encadrement classique par des rectangles comme dans la praxis P_2. Mais il est attendu des étudiants qu'ils exercent le niveau de rigueur de l'université, c'est-à-dire qu'ils justifient les propriétés habituellement lues sur la figure en s'appuyant sur l'axiomatique (voir partie empirique). La dernière tâche généralise le résultat aux fonctions continues quelconques, ce qui nécessite de raisonner en termes de ε et δ comme de coutume dans les praxéologies d'analyse à l'université.

La troisième partie vise à construire une mesure vérifiant l'axiomatique, ce qui nécessite de définir la notion d'ensemble quarrable du plan. Cette construction, proposée par Lebesgue (1935), s'appuie sur des quadrillages du plan, de plus en plus fins, pour définir des mesures "extérieures" et "intérieures" des sous-ensembles du plan, lesquels sont dits quarrables en cas d'égalité. A l'aide des praxéologies du domaine des suites numériques, on étudie la quarrabilité de certains ensembles (les rectangles, l'ensembles des points à coordonnées rationnelles du carré unité) puis démontre que la mesure satisfait les propriétés que pose l'axiomatique. La dernière tâche de cette partie consiste à prouver un critère de quarrabilité : une surface S est quarrable si et seulement si il existe deux suites de polygone (P_n) et (Q_n) telles que :

$$(1)\ \forall\ n\in\mathbf{N},\ P_n \text{ et } Q_n \text{ sont quarrables;}$$

$$(2)\ \forall\ n\in\mathbf{N},\ P_n\subset S\subset Q_n;\ (3)\ \lim \mu(Q_n)\text{-}\mu(P_n) = 0.$$

La fonction de ce critère apparaît par la suite.

En effet, on montre dans la dernière partie, en appliquant ce critère, que Ω_x est quarrable (sous l'hypothèse $f\geq 0$). Comme précédemment, on commence par le cas f monotone. On effectue une subdivision de l'intervalle $[a,b]$ de pas $(b\text{-}a)/n$: les unions des rectangles que l'on obtient par la praxis P_2 donnent deux suites qui répondent au critère. En fait, il s'agit de l'analogue de la preuve que les fonctions monotones sont Riemann-intégrables. Cette tâche permet donc de relier le logos Λ_o au logos de Riemann Λ_R.

ETUDE EMPIRIQUE

Nous avons donc soumis ce problème à 19 étudiants de Master MEEF, dans les conditions du concours (épreuve de 5 heures sans document). Nous analyserons les productions des étudiants relativement à des tâches issues des deux premières parties du problème : il s'agit de tester la capacité des étudiants à mobiliser le logos Λ_o pour justifier d'une part des calculs d'aires (première partie), d'autre part les inférences qui permettent d'établir le TFA (deuxième partie) dans le cas d'une fonction monotone positive.

Etant donnée une mesure μ vérifiant l'axiomatique, la première tâche consiste à démontrer l'assertion suivante : ($X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$) pour toutes parties quarrables X et Y. Si l'axiomatique n'est pas celle d'une mesure stricto sensu (au sens de ω_L), cette tâche peut néanmoins être considérée comme une instanciation du type de tâche "montrer qu'une mesure générale est croissante". La technique privilégiée consiste à faire un découpage classique (en théorie de la mesure, comme en théorie des probabilités) : $Y=X\cup(Y\backslash X)$, ce qui donne directement le résultat par additivité de la mesure, puisque $\mu(Y\backslash X) \geq 0$. Il s'agit donc essentiellement d'un bloc praxique relatif à ω_L. Sur les 19 copies, 5 étudiants ont utilisé la technique décrite ci-dessus pour réaliser la tâche, 5 ont proposé une preuve partielle en affirmant que Y pouvait être partitionnée en deux parties disjointes, sans expliciter davantage le raisonnement, et deux étudiants manifestent des conceptions erronées (confusion de la mesure avec la cardinalité, introduction d'une relation d'ordre sur les éléments, ce qui n'a aucun sens). Nous considérons que 10 étudiants sur les 19 ont su mobiliser la praxis relative à ω_L.

La deuxième tâche étudiée est : "montrer que la mesure d'un carré de côté 3 est 9". Là encore, l'axiomatique de μ est sollicitée : les étudiants peuvent soit décomposer le carré en 9 carrés disjoints ou bien utiliser une homothétie et la propriété d'homogénéité de la mesure. On remarque que la technique de découpage est peu utilisée puisque seulement 3 étudiants l'utilisent comparativement à 12 étudiants qui utilisent une homothétie. Enfin, 2 étudiants appliquent une formule d'aire, ce qui revient à admettre le résultat. A part ces derniers qui n'ont pas cerné l'enjeu de l'axiomatique, nous pouvons donc considérer que les étudiants ont su s'approprier la définition formelle de mesure d'une partie quarrable.

La troisième tâche étudiée consiste à démontrer le TFA, ce qui revient essentiellement à reprendre la preuve vue en lycée en justifiant soigneusement, à l'aide de l'axiomatique, les propriétés de l'aire habituellement lues sur la figure.

B.3. Soit f est une fonction définie, positive et continue sur $[a,b]$ $(a < b)$ et $\Omega = \{(x,y), x \in [a;b],\ y \leqslant f(x)\}$. Pour chaque $t \in [a,b]$, on note $S(t) = \mu(\Omega_t)$ où

$$\Omega_t = \{(x,y) \in E,\ a \leqslant x \leqslant t,\ 0 \leqslant y \leqslant f(x)\}.$$

B.3.1. On suppose f croissante sur $[a,b]$. Soit $x_0 \in]a,b[$.

B.3.1.1. Soit $h > 0$. Que représente $S(x_0+h) - S(x_0)$? En déduire un encadrement de $S(x_0+h) - S(x_0)$

B.3.1.2. Montrer alors que S est dérivable à droite en x_0.

Figure 1. Enoncé des tâches relatives à la preuve du TFA

Plusieurs sous-tâches sont à réaliser : d'abord remarquer que $\mu(\Omega_{x0+h})$-$\mu(\Omega_{x0})$ représente la mesure de $\Omega_{x0+h}\backslash\ \Omega_{x0}$ grâce à l'axiome d'additivité de μ , puis encadrer $\Omega_{x0+h}\backslash\ \Omega_{x0}$ par deux rectangles de largeur h et de longueurs respectives $f(x_0)$ et $f(x_0+h)$.

Bien que la justification de cet encadrement soit possible (et attendue) à l'aide de l'algèbre, le passage au cadre graphique permet surement d'obtenir plus facilement cet encadrement. On peut faire l'hypothèse que si l'encadrement est obtenu par l'étudiant, le registre graphique a été mobilisé au moins au brouillon. L'avant dernière étape mobilise à la fois la croissance de μ et la formule de la mesure d'un rectangle (de façon similaire à la praxis P_2 des sommes de Riemann) afin d'aboutir à l'encadrement $hf(x_0) \leq \mu(\Omega_{x0+h} \setminus \Omega_{x0}) \leq hf(x_0+h)$. Enfin, la dérivabilité (à droite) de la fonction $t \mapsto \mu(\Omega_t)$ en x_0 est une conséquence du théorème des gendarmes.

Sur les 19 copies, 10 étudiants ont réussi à obtenir l'encadrement attendu $hf(x_0) \leq S(x_0+h)\text{-}S(x_0) \leq hf(x_0+h)$. Seulement 3 étudiants ont effectivement tracé une figure. On reconnaît le dessin que l'on trouve habituellement dans les ouvrages de terminale, lequel constitue un bon support à la preuve (figure 2). Nous faisons néanmoins l'hypothèse que tous les étudiants qui ont su donner l'encadrement de $S(x_0+h)\text{-}S(x_0)$ ont mobilisé le cadre graphique, ne serait-ce qu'à travers une image mentale d'un tel dessin. Par exemple, l'étudiant dont le travail est présenté dans la figure 3 note R_1 et R_2 les deux domaines qui permettent d'encadrer l'aire sous la courbe, notations qui font directement référence à la méthode des rectangles.

Parmi les 10 étudiants à avoir proposé le bon encadrement, seulement 3 d'entre eux ont évoqué l'axiomatique, par exemple en utilisant la notation μ (figure 3). Cependant, la propriété de croissance de μ n'est jamais explicitement utilisée et il n'est pas fait référence au résultat précédemment établi sur la mesure des rectangles, ce qui laisse supposer que ce dernier a également été mobilisé sur une base essentiellement intuitive.

Parmi les 9 étudiants restants et qui n'ont donc pas su donner l'encadrement attendu de $S(x_0+h)\text{-}S(x_0)$, cinq étudiants n'ont pas abordé cette question. Les autres apparaissent bloqués ou mis en défaut par l'interprétation de l'ensemble $\Omega_{x0+h} \setminus \Omega_{x0}$, sans doute face au grand nombre de variables et paramètres dans les écritures algébriques. Par exemple, un étudiant a considéré l'ensemble Ω_{x0} comme étant un rectangle. Il n'a visiblement pas perçu la dépendance en x dans l'expression des ordonnées, ou mal interprété cette dépendance dans la conversion vers le registre graphique. On peut donc faire l'hypothèse que le formalisme a été un obstacle majeur pour ceux qui n'ont pas interprété directement $S(t)$ comme une aire.

Au final, aucun étudiant n'a justifié les inférences à l'aide de l'axiomatique ce qui était l'enjeu de cette question. En particulier, ni la croissance ni l'additivité de μ n'apparaissent comme des éléments clés de la preuve. Nous faisons donc l'hypothèse que les étudiants qui ont réalisé la tâche se sont appuyés sur un point de vue intuitif de la notion d'aire et n'ont pas compris le rôle joué par l'axiomatique dans la démonstration. Ceci est à mettre en perspective avec les attendus du programme de Terminale S (2011) : « On s'appuie sur la notion intuitive d'aire rencontrée au collège et sur les propriétés d'additivité et d'invariance par translation et symétrie. »

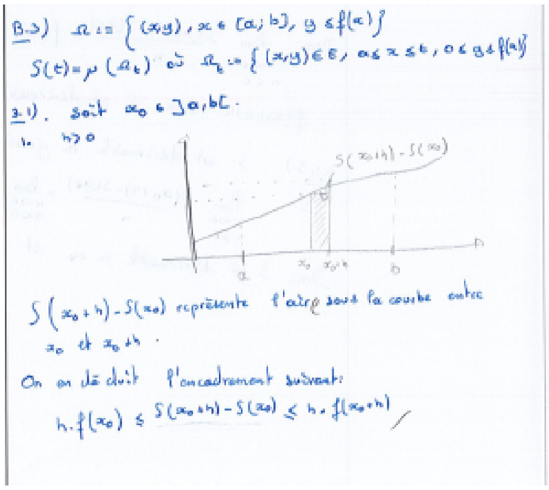

Figure 2. Exemple de réalisation de la preuve du TFA avec usage explicite du registre graphique

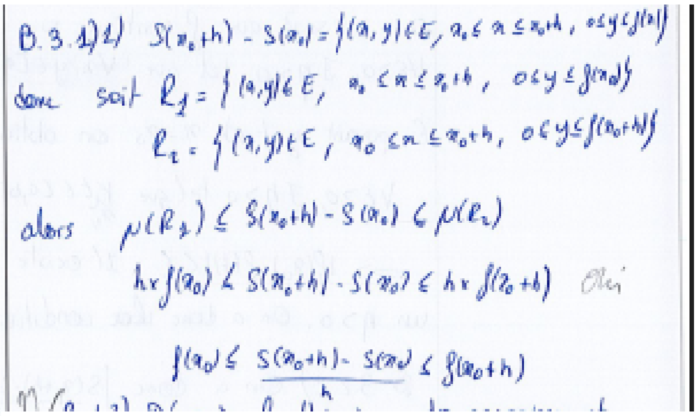

Figure 3. Exemple de réalisation de la preuve du TFA avec mobilisation de la notation μ

CONCLUSION ET PERSPECTIVES

Dans le contexte institutionnel de la formation des enseignants du secondaire en France, nous avons élaboré un problème de CAPES qui vise à créer des liens entre l'intégrale du Lycée o et les théories de Riemann ω_R et de Lebesgue ω_L enseignées à l'Université. Ce problème répond à une demande institutionnelle, notifiée dans les programmes du concours de recrutement, d'aborder les objets de savoirs du secondaire avec le recul des connaissances de Licence (Bachelor). Interprétant cette injonction dans l'esprit du plan B de Klein (comme remède à la seconde discontinuité), nous avons suivi la méthodologie proposée par Winslow et ses collaborateurs. L'élaboration du problème se fonde ainsi sur un modèle praxéologique de référence pour o, ω_R et ω_L. Ceci a permis de construire des tâches autour d'un nouveau logos Λ_o pour l'intégrale du lycée, à destination de l'enseignant. Du point de vue historique, ce logos correspond à une formalisation de la mesure antérieure à celle de la théorie contemporaine de la mesure (celle-ci inclut la σ-additivité), mais reformulée en partie en termes de transformations du plan (rôle des isométries et homothéties). Les liens que le problème cherche à établir ont été décrits en termes de relations entre blocs du logos et de la praxis de praxéologies mobilisant les différents objets o, ω_R et ω_L et de différentes praxéologies de calculus et de théorie des ensembles.

L'étude empirique a montré que les étudiants, dans leur majorité, ont réussi à se saisir de l'axiomatique dans les premières questions du problème. On peut y voir l'impact du travail sur les axiomatiques effectué à l'université, autant en algèbre qu'en analyse (topologie, théorie de la mesure) ou en probabilités, mais aussi un transfert réussi de techniques élémentaires travaillées à propos de ω_L. Le contraste est alors saisissant lorsque ces derniers s'engagent dans la preuve du TFA en restant à un niveau de justifications faisant appel à des propriétés de l'aire lues sur la figure plutôt que de mobiliser l'axiomatique. Dans ce contexte, la fonction de l'axiomatique semble non perçue par les étudiants. Outre la persistance du contrat didactique du second degré, ce phénomène peut également s'expliquer par l'obstacle du formalisme et un déficit de flexibilité cognitive dans l'articulation entre les registres graphiques et formels.

En conséquence, notre étude confirme que le transfert des connaissances universitaires en des connaissances utiles pour l'enseignant est très loin d'être automatique. La modalité en termes de problème de CAPES, si elle répond à une demande institutionnelle, s'avère insuffisante pour produire les effets recherchés sur les apprentissages, suite à une insuffisance de rétroactions du milieu. Ceci suggère qu'une modalité sous forme de situation, dans l'esprit de la théorie des situations didactique de Brousseau, ou bien un processus d'étude permettant la mise en place d'une dialectique fertile entre médias et milieux, dans la lignée des travaux récents en TAD, seraient davantage appropriés. C'est ce travail que nous nous proposons d'entreprendre, comme prolongement de cette première expérimentation.

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TWG2: Mathematics for engineers; Mathematical Modelling; Mathematics and other disciplines

TWG2 report: Mathematics for engineers, Mathematical Modelling, Mathematics and other disciplines

Berta Barquero[1] and Nicolas Grenier-Boley[2]

[1]Faculty of Education, University of Barcelona, Spain, bbarquero@ub.edu; [2]INSPE, Université de Rouen Normandie. nicolas.grenier-boley@univ-rouen.fr

INTRODUCTION TO TWG2

The purpose of this report is to give an account of the work conducted within TWG2 during the INDRUM2020 conference, which was held virtually from Bizerte (Tunisia), in September 2020. Initially, the group was composed of 43 registered participants from 16 different countries, with up to 28 participants simultaneously connected during the sessions. We had a total of 13 presentations, 11 papers and 2 posters, all of them addressing essential issues related to the teaching of mathematics for engineers, mathematical modelling or mathematics and other disciplines. In the three sessions of TWG2, the discussions were organised around three main topics covering five leading themes (Table 1). The three main topics delimited were about: (a) students and/or teachers-researchers' practices (Theme 1 and 2); (b) the prevailing way to teach and learn university mathematics for engineers and for non-specialists (Theme 3 and 4); and, (c) looking for a change of paradigm in maths university teaching and learning (Theme 5 and 6).

Topics	*Leading themes*	*Papers*
(a)	T1. Teaching and learning strategies for engineering students	2
	T2. Teaching practices of teachers-researchers at the university	2
(b)	T3. Analysing the prevailing conditions for the teaching and learning of Calculus for engineers	2
	T4. Institutional analysis of mathematical modelling for non-specialists	2
(c)	T5. New perspectives for a renewed teaching and learning of mathematics: problem-posing activities and interdisciplinary projects	3
	T6. Instructional proposals to move towards the paradigm of 'questioning the world'	2

Table 1: Overview of the leading themes related to the main topics in TWG2

This delimitation of the leading themes facilitated to group papers with similar aims and to make researchers interaction easier. The thematic group sessions were organised in five phases in order to make the discussions as fruitful as possible: presentation of the session, work in small parallel groups to discuss the themes and formulate questions, report of the groups, authors' answers and general discussion. In light of the quality of submissions, the substance and relevance of exchanges during the sessions, we can point important contributions to the development of research in the different topics and themes. The next sections summaries the contribution we had concerning each main topic and wants to report on the main issues and questions raised and discussed within TWG2. We conclude the report by highlighting some of the main open questions for future research that deserve more attention in the years to come.

Focus on students or teachers-researchers

The two first papers *Liebendörfer et al.* and *Zakariya et al.*, in Theme 1, focus on teaching and learning strategies for engineering students. Several common issues have emerged from the corresponding presentations and discussions. They are both concerned with the specificities of studying the teaching and learning of mathematics for engineering students, and consequently, with the design of appropriate interventions to support these students. Both investigations propose relevant tools in order to differentiate engineering students' learning strategies or attitudes in relation to their approaches to learn mathematics. Some are statistical tools used to interpret the quantitative and qualitative analyses and to locate correlational patterns among the data.

Concerning Theme 2 about the teaching practices of teachers-researchers at the university level, we discuss two connected papers from *Bridoux et al.*, *Bridoux, de Hosson* and *Nihoul*. Because their authors are involved in the same research project, they share several common aspects. Both investigations address theoretical and methodological issues to analyse teachers-researchers' teaching practices, for ideal or declared practices (first paper) or *in situ* practices (second paper). In both cases, the purpose is to compare the practices of teachers-researchers from different disciplines (mathematics and others) in order to measure the influence of the disciplines involved, their epistemology or their didactics upon these practices. The second research also addresses the issue of detecting the possible effects of these practices on students' learning during courses or tutorial sessions.

Among other issues, the following questions give a pertinent account of the discussions that took place in relation to Theme 1 and Theme 2:

- What are the specific mathematical needs of engineering students? In which way mathematics teaching for future engineers is adapted to their future professions? What balance between proof-based teaching and applications in mathematics?
- Are there differences between applied and theoretical mathematicians' teaching practices? How can we compare the practices of teachers from different disciplines? What is the influence of the discipline and its epistemology?
- What can theoretical and methodological tools help give an account of the restrictions experienced by teachers-researchers concerning their teaching? What are the conditions and constraints to use statistical tools to interpret quantitative and qualitative results in didactics research properly?

Focus on the prevailing way to teach and learn university mathematics for engineers and for non-specialists

The second topic focuses on analysing the prevailing way of how mathematics is taught and learnt at the university for engineers and non-specialists. Different empirical data is here considered, from textbook analysis, course content analysis, students' attitudes, among others, to analyse how some particular mathematical topics and taught and learnt for mathematics undergraduate courses. About the papers here discussed we have, on the one hand, two papers that analyse the specific conditions and constraints for the teaching and learning of Calculus for engineers, in particular, of integration. The paper from *Nilsen* analyses a group of first-year university engineering students and their sensemaking of integration and its symbolism. Through a semiotic approach, special attention is made on how students use and interpret symbols for integration. By their side, *González-Martín* and *Hernandes-Gomes* focus on developing a praxeological analysis, in the sense of the ATD, for analysing a course's reference book, of a Strength of Materials course, to show the role that integrals have in *logos* block. This analysis is complemented by interviews with an engineering teacher to understand the dominant way how integrals are planned to be taught and learnt in a Calculus course for engineers.

On the other hand, the other two papers are more specifically focused on the role of mathematical modelling for university mathematics teaching and learning for non-specialists. In particular, the paper from *Doukhan* focuses on probabilistic modelling in the transition between secondary and tertiary education with first-year biology students. Through the analysis of students' responses to a test, the paper shows the diversity of difficulties in the secondary-tertiary transition concerning probability and probabilistic modelling. *Job* discusses the prevalence of "applicationism" as the dominant way to understand mathematical modelling for economics. In particular, the paper describes a peculiar epistemological standpoint about the relationship between mathematics and economics, namely that of subordinating economics as an application of mathematics, may impact students' views about the interplay between mathematics and economics.

In the general discussion about this topic, we address relevant questions about the aims, contents' selection and lack of specificity of the mathematical knowledge to be taught in the different specialities. In particular about:

- What is the main goal with the first-year Calculus courses? What kind of conceptual understanding is needed in different engineering specialities?
- What is important about calculus (integrals, sums, derivatives, …) for engineers, mathematicians, other university degrees? Could we find different rationales for their teaching and learning, depending on the university context?
- What elements of calculus, probability, mathematical modelling, etc., does each profession need? What elements have to be included in each undergraduate programme?

Focus on looking for a change of paradigm in maths university teaching

Contributions related to the last topic refer to some instructional proposals for university mathematics to move towards a change of paradigm, such as problem-posing activities, interdisciplinary projects or study and research paths. The pursued aims are varied, but when looking at their complementarities, we found some common aspects. They all refer to the detection of conditions and necessities concerning the change of paradigm through the analysis of the student's attitudes and competencies; the impact of alternative teaching proposals; and the viability of their implementation and long-term dissemination.

Radmehr et al. explore engineering students' mathematical problem posing competencies in relation to integral calculus, and their attitudes towards mathematical problem posing. Answers from students to some tasks related to the Fundamental Theorem of Calculus and the notions of integral and area are explored, complemented by a questionnaire that explores students' attitudes towards problem posing. The poster from *Gaspar Martins* presents an interdisciplinary project for computing engineering students about a car race, where Python appears as a means for programming language. The paper from *Cumino*, *Pavignano* and *Zich* presents an interdisciplinary project for first-year students in Architecture about the visualisation of mathematical objects through physical and digital models. The authors explore the appearance of varied models to improve the accessibility of interdisciplinary elements, building a common language for students with different backgrounds.

About the proposal of study and research paths (SRPs) within the ATD, *Barquero et al.* focus on the analysis of several implementations of SRPs as an inquiry-oriented instructional proposal at the university level. This paper focuses on the different modalities of integration of SRPs into current university teaching, by linking inquiry to the study of knowledge organisations, without considering it only as a means to better learn the curricular content. The poster presented by *Quéré* presents a particular SPR guided by a chemistry lecturer about how we can be sure that a medical product meets the dosage as it is described on the package. The implemented SRP is discussed in terms of the usefulness of the developed praxeologies for the engineer's professional context.

Some critical questions discussed refer to the inherent assumptions in the design and implementation of the different teaching proposals, in particular:

- About problem posing: What do we consider a "good problem"? What does it mean to pose a problem for teachers-lecturers? For the students?
- About interdisciplinarity: What epistemological limitations appear when working in co-disciplinary or interdisciplinary contexts?
- About SRPs: How to find a "good" generating question for an SRP? Can the design and implementation of SRPs help us to rethink the contents of the course? Does the context of engineers' university training offer better conditions to implement SRPs than others, due to their proximity to the profession?

CONCLUDING REMARKS AND FUTURE PERSPECTIVE FOR TWG 2

The topics and leading themes addressed in TWG2 show the variety of research approaches and questions addressed. Since all those papers that are more focused on making the different "agents" (students and lecturers) react, to the ones working on the proposal of alternative teaching proposals to investigate about a change of the dominant pedagogical paradigm for the teaching of mathematics at university. Furthermore, the variety of theoretical frameworks provide a fruitful interaction to collectively understand phenomena related to the teaching and learning of mathematics for engineers, for other disciplines, and about the role that mathematical modelling can play to build bridges between disciplines. We want to finish this presentation by sharing some questions that were discussed in TWG2 about the future lines of development of our working group:

Mathematics for engineers and other disciplines

- Is "mathematics for engineers" a too general term? How do we approach the specificities of each engineers' context?
- What mathematics do university students need for their future professions? What communities may participate in the discussion of this crucial question?
- How has the use of technology (at university and in the workplace) accelerated the distancing between what is actually taught from actual professional needs?
- What are the theoretical and methodological possibilities to study teacher-researchers' practices and detect their possible effects on students' learning?

Need to rethink university mathematics curriculum

- How to make such a significant change in curriculum design at university (where we have a very "stable" curricula)?
- How to look at university mathematics curricula from an interdisciplinary approach? How can the perspective of mathematical modelling contribute to it?
- How to consider professional and workplace needs when designing mathematics curricula and defining its role in the different undergraduate programmes?

Last but not least, a crucial issue is a necessary collaboration between different communities (mathematicians, didacticians, engineers, among other) to rethink university curricula, not only in terms of contents, but also thinking about the new needs, competencies and abilities that may be integrated. Who may participate in this discussion and what is the role of didactics research are also questions that may be addressed. And a more complicated and yet essential one can be the long-term collaboration of didacticians with researchers-lecturers from different disciplines to make this curriculum questioning evolve productively.

How to integrate study and research paths into university courses? Teaching formats and ecologies

Berta Barquero[1], Marianna Bosch[2], Ignasi Florensa[2], and Noemí Ruiz-Munzón[4]

[1]Universitat de Barcelona; [2]Universitat Ramon Llull, Barcelona, Spain, marianna.bosch@iqs.edu; [3]Escola Universitària Salesiana de Sarrià, Barcelona, Spain; [4]Tecnocampus – Universitat Pompeu Fabra, Mataró, Spain.

The implementation of inquiry-oriented instructional proposals at university level collides with many constraints related to the current pedagogical paradigm based on the transmission of previously established knowledge organisations. One of them is the fact that subject's agendas are determined by a set of topics to study, not of questions to inquire about. Our research team has worked during this last decade in the design and implementation of an inquiry-oriented instructional format called study and research paths (SRPs) within the Anthropological Theory of the Didactic. This paper focuses on different ways of integrating SRPs in current university teaching, by linking inquiry to the study of knowledge organisations, without considering it only as a means to better learn the curricular content.

Keywords: Teachers' and students' practices at university level, Novel approaches to teaching, curricular and institutional issues concerning the teaching of mathematics at university level, anthropological theory of the didactic, inquiry-oriented instruction.

INTRODUCTION

Current university instruction is mainly embedded in what has been called the *paradigm of visiting works* (Chevallard 2015), where teaching and learning goals always include a list of works of knowledge students have to study. Teachers know in advance the knowledge works or topics that determine the course content. Their responsibility consists in organising productive visits to these works for the students. The students, in turn, are required to encounter the works that shape the syllabus of the course and be able to activate them in certain situations, for instance, to solve problems, make analyses and elaborate on them. The paradigm of visiting works is nowadays evolving towards a new pedagogical paradigm that is characterised in the ATD as the *paradigm of questioning the world*. Instead of works to visit, this paradigm is based on the approach of open questions students carry out under the guidance of the teachers, thus developing inquiry processes characterised as *study and research paths* (SRP).

As happens with other forms of inquiry-oriented instructional proposals, the implementation of SRPs at university level collides with many constraints linked to the current pedagogical paradigm of visiting works. One of them is the fact that university subjects are determined by a set of works of knowledge to study, not of questions to inquire about. In this situation, inquiry activities need to be related to previously established topics. Otherwise, they run the risk of appearing as "extracurricular" and end up disappearing. Relevant investigations about inquiry-oriented teaching in undergraduate mathematics focus on given bodies of knowledge (differential calculus, linear and abstract algebra, differential equations, etc.) and show how inquiry processes help students create a better approach to them (Larsen, 2013; Rasmussen, Kwon, Allen, Marrongelle, & Burtch, 2006; Wawro, 2015; Zandieh, Wawro & Rasmussen, 2017). As stated by Kuster, Johnson, Keend and Andrews-Larson (2017, p. 3):

> [Inquiry-oriented] activities promote the emergence of important student-generated ideas and solution methods, which one can think of as the mathematical "fodder" available to the teacher for the progression of the mathematical agenda.

This *curriculum constraint* can have as a consequence the consideration of inquiry activities as a means to better acquire a given content, without letting them being the core of the content to be taught and learnt. This reduction can explain why many investigations about inquiry-based teaching at the university seem to oppose it to more transmissive (or traditional) instructional formats as if one approach could only exist to the detriment of the other (see, for example, Khalaf, 2018).

The approach in terms of SRPs do not oppose inquiry with transmission, but on the contrary relate them dialectically through the notions of "study" (consulting existing knowledge, attending lectures where the teacher acts as the main media to provide mathematical knowledge, etc.) and "research" (inquiry, problem solving, problem posing, etc.) (Winsløw, Matheron, & Mercier, 2013). As has been illustrated in (Barquero, Monreal, Ruiz-Munzón, & Serrano, 2018), SRPs need the interaction and combination of inquiry-based learning devices with others more based on the transmission of knowledge. This dialectic can also be seen as a reverse in priority: instead of using inquiry activities to help students learn a given work of knowledge, in an SRP, students have to learn works of knowledge to carry out their inquiry better.

Our research team has worked during this last decade in the design and implementation of SRPs at university level (Barquero, Bosch, & Gascón 2011, 2013; Barquero et al., 2018; Bartolomé, Florensa, Bosch, Gascón, 2018; Bosch, 2019; Florensa, Bosch, Gascón & Mata, 2016; Florensa, Bosch, Gascón, & Winsløw, 2018; Serrano, 2013; Serrano, Bosch & Gascón, 2010). We used different strategies to deal with the curriculum constraint and integrate SRPs into the current instructional organisations as official teaching and learning activities. This paper aims to examine these strategies and highlight some of their drawbacks and advantages. The research questions underlying the analyses can be formulated as:

> *RQ1. How can SRPs be integrated into current university instructional formats that are subject to specific constraints of the paradigm of questioning the world? What elements of the SRPs' organisation foster their sustainable implementation? What elements hinder it?*
>
> *RQ2. How does the implementation of SRPs affect the content organisation of the subject? What aspects of the knowledge works that define the agenda of the course evolve and how? Which other remain as constraints difficult to overcome?*

These questions are addressed by considering seven implementations of SRPs as case studies. The research here presented is still in an exploratory stage, where the main indicators of the different cases are identified to formulate hypotheses about the problem of the ecology and sustainability of SRPs and, more generally, about the possible ways to foster the transition between the paradigm of visiting works and the paradigm of questioning the world. The next sections present the main elements of SRPs and the courses where they were implemented, considering three main modalities. A summary of these elements will be presented in the conclusion sections as preliminary propositions to be tested with further research.

THE IMPLEMENTATION OF SRP: DIFFERENT MODALITIES

SRPs are initiated by a *generating question* (Q_0) addressed by a community of study (a set of students X and a set of guides of the study Y) that form a *didactic system* $S(X, Y, Q_0)$. The aim of the didactic system is to generate a final answer $A^{\heartsuit}$ to question Q_0. The work of the community of study and the knowledge involved can be described as a concatenation of derived questions and the development of their associated answers that will lead to the elaboration of $A^{\heartsuit}$. The inquiry process will combine moments of *study* of information available in different sources – the *media* – with moments of *research* and creation of new questions and answers, including the adaptation of the information to the specific (initial and derived) questions addressed. Any kind of tools and knowledge productions used in the study and research process is progressively integrated into the *milieu*, which provides the resources needed to answer the initial question Q_0. This process can be summarised in the following schema, where the $A_i^{\lozenge}$ are bodies of knowledge accessible to the study community that help to answer the derived questions Q_k using the already available works W_j:

$$[S(X; Y; Q_0) \Rightarrow \{ A_1^{\lozenge}, A_2^{\lozenge}, \ldots, A_i^{\lozenge}, W_{j+1}, W_{j+2}, \ldots, W_j, Q_{k+1}, Q_{k+2}, \ldots, Q_n \}] \Rightarrow A^{\heartsuit}$$

In the paradigm of visiting works, a predominance is given to the set of bodies of knowledge $A_i^{\lozenge}$ students are required to study, while the questions Q_k usually arrive at the end of the instructional process, as possible applications. In an inquiry process, what is important is the question Q_0 that is to be answered, while in the paradigm of visiting works learning processes focus on some pre-existing answers $A_i^{\lozenge}$. Of course, behind any work to visit there exist questions that were once important to elaborate the answers, but what is put forward are the works – the answers – not the questions.

The contra posed role between questions and answers appears as an important constraint for a sustainable implementation of inquiry-oriented teaching processes. SRPs are not an exception.

Table 1 presents a brief account of some SRPs that were implemented in university courses of mathematics and engineering subjects. We will focus on the different modalities adopted to integrate them into the traditional organisation of the courses.

SRP	Subject / SRP questions	Level and degree	Period	References
SRP1	*Mathematics* Population dynamics	1st-year Chemical engineering	2005-2009	Barquero et al (2011; 2013)
SRP2	*Mathematics* Sales predictions; Organising a bike-sharing system	1st-year Business administration	2006-2014	Serrano (2013), Serrano et al (2010)
SRP3	*Mathematics* Evolution of Facebook users	1st-year Business administration	2015-2017	Barquero et al (2017)
SRP4	*Strength of materials* How to make a slatted bed base?	3rd-year Mechanical engineering	2015-2018	Bartolomé, Florensa, Bosch, Gascón (2018)
SRP5	*General elasticity* How to make a bike part?	2nd-year Mechanical engineering	2015-2018	Florensa et al (2016) Florensa et al (2018)
SRP6	*Statistics* Data analysis for an investigation on consumer behaviour	2nd-year Business administration	2018-2020	(in process)
SRP7	*Mathematics* Progressive discounts	1st-year Business administration	2018-2019	(in process)

Table 1. List of experienced SRPs

SRP1 and SRP2 were organised as workshops that run in parallel to the regular annual course, as weekly 2-hour sessions for a total of 60 hours both, thus complementing the lectures and problem sessions. SRP3 was organised in a similar way but for a quarterly subject and run for a shorter time (9 sessions of 2 hours). In the three cases, the activities were inserted in an instructional device called "Modelling workshop", to differentiate them from the lectures and problem sessions or tutorials. The subject of SRP4 was fully organized as an SRP lasting a whole 6 ECTS subject (17 weeks, 4 hours per week). The last ones, SRP5-SRP7 were implemented at the end of the courses, after 8-10 weeks of traditional lectures, labs and problem sessions. SRP5 run for the seven last weeks of the course, thus covering a total of 28 h, while SRP6 and SRP7 only run during the three last weeks (12 h in total). In these three cases, the work done in the SRP was named as the "Final project" of the subject.

We can, therefore, distinguish three modalities of integration: SRPs running in parallel to the traditional organisation of the subject (SRP1-SRP3, Modelling workshop); SRPs organised at the end of the subject (SRP5-SRP7, Final project) and a subject totally organised as an SRP (SRP4).

SRP IN PARALLEL TO THE COURSE: "MODELLING WORKSHOPS"

SRP1 is fully presented in Barquero, Bosch and Gascón (2013). It was the first SRP experienced by our research team at the university level. It was implemented as a "Workshop of mathematical modelling" along the annual course "Mathematical Foundations for Engineering" for students of Industrial Chemical Engineering. The workshop was aimed to complement the regular lectures and problem sessions with connected activities focused on a vision of mathematics as a modelling activity. Two teachers were responsible for the subject: one preparing the lectures and the other one (first author of this paper) leading the problem sessions and the workshop. The syllabus was organised in a traditional way: one-variable calculus, several variable calculus and linear algebra. The design of the SRP considered this organisation and proposed a longitudinal inquiry starting with the generating question "How can we predict the long-term behaviour of a population size, given the size of a population over some previous periods of time? What assumptions should be made? How to forecast the population size's evolution and how to test its validity?". The SRP was divided into three subquestions depending on the assumptions made about the population (separated or mixed generations) and the time (discrete or continuous). In the first part, the forecast question for one population with separated generations motivated the use of numerical sequences, one-variable elementary functions (discrete time), derivatives and elementary differential equations (continuous time) that were introduced in the lectures and worked in the problem sessions in a rather synchronic way. In the second part, the case of mixed generations required tools from linear algebra (transition matrices, discrete time) and of systems of differential equations (continuous time). The workshop was implemented during four academic years. The alignment between the contents of the three instructional devices (lectures, problem sessions and workshop) appeared progressively. During the first ones, the workshop could be seen as an application or illustration of the tools introduced in the lectures and worked in the problem sessions. However, progressively, the workshop started leading the dynamics of the course: its needs were supplied by the lectures and problem sessions as soon as they appeared. We can consider this as an ideal situation in the paradigm of questioning the world: a course mainly based on an inquiry (or several ones) carried out by the students under the guidance of a teacher, the demands of the inquiry being supplied by other forms of knowledge dissemination (lectures, experts' talks, readings, etc.).

A similar structure was implemented a few years later in the annual subject of Mathematics of a 1st-year degree in Business Administration (SRP2). In this case, all the teachers were members of the research team. Therefore, it was easy to integrate the workshop into the global organisation of the subject. Lectures and problem sessions were not separated. The syllabus was organised in topics (one-variable functions, derivatives, integration, two-variable functions, optimisation, systems of equations, linear applications) and the topics introduced through modelling activities based on business or economical situations (incomes and costs, demand-supply equilibrium, optimisation of benefits, etc.). An SRP was organised during each term and was linked to one of the main modelling situations of the course as if one of the problems approached in the course was devoted to a more in-depth inquiry. For instance, the first term SRP (one-variable calculus) was proposed from the generating question: “A firm registers the term sales of its seven main products for three years. What amount of sales can be forecasted for the next terms? Can we get a formula to estimate the forecasts? Which are its limitations and guarantees?” In the third term (linear algebra), the generating question was about the managing of a bike-sharing system in Barcelona, considering 3 park locations and the transfer of bikes from one to another. In both cases, one of the three-week sessions was devoted to work on the SRP (in a Modelling Workshop), one week for autonomous group work under the lecturers’ supervision and another to share the results obtained and collectively validate them. An important difficulty found was to organise an SRP for the second term (two-variable calculus) that could start from a question interesting to the students and involving some of the core contents of the term.

SRP3 was implemented considering our previous experience with SRP1 and SRP2. It was initiated by a forecast question about the evolution of the number of Facebook users (Barquero et al., 2017). In this case, however, the Modelling Workshop was run independently of the course content, as a volunteer extracurricular activity that could give students an extra point (up to 10) to their final mathematics grade. The SRP combined online sessions (autonomous work) and face-to-face sessions (sharing results and deciding the new derived questions to follow). The workshops sessions were devoted to students’ presentations and the debate about the questions posed, new questions to inquiry and the models, tools and answers found out.

AN SRP AT THE END OF THE COURSE: “FINAL PROJECTS”

The second modality of integration is to incorporate the SRP at the end of the subject organisation. Of course, this option forces the researchers to reduce some of the activities implemented in the previous organisation of the course content. SRP5 was the first to follow this format: implementing the SRP in the last seven weeks of the semester and leaving a traditional organisation in the first 9. This SRP was implemented in an Elasticity course (6ECTS; 1 semester) in a Mechanical Engineering Degree in an Engineering School in Barcelona. To study the best options to fit the new teaching format, researchers conducted an epistemological analysis of the prevailing epistemology. This analysis revealed two main facts: on the one hand that the analytically solvable problems in this domain were only the ones including ideal situations detached from real workplace activity. On the other hand, the lab sessions were proposed because of the traditional character, and they presented isolated and non-functional knowledge. The SRP implementation initiated by the generating question “How to design (its shape and material) a bike part?” was intended to enable an engineering activity, closer to the workplace tasks and including the finite element method as one of the main mobilised tools. Even if the students only started the SRP at the final part of the course, the SRP was presented at the beginning of the semester, and it was explicitly used as the raison d’être of taught knowledge in the traditional instructional organisation. This organisation caused the contents taught in the classical organisation to be presented as necessary for the SRP. However, once the SRP was initiated, the question became central, and the goal was to provide a suitable solution for the part of the bike while the contents were incorporated within the engineering activity. The SRP was assessed by means of weekly reports to describe the inquiry activity and a final report including the design and justification of the part. This first experience, implemented during the academic year 2014-15, has already been implemented in 5 editions with different generating questions around the design of machine parts. This modality seems to be one of the most sustainable proposals to incorporate SRP because it represents a balance between the two paradigms.

SRP6 and SRP7 followed a similar modality: devoting the last weeks of the course to a final project to be carried out in the form of an SRP. SRP6 was implemented in the subject of Statistics (6 ECTS) for second-year students of a degree of Business Administration. It was organised as a three-week activity carried out by three groups of 30-40 students with a weight of 30% of the final grade. The generating questions of the SRP were linked to two final research projects of a Master in Marketing and a PhD work in the area of Marketing. The three MSc and PhD students were conducting an investigation on consumer behaviour and had prepared a survey to collect data about different issues: Barcelona residents' attitudes towards tourism, streaming TV consumers and employees' competences in service recovery processes. The generating questions of the SRP were related to each investigation and were to be approached through the analysis of the data collected with the survey. Students used all the statistical tools introduced during the course to elaborate a report for the Master and PhD students and present the results in a poster session on the last day of the course. During the sessions, students worked autonomously in teams of 3-4, with some collective activities to share their main results and difficulties with the rest of the teams. They had to present weekly reports to get feedback on the progress of their work. The teachers intervened to propose guidelines for the final reports and posters, to introduce new tools when necessary and to organise the dynamics of the SRP. The effect of the SRP in the global content organisation of the course has appeared this academic year, the teachers proposing an organisation of the subject centred on case studies to be carried out during two weeks (that can be seen as "micro-SRPs") aimed at preparing the future work of the students in the final project. The results of this second edition of the SRP will be presented during the conference.

SRP7 took place during the second quarter of the 2018-19 academic year in a mathematics course of 6 ECTS of the first year of the degree in Marketing and Digital Communities with a group of 70 students with a weight of 15% of the final grade. In the first seven weeks, the course was organised in lectures and problem sessions covering the usual agenda of one-variable differential calculus (families of functions, domains, derivation, limits, etc.). The last three weeks took the form of a modelling workshop with students working in small teams under the guidance of three teachers in sessions with half of the group-class alternated with sharing sessions with the whole group-class. The generating question was an adaptation of the one proposed by Ruiz-Olarría (2015) on progressive discounts. Starting from a press release on the impact of the marketing strategy of offering progressive discounts on the price according to the volume of purchase, the question was: “Are progressive discounts a good marketing strategy? Each team looked for an online company that offered discounts of this type with a minimum of two purchase tranches, and different questions were asked, for example: What is the final amount when the number of units increases? What percentage of discount is there when the number of units increases? As a client, I am interested in buying 100 or 101 units? And 500 or 501? As an entrepreneur, how much money am I losing? Etc.” In the last session of the course, the teams made a poster presentation in a public exhibition to explain the results to students of upper courses. The visitors could reward the team with the best poster and the best explanation.

A COURSE ORGANISED AS AN SRP

The third modality of integration is to incorporate the SRP to all the sessions of the course. SRP 4 that is presented in Bartolomé et al. (2018) was implemented for the first time in 2016/17 in a Strength of Materials course of the same Mechanical Engineering Degree as SRP5. The epistemological analysis of the prevailing epistemology of the course revealed that the course contents were presented isolated and that the problems proposed were far from real problems, leaving tasks such as the load estimation or geometry definition out of the students’ scope. The question that initiated the first edition was “You are working as an engineer in a company manufacturing slatted-beds. Your company supplies beds to an American client (a chain of motels). Recently, you have been commissioned to provide them with single slatted-beds, capable of supporting the weight of a 120 kg person.” The course – and the SRP – lasted 17 weeks in two 2-hour sessions per week. Sessions were structured into four parts. One to check the status of the project and decide the questions Q_i that were considered relevant to the problem. Questions were distributed among the teams during the second part and addressed in autonomous work in the third part. Finally, in the fourth part, each team presented the answers obtained to the whole group. Students were assessed during their work in class and through the weekly reports submitted to the teacher.

CONCLUSIONS: THE COEXISTENCE OF THE TWO PARADIGMS

Our review describes three kinds of implementations of SRPs in different university institutions. The first modality (SRP1, SRP2 and SRP3) requires creating a specific timeframe to implement the SRP without modifying the previous organisation of the course. In this setting, the first moment of the implementation have little effects on the contents of the traditional course. However, the evolution of the SRP along the course modifies the way knowledge is presented in the lecture-problem sessions.

The second modality incorporates the SRPs at the end of the course (SRP4, SRP6 and SRP7) causing important and faster changes in the traditional structure. The epistemological analysis of the previous organisation of knowledge is crucial to decide how to shorten the time devoted to the lecture-problem structure.

The final modality (SRP7) is the one requiring deeper changes in the organisation. In this setting, the institutional conditions play a crucial role and should involve not only aspects regarding the course but also the support of the university institution, including material aspects such as availability of different spaces and flexibility on the assessment method.

We finish by stating some initial hypothesis regarding RQ1 and RQ2:

- An important fragility on the SRPs implementation exists depending on the teacher being familiar with didactics or not. When non-didactician teachers take the responsibility of implementing SRPs, some constraints become more explicit and specific support seems necessary.
- The implementation of SRPs can be seen as a process requiring deep changes into the didactic contract. Both teacher and students need to accept new roles and responsibilities that take time and effort. Research on new teaching and learning strategies and devices seems necessary at this respect.
- To guarantee long-term incorporation of SRPs, non-researcher teachers need explicit training on the didactic design and on the use of tools to manage and describe the knowledge involved in inquiry processes (Florensa et al., 2019).
- The more integrated the SRP is (such as SRP4 and SRP5), the stronger is the evolution of the taught knowledge. Consequently, a deeper epistemological analysis of the content at stake is needed, together with an explicit description of the new way to organise it.

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Pratiques *in situ* d'enseignants universitaires et confrontation avec le vécu des étudiants : une étude de cas

Stéphanie Bridoux[1], Cécile de Hosson[2] and Céline Nihoul[3]

[1]Université de Mons, Belgium, stephanie.bridoux@umons.ac.be; [2]Université de Paris, France, cecile.dehosson@univ-paris-diderot.fr; [3]Université de Mons, Belgium, celine.nihoul@umons.ac.be

In this contribution, we investigate which dimensions of the professional identity of the academic profession are reflected in the practices of two university teachers (UT), one in mathematics and the other in physics, and what are the consequences for the students' experience at the end of a course delivered by each UT. Our analyses show that the teaching practices of each UT are organized according to the representations they have of their discipline and the understanding expected from students. The analysis of the questionnaires administered to the students reveals regularities but also gaps between the objectives of the UTs and the way in which the students have experienced each course.

Keywords : teachers and students practices at university level, preparation and training of university mathematics teachers, professional identity, epistemology.

INTRODUCTION

L'enseignement supérieur universitaire est devenu depuis plus de deux décennies et à l'échelle internationale, un objet d'étude investi par des chercheurs d'origines disciplinaires variées et relevant majoritairement des sciences humaines et sociales (sociologie, sciences politiques, sciences de l'éducation, etc.). Dans ce contexte, les politiques éducatives de même que les identités étudiantes et enseignantes ont été la cible de travaux divers visant à éclairer les dynamiques sociales, économiques, pédagogiques à l'œuvre dans un espace singulier où des enseignants peu ou pas formés pour enseigner sont pour la plupart chercheurs. Les pratiques d'enseignement des universitaires se sont vues investies elles aussi par des enquêtes issues principalement de la recherche en éducation dont peu d'entre elles prennent ou ont pris en charge la dimension disciplinaire de ces pratiques. Certaines études soulignent pourtant que la communauté des enseignants universitaires, particulièrement celle des enseignants-chercheurs (noté EC dans la suite du texte), est « modelée » par la discipline académique dont elle se réclame, et qu'elle partage de ce fait « un même ensemble de valeurs intellectuelles, un même territoire cognitif » (Becher, 1994, notre traduction). Il apparaît donc nécessaire que des recherches sur les pratiques enseignantes universitaires se développent avec pour entrée la spécificité disciplinaire des acteurs et des savoirs (Berthiaume, 2007 ; Trede, Macklin, & Bridges, 2012).

Le travail que nous présentons ici rencontre ce besoin de contextualisation disciplinaire. Il se déploie au sein d'un espace d'enseignement – le cours magistral, et pour deux disciplines académiques spécifiques : les mathématiques et la physique. Au-delà des seules pratiques des enseignants concernés, nous interrogeons également la manière dont ces pratiques et les intentions qui les soutiennent sont perçues par les étudiants. Plus précisément, nous cherchons à caractériser les choix opérés par les EC lorsqu'ils enseignent, à en comprendre les raisons et à les confronter au vécu des étudiants.

OUTILS THÉORIQUES ET PROBLÉMATIQUE

Notre recherche repose sur le postulat suivant : la manière dont un EC organise son discours pédagogique, la manière dont il « fait cours », dont il expose les connaissances à ses étudiants, est portée par un ensemble de convictions sur ce qui doit être su (et donc enseigné) et sur la façon dont cela doit être fait.

Ce postulat se veut volontairement restrictif. Il exclut en effet un ensemble d'autres facteurs susceptibles d'influencer « l'agir pédagogique » (Leclercq, 2000) de l'EC et qui façonnent ce que les chercheurs en sociologie du travail désignent sous le concept « d'identité professionnelle ». Ce concept renvoie au sentiment d'appartenance d'un travailleur à son groupe professionnel (Dubar, 1996 ; Blin, 1997).

La littérature scientifique concernant l'identité professionnelle des enseignants universitaires est assez abondante (Tickle, 2000 ; van Lankveld, Schoonenboom, Volman, Croiset, & Beishuizen, 2017) mais les recherches se distribuent selon des axes au sein desquels l'appartenance disciplinaire de l'EC est peu (voire pas) travaillée. Tout se passe (presque) comme si l'identité professionnelle des EC pouvait être étudiée indépendamment de la discipline qui porte à la fois leur activité de recherche et d'enseignement. Pourtant, et toujours selon Becher (*op. cit.*), « (...) les cultures disciplinaires sont la principale source d'identité et d'expertise des membres du corps professoral et comprennent des hypothèses sur ce qui doit être connu et comment les tâches doivent être effectuées ».

Dans la mesure où le concept d'identité professionnelle a d'ores et déjà fait ses preuves pour éclairer les raisons pour lesquelles « les enseignants font ce qu'ils font » (Kogan, 2000 ; Trickle, 2000) nous avons choisi d'inscrire notre démarche au sein de cet environnement théorique. Mais la prise en compte de la « culture disciplinaire » des EC, pointée comme un incontournable par Becher, par exemple, et qui intéresse également les chercheuses en didactique (des mathématiques, de la physique) que nous sommes, nécessite que le concept d'identité professionnelle soit spécifié au regard de cette culture, c'est-à-dire au regard du rapport que les EC entretiennent avec :

1. la discipline dont ils sont issus (rapport de nature « épistémologique ») : ce que sont les mathématiques, la physique, par exemple mais également la manière dont les savoirs dans ces disciplines s'élaborent dans les laboratoires de recherche.

2. la manière dont cette discipline (ou les savoirs qui en sont issus) doit s'enseigner (rapport de nature « pédagogique »), ce qui n'est pas indépendant des besoins qu'ils projettent sur leurs étudiants.

Ce rapport sera d'autant plus facile à cerner que les chercheurs en charge de son exploration sauront « de quoi il est question », c'est-à-dire entretiendront une certaine familiarité avec les savoirs de la discipline concernés. La recherche en didactique disciplinaire apparaît, pour cela, relativement bien outillée. Ainsi, considéré au prisme disciplinaire, le concept d'identité professionnelle devient un outil opérant pour entrer dans les espaces d'enseignement par la porte des savoirs et enrichir, ce faisant les connaissances sur l'identité professionnelle des EC (de Hosson, Décamp, Morand, & Robert, 2015). De manière plus opérationnelle, l'identité professionnelle d'un EC spécifiée selon sa culture disciplinaire peut se voir inférée à partir :

- des normes assignées à son métier et qu'il reconnaît comme telles (cela inclut les normes institutionnelles liées à son statut, à l'organisation des enseignements, à la manière dont les étudiants sont évalués, ou des normes plus tacites comme les types de savoir qu'il est nécessaire de connaître à tel ou tel niveau d'enseignement, etc.) ; plus généralement, cette dimension renvoie à ce que l'EC juge légitime / illégitime dans l'exercice de son métier.
- des qualités jugées nécessaires pour exercer son métier (il peut s'agir de qualités pédagogiques – être à l'écoute des étudiants ou plus disciplinaires – bien maîtriser les savoirs que l'on enseigne, faire de la recherche, etc.) ;
- des valeurs (ce que l'on apprécie particulièrement dans son métier, que l'on ne serait pas prêt à déléguer à d'autres – une thématique d'enseignement, par exemple, et inversement, ce que l'on déléguerait volontiers, que l'on considère comme ne faisant pas partie du cœur de son métier).

En regard de ces éléments spécifiques (qui façonnent selon nous les choix et les pratiques d'enseignement des EC), la question des effets de ces pratiques sur les étudiants reste encore très peu étudiée (Duguet, 2015). Cela dit il est intéressant de mentionner que les chercheurs qui s'y sont intéressés soulignent de manière convergente qu'il existe un écart assez systématique entre ce que l'enseignant considère comme central dans son enseignement (ie : ce qu'il valorise le plus) et ce que les étudiants en retiennent (Lizzio, Wilson, & Simons, 2002). Les exemples mobilisés, considérés comme moteur de compréhension par les enseignants ne sont, par exemple, que rarement pris en note par les étudiants, les anecdotes passent inaperçues (Titsworth, 2004 ; Clanet, 2004) et les innovations n'ont pas nécessairement la portée motivationnelle attendue.

Ce travail s'inscrit dans le prolongement de ces recherches. Nous formulons pour hypothèse qu'une bonne adéquation entre les intentions qui portent la pratique d'un enseignant et la manière dont celles-ci sont perçues par ses étudiants peut constituer l'un des moteurs de la réussite étudiante. À l'inverse, une inadéquation documentée pourrait permettre à l'enseignant de porter un regard différent sur les causes des difficultés de ses étudiants. Finalement, la question de recherche que nous nous proposons d'étudier est la suivante : quelles dimensions de l'identité professionnelle des enseignants-chercheurs sont perceptibles à travers leurs pratiques et quelles en sont les conséquences sur le vécu des étudiants ?

MÉTHODOLOGIE

Notre travail prolonge les travaux du groupe « enseignement supérieur » du LDAR (Bridoux, de Vleeschouwer, Grenier-Boley, Khanfour-Armalé, Lebrun, Mesnil, & Nihoul, 2019). Nous nous inscrivons donc dans une visée comparatiste des pratiques, en choisissant comme disciplines les mathématiques et la physique. Notre problématique nous amène à étudier à la fois les pratiques d'EC lorsqu'ils enseignent mais aussi le vécu des étudiants concernés par ces enseignements. Pour ce faire, nous avons ciblé deux cours magistraux, un cours en mathématiques et un cours en physique à l'Université de Mons (Belgique), chacun donné par un EC de la discipline. Ces deux cours sont suivis par 17 étudiants d'une filière mathématique en première année d'université.

Afin de saisir les normes, les qualités et les valeurs que chaque EC se donne pour exercer son métier d'enseignant (et caractériser ainsi les rapports épistémologiques et pédagogiques qu'ils entretiennent avec la discipline qu'ils enseignent), nous avons mené avant chaque cours observé un entretien individuel de trente minutes environ structuré autour des questions du tableau 1. Nous confrontons ensuite ces dimensions au vécu des étudiants recueilli à l'aide d'un questionnaire (voir tableau 1). À noter, certains éléments des cours *in situ* (supports utilisés, exemples cités, questions, etc.) viennent compléter l'analyse des perceptions étudiantes.

	Questions aux EC	Questionnaire Etudiants
Normes	Que faut-il que les étudiants aient appris / compris ? Qu'est-ce qui est difficile / facile ? Que penses-tu de la séparation théorie/exercices ?	Qu'est-ce qui t'a semblé facile / difficile ? Pourquoi ? Apprécies-tu le cours ? Pourquoi ?
Qualités	Qu'est-ce qu'un bon cours de maths / de physique ?	Apprécies-tu le cours ? Pourquoi ?
Valeurs	Quelle approche vas-tu adopter ? Quels supports vas-tu utiliser ? Quels exemples ?	Les exemples, les dessins, t'ont-ils aidé à comprendre le cours ?

Tableau 1 : mise en relation des dimensions (normes, qualités, valeur) de l'identité professionnelle à l'étude avec les questions posées aux EC et aux étudiants.

QUELQUES ASPECTS DE L'IDENTITÉ PROFESSIONNELLE DES EC

L'enseignant-chercheur de mathématiques

Dans l'entretien qui s'est tenu avant le cours, l'EC explique qu'il va démarrer un nouveau chapitre portant sur les équations différentielles ordinaires. Il s'agit de présenter les notions d'équations différentielles et de solution. Selon cet enseignant, ce cours consiste en la présentation de nouveaux concepts et ne devrait pas poser de difficulté particulière aux étudiants. Dans l'entretien, l'enseignant confie viser une compréhension approfondie de la part des étudiants. Il se donne comme objectif que ceux-ci donnent du sens aux notions. L'enseignant vise aussi à ce que son cours ne se limite pas à des aspects techniques qui pourraient amener les étudiants à retenir les définitions et les résultats sans essayer de les comprendre. Les extraits suivants issus de cet entretien témoignent de ce fait :

M : … un cours de maths a priori ça doit être différent avant et après… ta manière de raisonner, ton intuition doit avoir évolué dans la pratique.

M : … ce qu'il y a d'important c'est quoi, comment est-ce qu'on pense le truc, c'est quoi les voies directrices, comment est-ce qu'on interprète les formules un petit peu absconses, comment est-ce que on essaie de développer l'intuition, qu'est-ce qu'on fait quand on est bloqué dans une preuve ou qu'il y a un cas qu'on a pas vu, etc.

Chez cet enseignant, il y a une tension entre ce qu'il attend des étudiants en termes de travail personnel et de compréhension et ce qu'il constate réellement (entre ses valeurs et les normes du métier) :

M : … pour eux [les étudiants] c'est qu'est-ce que c'est la matière, qu'est-ce qu'on a à l'examen et comment je fais pour réussir. C'est pas comment je vais améliorer mes capacités de raisonnement, ma finesse, … comment je vais affiner mon intuition.

Pour tenter d'amener les étudiants à donner du sens aux nouvelles notions, l'enseignant a l'intention de les introduire avec des exemples issus de la physique. Ce choix est lié au fait que l'enseignant estime que le cours de physique ne montre pas suffisamment rigoureusement les mathématiques cachées dans les équations différentielles qui décrivent l'étude de certains mouvements. Du coup, il veut combler ce manque en faisant tous les détails mathématiques pour montrer aux étudiants d'où viennent les équations qui décrivent ces mouvements :

M : Ce que je vais faire, c'est d'abord présenter quelques exemples d'équations en essayant de lier au cours de physique, si possible, parce que la vérité ici c'est que dans le cours de mécanique, ils voient pas… enfin le concept d'équation différentielle ne ressort pas. Ils ressortent du cours sans savoir que $F = ma$ est une équation différentielle.

L'enseignant-chercheur de physique

Dans l'entretien pré-enseignement, cet EC explique qu'il va démarrer un nouveau chapitre portant sur le flux électrique et le théorème de Gauss. Selon lui, c'est un chapitre très difficile pour les étudiants. Les notions de physique à introduire sont conceptuellement difficiles et les mathématiques nécessaires pour les aborder posent également des difficultés aux étudiants.

En termes de conceptualisation visée, les objectifs de cet enseignant sont de préparer les étudiants à l'évaluation et qu'ils retiennent les grandes idées développées dans le cours :

P : … les gens qui n'avaient absolument pas envie de venir faire de la physique repartent avec quelque chose comme un bagage… même s'ils n'ont pas compris le détail ça c'est l'examen je vais dire. Moi finalement le bon cours de physique c'est ce qui reste deux mois après l'évaluation.

En ce qui concerne les difficultés liées aux mathématiques, l'enseignant souligne qu'il n'attend pas une utilisation rigoureuse des mathématiques. L'enseignant sait qu'il ne les utilise pas « proprement », il ne s'en cache pas et le dit aux étudiants :

P : … j'essaie de mettre les balises en leur disant attention c'est pas très propre d'un point de vue mathématiques… je pense que les maths élémentaires, le calculus leur pose vraiment des problèmes… vraiment.

Quelques éléments comparatifs entre les deux EC

Chez l'EC de mathématiques, l'ancrage épistémologique est fort et se transpose sur ce que l'enseignant attend des étudiants. Il souhaiterait que toute la rigueur mathématique dont il fait preuve se transmette aux étudiants. L'EC de physique cherche plutôt à être structuré et à développer des méthodes que les étudiants pourront appliquer seuls, y compris le jour de l'évaluation, mais sans forcément les comprendre en profondeur. Chez cet enseignant l'ancrage épistémologique apparaît moins marqué.

Ainsi, l'EC de mathématiques a un profil que nous qualifions de « profil épistémologique » alors que l'enseignant-chercheur de physique a un « profil méthodologique ». Dans leur recherche sur les EC de physique, de Hosson, Manrique, Regad et Robert (2018) avaient repéré des profils semblables. Nous allons maintenant regarder comment ces profils peuvent influencer les pratiques des deux enseignants en relatant quelques aspects frappants dans les deux cours que nous avons observés.

DÉROULEMENT DES COURS

L'étude exploratoire que nous menons ici ne vise pas une analyse fine du discours des deux enseignants pendant le cours. Nous décrivons dans les grandes lignes le déroulement de chaque séance, de manière à montrer jusqu'à quel point il est conforme aux propos des EC recueillis dans les entretiens.

Dans les deux cours, les enseignants vont mobiliser à la fois des mathématiques et de la physique, principalement dans les exemples qu'ils présentent. Le recours à ces derniers était d'ailleurs considéré par chacun des enseignants comme influencé par leur activité de recherche. Toutefois, le profil très différent des deux enseignants va fortement influencer le traitement des exemples dans les deux cours.

Dans le cours de mathématiques, l'enseignant présente, comme annoncé dans le pré-entretien, des exemples issus de la physique. D'une part, il fait l'hypothèse que les étudiants ont le bagage nécessaire en physique pour les comprendre et d'autre part, il intègre tous les détails mathématiques permettant de décrire les mouvements physiques avec des équations différentielles. Or, le traitement mathématique très rigoureux des exemples est trop éloigné des acquis en physique d'un étudiant générique d'une première année universitaire. Ainsi, tout se passe comme si l'extraction de la physique, par l'EC de mathématiques, lui conférait de nouvelles praxéologies (nouvelles théories, nouvelles tâches). Cet EC utilise le tableau, les étudiants prennent des notes, un polycopié est à leur disposition. Nous n'avons pas étudié dans quelle mesure le polycopié est conforme au cours magistral.

Les exemples abordés dans le cours de physique sont finalement très formels et les mathématiques prennent beaucoup de place. Toutefois, comme l'enseignant ne traite pas ces mathématiques rigoureusement, le sens physique des objets se perd derrière une préoccupation forte de l'enseignant de faire acquérir des techniques de résolution de problèmes aux étudiants. Le cours est ainsi structuré en méthodes à retenir en fonction des cas. Cette fois, le cours est beaucoup plus proche des acquis et des étudiants. L'enseignant projette un diaporama qu'il complète ponctuellement en direct, en soulignant certains mots importants, en complétant certaines figures ou en ajoutant certains détails de calculs. Les étudiants ont avec eux le diaporama déjà complété. Ils prennent très peu de notes complémentaires.

Nous allons maintenant confronter ces déroulements au vécu des étudiants juste après le cours.

VÉCU DES ÉTUDIANTS

Nous avons ciblé ici trois questions en lien avec les aspects des déroulements précédemment relatés. Nous avons ainsi demandé aux étudiants s'ils appréciaient les séances de cours, ce qui leur avait semblé difficile et comme les deux EC valorisaient beaucoup la présence d'exemples, nous avons demandé si ces exemples les avaient aidés à comprendre le cours. Nous avons recueilli 17 questionnaires.

82% des étudiants qui suivent le cours de mathématiques apprécient peu l'exposition magistrale. Ils sont nombreux à évoquer que c'est durant les travaux dirigés qu'ils comprennent les notions vues dans le cours et à quoi elles servent. La rapidité du cours et une présentation qui a semblé peu claire aux étudiants sont cités comme des sources de difficultés pour suivre le discours de l'enseignant. Ce sont précisément les liens entre les mathématiques et la physique qui étaient difficiles à cerner dans le cours. De plus, des aspects liés à l'épistémologie de la discipline que l'enseignant avait déclaré mettre en valeur dans son cours induisent des difficultés chez les étudiants, comme l'exprime l'étudiant suivant qui apprécie peu les séances de cours tout en reconnaissant leur utilité :

E_1 : … le cours est quand même utile et nous apporte une démarche plus scientifique (prouver des choses, se poser des questions).

En physique, ils sont 47% à peu apprécier les séances de cours. Un point souvent évoqué comme étant difficile est la concentration. Rappelons que les étudiants disposent du cours complet qui est projeté par l'enseignant, les seules notes qu'ils pourraient prendre viennent de commentaires oraux et non écrits parfois ajoutés par l'enseignant. Cependant, le mode d'enseignement choisi est un point positif relevé par plusieurs étudiants :

E2 : Elles nous permettent de vraiment bien cibler la matière et de comprendre le pourquoi du comment on a introduit tel ou tel outil au fur et à mesure du cours.

87% des étudiants estiment que les exemples donnés par l'EC de mathématiques n'ont pas aidé à comprendre le cours. Les étudiants expliquent qu'ils ne voyaient pas les liens entre les exemples issus de la physique et les mathématiques qui en découlent, comme le montre l'extrait suivant :

E3 : … à part le fait qu'on utilise les différentielles en physique, les exemples n'ont servi à rien.

En physique, 70% des étudiants estiment que les exemples ont aidé à la bonne compréhension du cours. L'étudiant suivant en donne la raison :

E4 : Cela permet de voir comment on applique directement les notions qu'on a vues.

BILAN ET PERSPECTIVES

Cette première étude de cas montre que les pratiques des deux EC s'organisent en fonction de certaines dimensions de l'identité professionnelle telles que la représentation de leur discipline (norme), les valeurs qu'ils lui associent, comme par exemple la compréhension attendue des étudiants, des besoins qu'ils projettent sur les étudiants mais aussi des qualités, comme par exemple la présentation d'exemples et de dessins pour illustrer les notions introduites. Ces résultats confirment l'étude menée par de Hosson et al. (2018) dans laquelle elles mènent une analyse du discours de deux EC de physique en cours magistral.

En mathématiques, les choix qui portent sa pratique semblent contraints par le rapport qu'il entretient avec sa discipline. L'enseignant souhaitait partir d'exemples issus de la physique et montrer rigoureusement les mathématiques qui se cachent dans l'étude des mouvements. De ce fait, il donne énormément de détails oralement sur les notions mathématiques qu'il introduit et fait comme si les notions de physique dont il a besoin étaient totalement disponibles chez les étudiants. De notre point de vue, l'enseignant s'adresse à un étudiant « modèle » qui n'est sans doute pas l'étudiant physiquement présent dans la classe. Nous avons observé un décalage entre l'intention de l'enseignant de donner du sens aux notions mathématiques avec des exemples issus de la physique et le vécu des étudiants. Le décalage entre le discours de l'enseignant et les acquis des étudiants est trop important et les étudiants ne s'y trompent pas, les exemples ne les ont effectivement pas aidés à comprendre le cours.

En physique, les choix de l'enseignant semblent plus contraints par des préoccupations de nature pédagogique, le « sens » des concepts introduits pendant le cours disparaît derrière l'application de techniques présentées comme des méthodes à appliquer par les étudiants. Ces choix amènent l'enseignant à être plus proche à la fois des acquis des étudiants et de leur manière de travailler. Nous avons pu voir que les étudiants sont ainsi plus enclins à adhérer aux choix de l'enseignant et ils pensent avoir été aidés par les exemples pour comprendre le cours.

Cette étude de cas montre qu'il y a une certaine cohérence entre les objectifs des enseignants et le déroulement du cours. Toutefois, les étudiants sont davantage mis en difficulté lorsque la compréhension visée requiert une certaine profondeur et donc lorsque l'enseignement est le fruit d'une pratique de nature épistémologique. Les pratiques des deux EC sont ainsi influencées par les objectifs et la compréhension visés par l'enseignant. Enfin, nous n'avons finalement que peu d'éléments sur les apprentissages des étudiants, raison pour laquelle nous parlons plutôt de leur vécu et pas de leur compréhension.

Il s'agit maintenant de compléter notre corpus de manière à dépasser l'étude de cas pour parvenir à une spécification de portée plus générale des pratiques enseignantes, sans doute plus variées que celles décrites dans ce texte, et de les mettre en relation avec les conceptualisations visées.

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Architectural Heritage between Mathematics and Representation: studying the geometry of a barrel vault with lunettes at a first year Bachelor's in Architecture

Caterina Cumino[1], Martino Pavignano[2], and Ursula Zich[2]

[1]Politecnico di Torino, Department of Mathematical Sciences "G. L. Lagrange", Italy, caterina.cumino@polito.it; [2]Politecnico di Torino, Department of Architecture and Design, Italy, martino.pavignano@polito.it, ursula.zich@polito.it.

We present an interdisciplinary activity, directed to students of the first year Bachelor's in Architecture, about the visualization of mathematical objects through physical and digital models, with specific regard to surfaces generated by intersections of cylinders. We focus on the geometrical structure of barrel vaults with lunettes and propose the use of various kind of models to improve the accessibility of interdisciplinary elements and to translate specialistic knowledge, building a common language for students with different backgrounds. The work is a result of a broader research project, dedicated to enhancing the relationship between Mathematics and Architecture through Geometry and Representation.

Keywords: Geometry for Architects, Surfaces, Models, Representation, Visualization.

INTRODUCTION

In the field of Italian university Mathematics education, the need to set up a dialogue within other disciplines in which Mathematics is taught (Architecture, Engineering, Biology, etc.) is increasingly felt. Teaching Mathematics to non-Mathematics learners means teaching it as a service subject, hence as a tool to model systems of the domain and to solve the associated problems (Howson, Kahane, Lauginie & Tuckheim 1988).

We are interested in the role played by Mathematics in courses of the bachelor's degree program in Architecture. In this context, basic mathematical tools are conveyed by the first year Calculus course: here students learn topics which are preparatory and in support of the parallel course of Architectural Drawing and Survey Laboratory (and of the subsequent courses of building physics, real estate evaluation, as well as of the structural matters).

TEACHING PROBLEMS

One of the main teaching problems is that students, in their first year of academic study, come from diversified educational backgrounds and have different skills (or lack of them) about mathematical and graphic language; as for Mathematics, they may not understand where and when the proposed abstract mathematical topics will be concretely applied. It should be necessary to create a perception of their use value in the wider sense, in order to avoid the traditional insularity of Maths in the technical faculties (see e.g. Harris, Black, Hernandez-Martinez, Pepin, Williams, & TransMaths Team 2015; or Rasmussen, Marrongelle, & Borba 2014) and to promote a synergistic relationship that fills the gaps between the various languages. For example, the increased relevance of digital and parametric modelling in Architecture (see e.g. Calvano 2019) has created a need for developing the education of future architects through a greater integration of mathematics and disciplines which are specific of Architecture. Another problem is students' previous mathematical experience in facing university studies, depending on whether they were more or less oriented to the formation of spatial visualization capacity. Important difficulties appear in the transition from 2D to 3D analytical representations of geometrical objects (Cumino, Pavignano, Spreafico, & Zich 2019): for example, from the 2D analytical representation of a straight line or, more generally, of a plane curve to the representation of a plane or cylinder having that curve as a directrix. Students have the concept of cartesian equation based on their experience from high school Mathematics, but some of them may not accept that the same equation represents different set of points passing from 2D to 3D. Such phenomena can be related to those of «tacit models» mentioned by Fischbein (1989), who refers to representations of certain mathematical, abstract notions developed at an initial stage of the learning process which continue to influence, tacitly, reasoning and interpretations of the learner, hence the need to help students to control the (possibly negative) impact of these models.

So, a myriad of problems of cognitive nature (type of background, crystallized information, gaps between disciplinary languages) or of didactic nature (teaching strategies, disciplinary content and courses organization) are to be taken in consideration, before being able to concretize any proposal for teaching interventions.

Investigations in this direction, related to this particular context, seem to have been somewhat limited, mainly in connection with other disciplines, while correlations between spatial imagery information processing, spatial visualization and geometrical figure apprehension have been the subject of a number of studies, see e.g. the comprehensive review by Jones and Tzekaki (2016) and Kovačević, N. (2017), about recent research in Geometry education. On the other hand, it is fundamental to overcome the aforementioned problems, not only from the point of view of basic mathematical education: when approaching the study of the built form, the architecture student should acquire interdisciplinary knowledge -thus developing basic skills- for its analysis and it is important to underline how the knowledge that 'flows' into an architectural design is not only related to its specific disciplines but includes a variety of methodologies and interpretations derived from other subjects. Therefore, in students' educational path, it seems worthwhile to set up study activities having an interdisciplinary approach.

In this contribution, we present an attempt in this direction, focusing on the Geometry education of architecture students and its role in connection to geometrical comprehension of architectural shapes and spatial visualization ability and we highlight the importance of mathematical thinking in the formalization of architectural structures, in particular of roofing systems constituted by vaults.

We also refer to Duval's analysis of visualization process and its interactions with geometrical reasoning, adapting it to the particular educational context; in this sense we follow the idea of construction as a process dependent only on the connections between mathematical properties and technical constraints of the used tools.

VAULTS AND GEOMETRY

Most of the Italian Architectural Heritage was built with masonry structures and vaults are roofing elements mainly used for covering rooms of a building. Original vaults were as simple as the surfaces of rotation and/or of translation that were used to design them. Then they became more complex systems, reaching different forms (subtended by as many geometries).

In Architecture, the use of Geometry and its elements, such as points, lines, planes, then surfaces and solids [1], as the result of a learning path has its roots in the ancient past; i.e. Leon Battista Alberti, Francesco di Giorgio Martini, Andrea Palladio, Vincenzo Scamozzi and Guarino Guarini wrote about these issues in their treatises (Spallone-Vitali 2017, pp. 88-90), see Figure 1. The variety of the vaults compositions is briefly exemplified in Figure 2a, b.

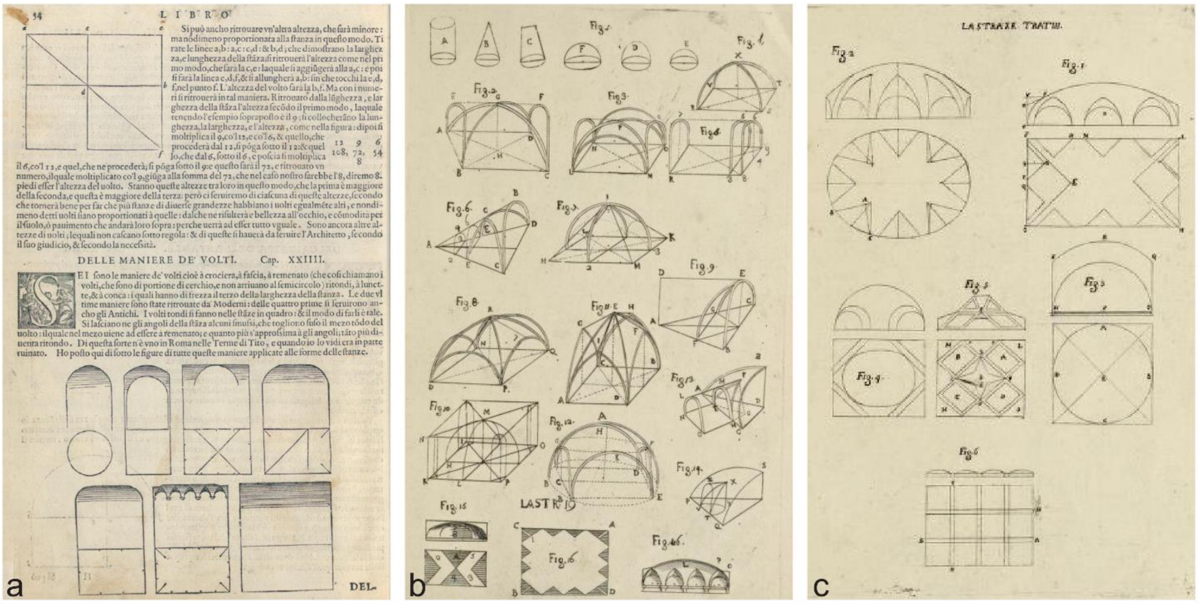

Figure 1: Historical examples of geometric and graphic description of some vaults. Palladio 1570, p. 54; b) Guarini 1737, tav. XXVII, c) Guarini 1737, tav. XXVIII.

The study of these constructive elements plays an important role: we should provide students with some critical tools useful for the conceptual investigation of these structures, even for starting from their graphic formalization.

During the 17th century Guarino Guarini, architect and mathematician, in his treatise, *Architettura Civile*, states that: «vaults […] are hardest to be invented, to put in drawing and to build» (Guarini 1737, p. 183). Again, Guarini states that: «in each of its operations Architecture uses measures, then it relays on Geometry and deserves to know at least its fundamentals» (Guarini 1737, p. 3). His ideas clearly highlight the close relationship between geometry and the architectural artefact.

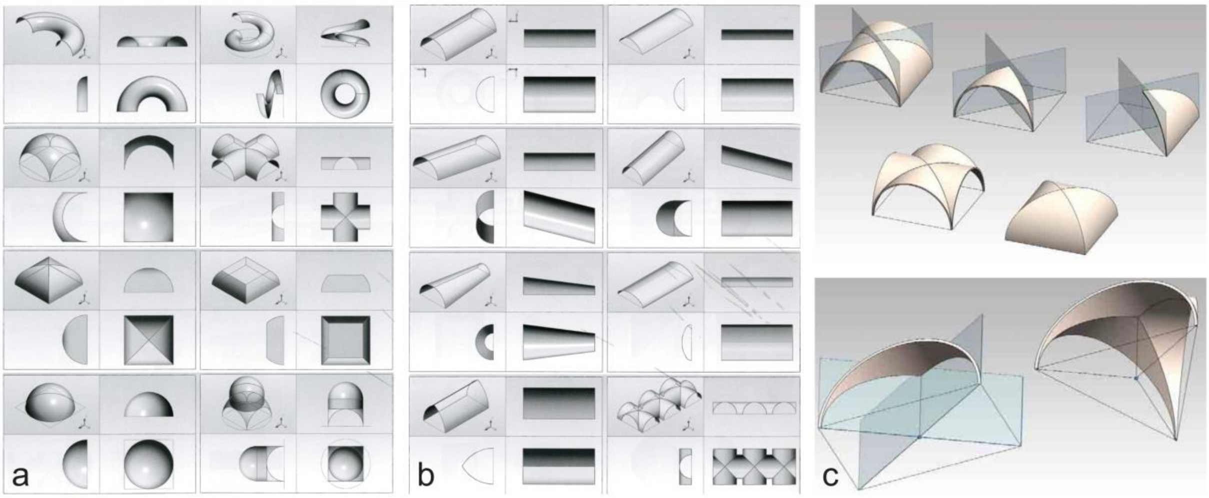

Figure 2: simple and compound vaulted systems: a), b) Bertocci, Bini 2012, pp. 265, 266; c) Fallavollita 2009, pp. 453, 455.

For example, Guarini sometimes used to describe the composition of a vaulted system by junction of parts of different surfaces cut by the same plane (Spallone-Vitali 2017). This idea is still used today. Figure 2c shows this kind of construction applied to square based groin and cloister vaults and to triangle-based groin vault, without caring about the problem of intersections. Recent bibliographical references for the geometric-conceptual study of vaults offer an almost always 3D or pseudo-3D set of views, with textual descriptions. Many authors represent these elements with axonometric views (see e.g. Docci, Gaiani, Maestri 2011; Fallavollita 2009) which immediately convey the idea of the three-dimensionality of these elements, but often neglect its geometrical genesis.

METHODOLOGY TO STUDY A BARREL VAULT WITH LUNETTES

In light of previous considerations, the complexity of the form needs a discretization of its characteristic and descriptive elements through the graphic language that, in being an expression of synthesis, risks to become an excessive simplification of its features. Consequently, at the base of the training path of the architectural student, there is the study conducted between the representation of a theoretical model and its analytical description.

We propose four teaching tools to enhance students' critical shape-reading skills referring to architectural heritage: a tangible model to introduce the problem; a graphic representation to investigate the relationship between drawing and shapes; a virtual model obtained by a dynamic geometric software (DGS), constructed through an analytical description; a physical model as outcome of the preceding three steps. These tools are in order to promote students' ability in switching between different registers of representation (see Duval 1999).

We identified a family of composite vaults -which can be traced back to intersections of barrel vaults with coplanar and orthogonal axes, see Figure 3- as having an explicit formative value in this context.

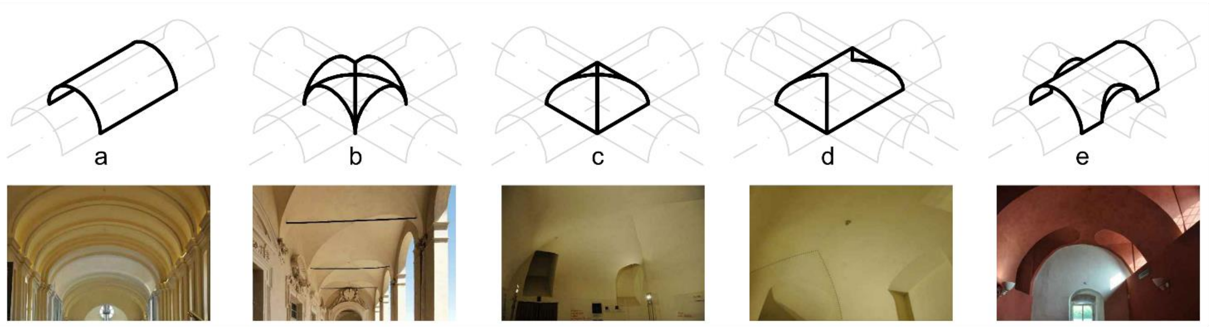

Figure 3: Cylinders intersections. a) barrel vault; b) groin vault; c) cloister vault; d) barrel vault with cloister heads; e) barrel vault with lunettes. Pictures of Royal Residence of Venaria Reale (Torino).

Graphic Analysis

Figure 4 shows the graphic study of intersection between two circular right cylinders of different radius with coplanar and orthogonal axes. The random nature of the choice of which cutting planes could be used for this study leads to subjective results, affected also by different graphic tools (for example manual drawing or Computer Aided Architectural Design (CAAD) bring different typologies of error) [2]. The auxiliary planes, here, were chosen to respect the uniformly distribution of the information about the development of the surface starting from the angular subdivision of the circular section.

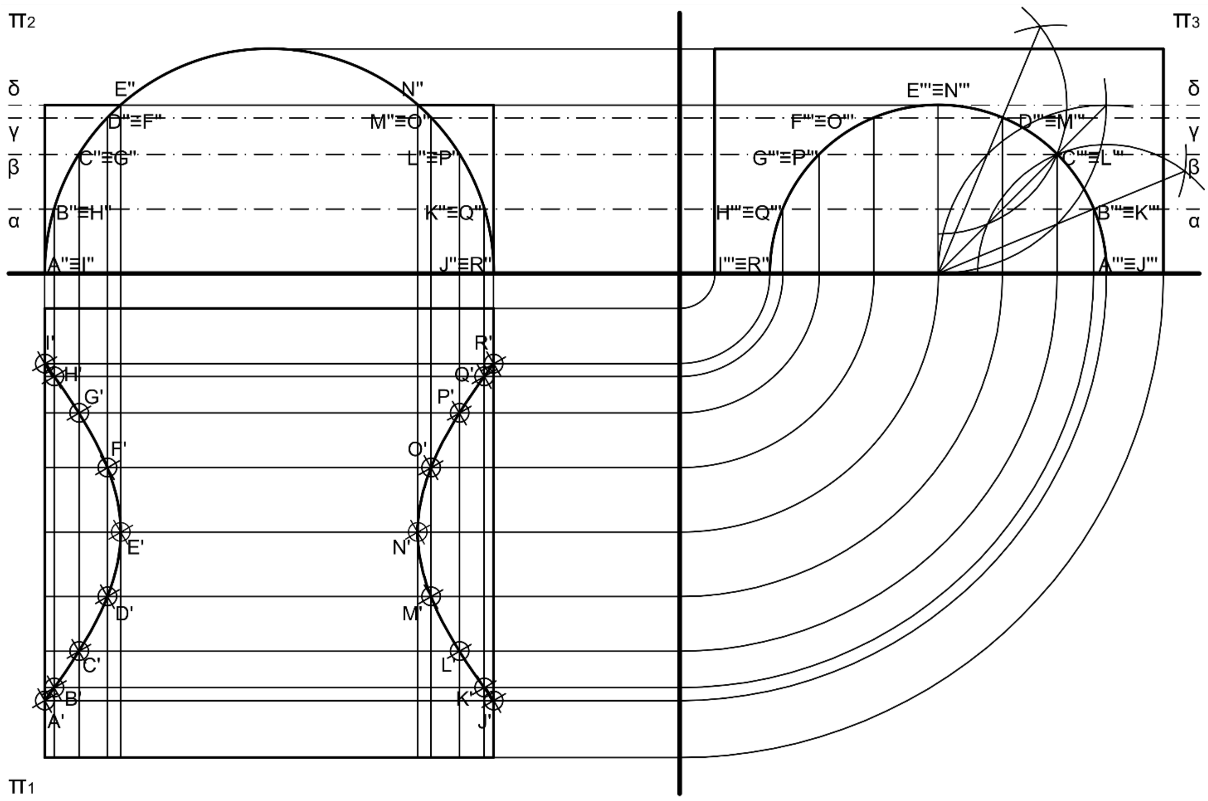

Figure 4: CAAD orthographic projection of the intersection between two barrel vaults.

DGS Analysis

Over the last few decades, the appearance of DGS has renewed Mathematics education providing computational and visual tools available in software environments like *Geogebra*, (Arzarello, F., Bartolini Bussi., M. G., Leung, A., Mariotti, M. A., & Stevenson, I. 2012).

In our context visualization is a crucial matter, in particular in 3D Geometry, but analytic methods may lead to heavy and unilluminating computations and students' mathematical background does not allow them to be autonomous in the visualization of mathematical objects. In the specific case, a surface generated by an intersection of cylinders can be studied in an interdisciplinary activity, making a joint usage of CAAD together with a DGS, which gives the possibility of approaching problems from different perspectives, connecting algebraic and geometrical views, facilitating constructions of mathematical objects and also the direct manipulation of them.

Due to the learners' small knowledge about 3D Analytic Geometry, a ready-made GeoGebra model is proposed, which is realized using an analytical description based on parametric equations of the involved geometric objects.

Let C_1 and C_2 be two semicircular cylinders with coplanar orthogonal axes and different radiuses $R_1 > R_2$. Let $\gamma = C_1 \cap C_2$ be the intersection curve of the two cylinders (see Figure 5), where C_1 is the cylinder generated by translation along the x-axis of the semi-circumference with center in the origin and radius R_1, in the plane (yz) and C_2 the cylinder generated by translation along the y-axis of the semi-circumference with center in the origin and radius R_2, in the plane (xz); taking parametric representations $C_1: (x, y, z) = (v, R_1 \sin(u), R_1 \cos(u))$ and $C_2: (x, y, z) = (R_2 \sin(u), v, R_2 \cos(u)), -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$, the quartic skew intersection curve $\gamma = C_1 \cap C_2$ may be represented by:

$$\gamma: (x, y, z) = \left(\pm\sqrt{R_2^2 - R_1^2 \cos^2(u)}, R_1 \sin(u), R_1 \cos(u) \right)$$

In order to visually understand the surface, GeoGebra dynamics are exploited, see Figure 5. As the software enables to rotate a still picture using the mouse, the drawing in Figure 6 shows how to visualize the orthographic projections of the intersection curve $\gamma = C_1 \cap C_2$.

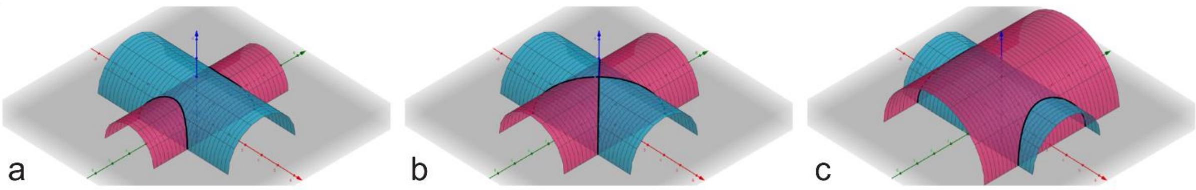

Figure 5: Frames from the dynamic DGS model using a slider to change the red cylinder radius.

We may compare the graphic outcome above with the GeoGebra one: it is clear that the DGS representation is univocal in contrast with the graphic one that is affected by approximations due to drawing choices.

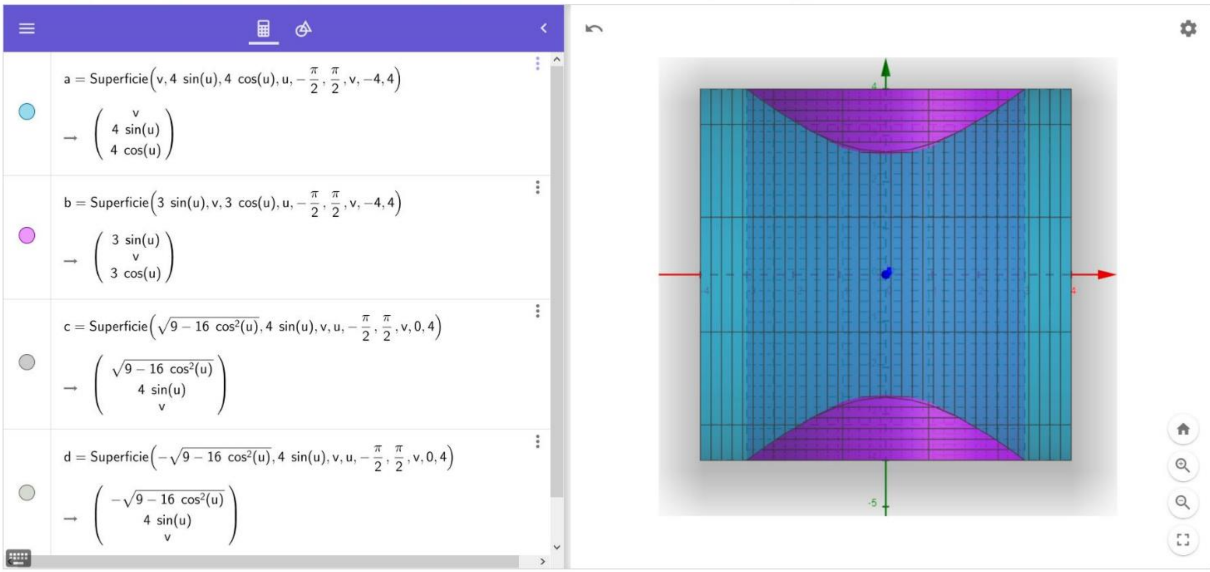

Figure 6: Top view of the barrel vault with lunettes obtained by GeoGebra.

Another way to enhance students understanding is to introduce a real parameter t, replacing the given cylinder C_2 with a 1-parameter family of cylinders: in this way, a slider bar enables to manipulate directly the surface thus generated (see Figure 5, 6) and to follow in particular the deformations of the curve γ, as the parameter varies, while, by suitably rotating this dynamic picture, as it was done in Figure 6, the orthographic projection of γ on the plane *(xy)* appears easily as subset of a 1-parameter family of hyperbolas.

Physical Models

Starting from the same analytic description as above, a physical model of a barrel vault with lunettes has been realized, using acetate or paper (Figure 7a, c).

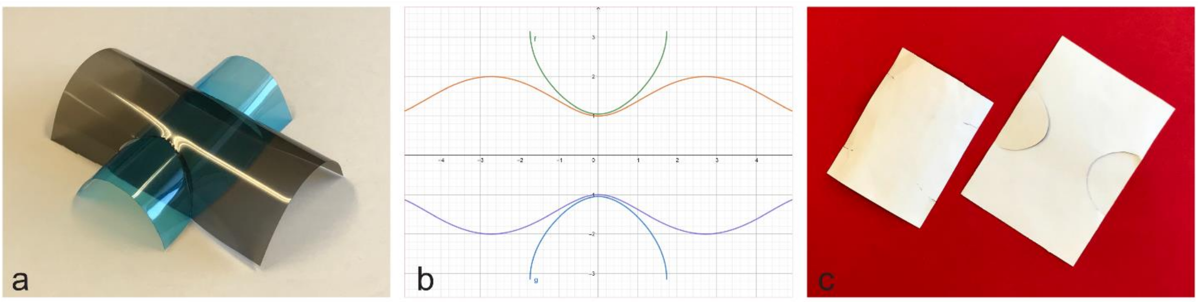

Figure 7: Physical model of a barrel vault with lunettes. a) acetate model; b) developments of the same intersection curve thought as belonging to C_1 or C_2; c) disassembled paper model (similar to the acetate one).

If the semicircular cylinders have the same radius, their intersection is a plane curve (generating a cloister or a groin vault, see Figure 3, b and c) and in this case students can realize that its development is a sinusoidal curve using basic elements of trigonometry (see Cumino et alii 2018); if the radiuses are of different lengths, the intersection curve of the cylinders is in general a skew curve, therefore, to obtain its development in the plane requires more sophisticated mathematical tools (e.g. the concept of development of a spatial curve on the plane and the arc-length calculation, see e.g. M. P. Do Carmo (1976), which do not belong to a standard Calculus course program. Nevertheless, the model may be used as a tangible object to communicate the particular shape of the surface under consideration, because it facilitates the geometric perception of the shape as a whole and also the understanding of the geometric properties of the intersection curve; in fact, disassembling the model one observes that this skew curve develops on the plane in two distinct ways (see Figure 8b), depending on whether it is considered belonging to the cylinder C_1 or to cylinder C_2. Then introducing the arc-length formula, the development of γ on the plane *(xy)*, as a curve on C_1, may be represented by

$$\gamma' : (x, y, z) = \left(\pm\sqrt{R_2^2 - R_1^2\cos^2(u)}, R_1\mathrm{u}, 0 \right)$$

In a similar way the development of γ as a curve on C_2, may be represented by

$$\gamma'' : (x, y, z) = \left(R_2\mathrm{u}, \pm\sqrt{R_1^2 - R_2^2\cos^2(u)}, 0 \right)$$

With respective cartesian equations

$$\gamma' : y = \pm\, R_1 \cos^{-1}\frac{\sqrt{R_2^2 - x^2}}{R_1} \quad \text{and} \quad \gamma'' : y = \pm\sqrt{R_1^2 - R_2^2\cos^2\left(\frac{x}{R_2}\right)}$$

CONCLUSIONS AND OUTLOOK

In the present paper, we considered how mathematical thinking may contribute in the formalization of architectural structures, using the specific case of barrel vault with lunettes. Our activity was born in an Architectural Drawing and Survey Laboratory, whose main purpose is to make students understand the use of drawing as a tool of analysis and synthesis and as a means to communicate and visualize geometrical objects. The dialogue between Mathematics and Architecture has brought to light a way of interpreting and using Geometry different from that usually practiced by Mathematics teachers.

We took into account Duval's analysis of visualization process and its interactions with geometrical reasoning, adapting it to the particular educational context in regard to 2D/3D graphical representation of a real object and exploration of geometrical situations via physical or digital model; we exploited the experience of teaching

Mathematics in order to make students aware of the variability of representations of an object (depending of the variety of physical or semiotic systems producing them) versus the invariance of the object itself. We present the geometrical reasoning that underpin strategic choices in the graphical description process. To do this, the teacher has to choose a specific set of points and lines that best describe the considered shape; to disclose how to make these choices, physical or digital models are employed, constructed according to a mathematical recipe: the same model which is used by the Mathematics teacher as a tool to teach Geometry. Therefore, the model appears to be, in a sense, a translator between the two disciplinary approaches. Further research in this direction may provide new means to promote a stronger interdisciplinary educational system and to enhance students' spatial and visualization abilities.

NOTES

1. There might be a connection between the idea of tacit model and the architectural design process: if we think of architectural artefacts as results of the creative application of Geometry and its basic elements. In this sense, such elements could act as tacit models for the first architectural composition/shapes recognition exercises.

2. «It is not possible to consider all the points of a surface when you need to represent it. The same is true for all the lines that belong to it. In order to proceed, it is necessary to transform the surface into a discrete set of lines: those that best lend themselves to describe its geometry», Migliari 2001, p. 160.

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Mathematical modelling in probability at the secondary-tertiary transition, example of biological sciences students at university

Camille Doukhan[1]

[1]CREAD, Université de Brest, France, camille.doukhan1@inspe-bretagne.fr

Abstract. The work presented here focuses on probabilistic modelling at the secondary-tertiary transition. Concerning the university level, I am interested in non-specialist students, more precisely first-year biology students. I have written and submitted a test on discrete probabilities to Grade 12 and to biology students. I present here the a priori and a posteriori analysis of one exercise of this test. The analysis of the students' responses to this exercise allowed me to establish initial results on possible difficulties in the secondary-tertiary transition. A first point is that students, in these modelling activities, have difficulty linking task and type of task. Another result is the very high use of probability trees by students in their modelling of the probabilistic situations proposed.

Keywords: mathematics in other disciplines, modelling, probabilities, secondary-tertiary transition, students' activity

INTRODUCTION

The study of the secondary-tertiary transition is not new in mathematics education research (Gueudet & Thomas, 2019). Many authors have focused on the difficulties encountered by students in the fields of calculus or linear algebra (e.g. Vleeschouwer & Gueudet, 2011). There are fewer studies on the topic of probability. However, probabilities are taught in many university courses, particularly for non-specialists such as biology students. My study takes place in France and I have chosen to focus on biology students who study probability from the first year of university.

In the following section, I present the context of my research and previous works about the teaching and learning of probabilities on which I based my study. Then in the third section I present my theoretical framework. In the fourth section I present the methodology I used in my research. In the fifth section I present my results. Finally, in the last section I present the conclusions of this study.

RELATED WORKS

I focus here on the secondary-tertiary transition in the particular context of mathematics in service courses, especially for biology students.

The relationships between biology and mathematics have been studied extensively (Lange, 2000) because they are quite complex and important. Biology has long been a mainly descriptive science and the recent development of new mathematical modelling tools has had an impact on this science by making it highly mathematical, especially giving to it a predictive and decision-making role. Mathematical modelling and probabilities are widely used by biologists (Duran and Marshall, 2019).

The good mastery by biology students of these mathematical contents is an objective of the biology studies at university. That's why general probability courses are designed for biology students from the early years of university to allow them to take courses in statistics or biostatistics in subsequent years.

Several issues are raised by the teaching of mathematics at university for non-specialist students. The difficulties encountered by non-specialist students in mathematics courses would be an important factor in dropping out of their programmes. Previous works have evidenced that the lack of links between these mathematics courses and the future professional practices of these non-specialist students. This lack of links leads them to consider mathematics as too abstract, and to have difficulties to mobilize mathematical tools (González-Martin, Gueudet, Barquero & Romo-Vazquez, to appear).

Concerning more precisely biology students, Viirman and Nardi (2018) highlighted that their involvement in mathematical modelling activities is a motivating factor in their learning of general mathematics courses.

I have therefore chosen to focus my work on mathematical modelling aspects and in particular the use of probability trees. In order to expose my analyses in the fourth part, I will present in the following my theoretical framework.

THEORETICAL FRAME

In this research, I choose an institutional perspective and consider that secondary school and university are two different institutions.

I use the Anthropological Theory of Didactics (ATD, Chevallard, 2006) and more particularly the concept of praxeology. A praxeology consists of four elements: a type of task; a technique to accomplish this type of task; a technology which is a discourse explaining and justifying the technique; and a theory. The comparison of praxeologies in secondary school and at university is very useful because it allows me to highlight possible difficulties for students during this transition. For example, it is interesting to consider the technique produced by the student when it is not explicitly requested; or to look at the technology used- or not- by the student in his/her solution of an exercise.

I also use Activity Theory and its adaptation to mathematics education (Vandebrouck, 2008) to allow me to look more closely at the complexity in the student's activity for a given task. I use here the notion of task as described in the Theory of Activity, i.e. referring to the object of the activity and its description. This theory has allowed me to highlight that the complexity, for a student, of linking a task proposed to a type of task is a feature of the secondary-tertiary transition. In my analyses below, I will give examples of this process.

MATHEMATICAL MODELLING IN PROBABILITY

I consider here the activity of mathematical modelling in general for the theme of probabilities (it can be the activity expected by the text of an exercise or the actual activity of students). This is what I call "mathematical modelling in probability".

Usually it starts with a random situation described in natural language. It is then necessary to identify the events at stake, name them and determine their probabilities. According to the situation, the use of a probability tree can be relevant or not. Mathematical probability modelling mixes recognition activity (recognizing the task to be accomplished and linking it to a certain type of task for which a technique is known), changes of register (moving from natural language to probability formalism) and entanglements of techniques (I will illustrate this probabilistic modelling activity later on).

I claim that associating concepts from the Activity Theory and the Anthropological Theory of Didactics contributes to a precise understanding of this mathematical modelling in probabilities and will illustrate it below.

Based on the theoretical tools developed above, I present as follows my research questions:

How do students build a probabilistic model for a situation from a biological context?

How to describe these probabilistic modelling activities using praxeologies?

METHODOLOGY OF THE RESEARCH

My research takes place in France, in secondary school in a rural environment and in a middle-sized university. Mathematics courses are offered to biology students from the first year, including a course entirely devoted to probabilities (14 hours of lectures with about 300 students and 14 hours of tutorials, which are sessions dedicated to exercises in groups of 30 students).

For this research, I was particularly interested in this course and attended two lectures and two tutorial sessions on the following probability topics: "independence and conditioning" and "continuous random variables". I also observed a class of Grade 12 in the science section of the secondary school (called "scientific section") for seven one-hour sessions on these same probabilities' themes.

I chose these two probability themes because they are part of the Grade 12 curriculum and are taught again at the university for biology students. Probabilities are present in the secondary school curriculum (there are discrete random variables in the Grade 11 curriculum, conditioning and independence in Grade 12 and continuous random variables in Grade 12 as well) (Ministère de l'Education Nationale, 2011). The probability course in biology at the university is quite general, it includes notions seen in secondary school (conditional probabilities and independence, discrete random variables, continuous random variables) but it also contains chapters devoted to new concepts (such as probabilistic model and probabilized space, independence of random variables, limit theorems and their applications, law approximation).

I have chosen to study the secondary-tertiary transition through the chapter of discrete probabilities "independence and conditioning" common to both curricula. In a previous study (Doukhan & Gueudet, 2019), I have evidenced through textbooks analyses that the same probability content (discrete random variables in this study) can lead to very different praxeologies in secondary school and at university. Applying similar methods here (analysis of the resources collected during my observations, such as films of the course, course handouts and textbooks) evidenced the interest of a focus on modelling and probability trees.

I designed a test and submitted it to secondary school students and to first-year biology students. This test (see Appendix) consists of three exercises on the theme "Independence and conditioning".

The test was administered to all the students (29) from the class of Grade 12 observed, during the fifth session on the topic of "Independence and conditioning", students had half an hour to do the test. The test was also offered to first-year biology students prior to their probabilities course of the second semester, with 25 of them participating for a similar duration.

I carried out a quantitative analysis of all the students' productions, which allowed me to highlight first results concerning mathematical modelling in probability at the secondary-tertiary transition, I also analysed students' answers. I have chosen here to focus only on the analysis of the second exercise, which reads as follows: "You are the director of the Minister's Office of Health. A disease is present in the population, in the proportion of one sick person out of 10,000. The manager of a major pharmaceutical company comes to you to tell you about his new screening test: if a person is sick, the test is positive at 99%; if a person is not sick, the test is negative at 98%. Do you authorize the marketing of this test?". This exercise is a classic application of the Bayes' Theorem with an important modelling work left to the student: statement is in natural language, events to be identified, etc. (more details in the a priori analysis in the following section)

I have chosen this exercise because it requires an important probabilistic modelling work, and then because is linked with a biology context.

MAIN RESULTS

A priori analysis

I present here the *a priori* analysis of the second exercise of the test.

The main types of tasks I have identified for this exercise are, first, to perform probabilistic modelling based on a natural language statement; second, to calculate a conditional probability; and finally, to interpret the numerical result in order to answer the question in natural language.

Each of these types of tasks contains several subtypes of tasks with which particular techniques are associated. The difficulty of each subtype of task depends on the precise teaching context. In order to analyse how students will link tasks and types of tasks, I have to associate AT and ATD because an analysis in terms of ATD would be insufficient for my purpose, for this reason I also use task analyses developed in Activity Theory.

The associated subtypes of tasks for the first type of task: "perform probabilistic modeling of a natural language statement", are as follows: identify probabilistic events, associate their probabilities to each of the events, identify contrary events and calculate their probabilities. All these types of tasks entirely within the scope of a probabilistic modelling activities as defined in a previous section.

For the second type of task "calculate the probability of being sick knowing that the test is positive", the associated subtypes of tasks are as follows: interpret the question in terms of probability, calculate the corresponding conditional probability.

The task is complex, as there are many subtypes of tasks for each of the task types identified above. In this exercise, identifying all these subtypes of tasks and organizing their reasoning are the student's responsibility. Here the combination of AT and ATD allows me to see how the complexity of the task impacts the praxeological organization and in particular the complexity of the technique to be implemented.

The techniques associated with these types of tasks are as follows: identify the events involved, associate the numerical data of the statement with events, calculate the missing probabilities (use of the probability of the complementary), identify the probability that must be calculated in order to respond, calculate this probability (for this it is necessary to calculate an intersection and use the Bayes formula); finally, interpret the result. The technique of representing the situation by a probability tree is a technique expected in secondary school but is no longer part of the praxeology at university. Here is a representation of the situation by a tree:

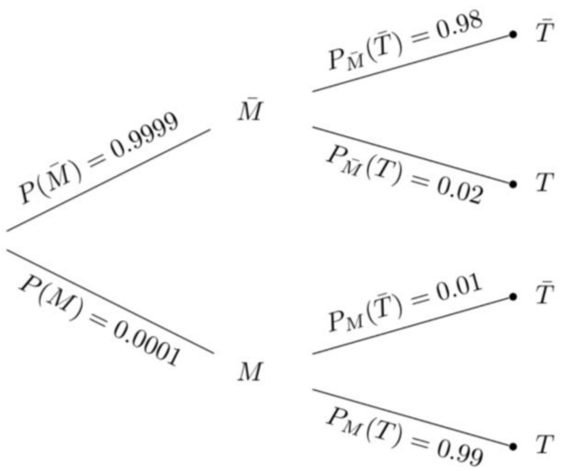

Table 1: probability tree

Here are some technological elements that justify the choice of the probability to be calculated, $P_T(M)$. What I am interested in here is whether the test is effective from the point of view of a caregiver. A patient comes for a test to find out if he/ she is sick or not. If his/her test is positive (respectively negative), it is important to know if he is really sick (respectively not sick), i.e. if the test is reliable. The test is reliable if the probability, knowing that the test is positive, that a person is indeed sick, is close to 1. The goal is therefore to calculate $P_T(M)$, for this it is necessary to calculate an intersection and use the Bayes formula, their uses can be justified by the associated theory or by the use of a probability tree. These technological elements go beyond mathematics, the fact of calculating $P_T(M)$ is entirely at the students' expense and is based on technological elements linked with a socio-medical context.

The context of the exercise, which is the study of the effectiveness of tests on sick and non-ill populations, is rather a context that is familiar to students. Indeed, there are many exercises in this context in secondary school textbooks. In this exercise they have to interpret the question by proposing the probability of an event themselves, then interpreting the numerical result obtained. In secondary school textbooks it is rather common to have in the first question "calculate the probability of such an event" and in the second question "interpret the result". Here, therefore, calculating $P_T(M)$ requires an important initiative from the student.

The techniques to be used by the students are all based on knowledge being acquired and already applied in other situations encountered in secondary school. On the other hand, recognizing this complex praxeological structure and organizing oneself accordingly is entirely at the student's expense because there is no intermediate subquestion associated with each of the subtypes of tasks described above. The very strong modelling activity left to the student is not something usual for them, so I expect to find difficulties for this exercise in the *a posteriori* analysis.

A posteriori analysis

I present here the *a posteriori* analysis of the second exercise of the test.

This exercise was tackled by 91% (49 students) of the students who answered the test (54 students). Both Grade 12 students and biology students encountered a lot of difficulties and proposed erroneous solutions. I expected these results from my a priori analysis above. The diversity of responses is very high, the probability most often calculated by Grade 12 students is P(M and T) (5 responses); there is no dominant answer among biology students. An example is presented in Figure 2.

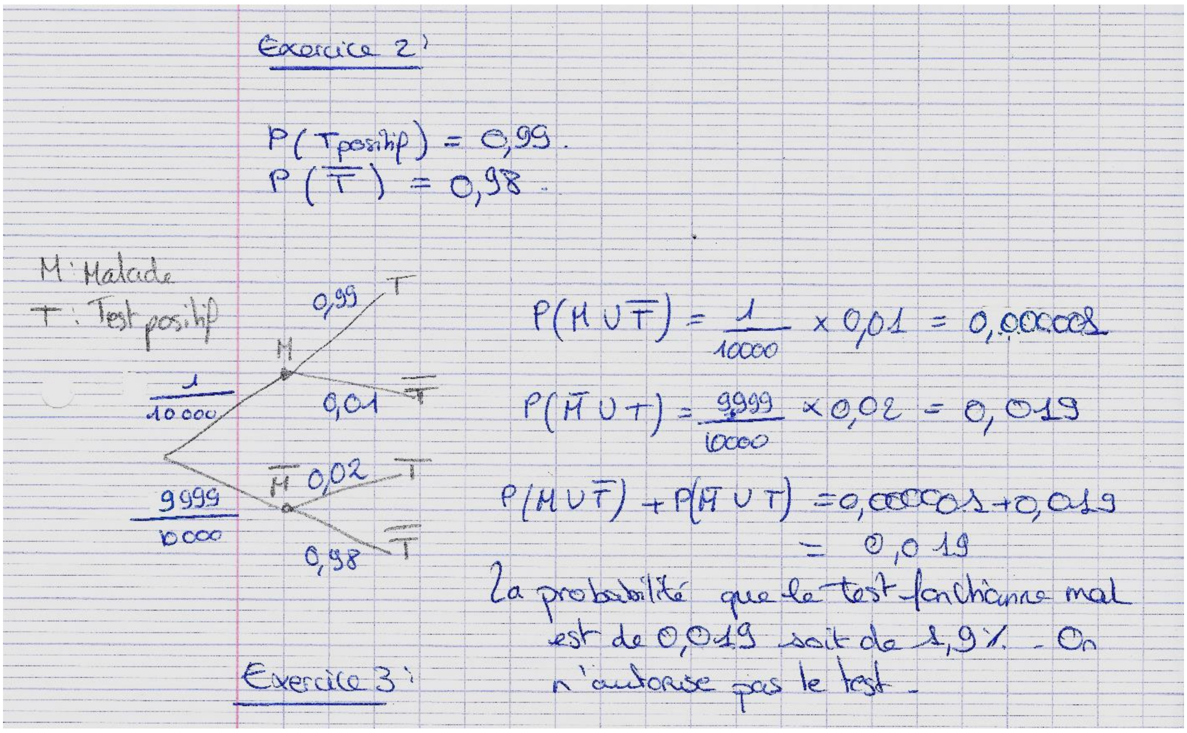

Figure 2: Example of a student's production. "The probability that the test goes wrong is of 0.019, meaning 19%. We do not authorize the test"

Even if the students did not answer correctly many of them have correctly identified the events at stake (32 of them, or about 65%). A slightly fewer amount of them represent the situation with a probability tree (26 of them, or about 53%). On the other hand, all the students who chose to represent the situation by a tree correctly identified the numerical values of the statement with the corresponding events and correctly calculated the probabilities of the complementary events. Considering here that correctly modelling the situation means identifying all the events at stake and associating their probabilities to them; therefore, in this exercise, only half of the students who responded did a correct modelling of the situation.

Concerning the interpretation of the question and the answer given by the students, 41 of them (or about 84%) have formulated a response in natural language. Of these, 17 relied on their previous probability calculations to answer. In contrast, 8 students answered the question in natural language based solely on their representation of the situation through a probability tree. Sixteen of them (about 33%), 10 Grade 12 students and 6 biology students answered the question without having previously made any probability calculations or probabilistic modelling (like probability tree). These students were unable to identify in the task prescribed to them the different subtypes of tasks to be performed. Here is an example of such a response:

> "No, because the margin of error is enormous for a population of 10,000. Out of 10000 there could be 200 people who are reported as sick when not at all. This is related to the 98%. If the disease was 1 in 100 people, it would have been more interesting."

More specifically, with regard to probabilistic modelling activities, students highly use probability trees in this exercise while there is no indication anywhere in the statement that a tree could be used to answer as can be seen on the student copy excerpt (see "table2"). This is an important observation which is linked with a secondary-tertiary transition issue.

Indeed, since secondary school, students have become accustomed to use this type of representation. The construction and use of a probability tree are skills that are widely developed in the official mathematics curriculum of the Grade 12 class (Ministère de l'Education Nationale, 2011).

Probability trees also play a very important role in the Grade 12 course I observed, the outline of the course handout distributed by the teacher for the chapter "Conditional Probabilities", consists of three main parts: "Conditional Probabilities", "Probability Trees" and "Independence of Two Events". The rules for building the tree are detailed with technological elements, here is an example: "Rule 3 (total probability formula): the probability of an event is equal to the sum of the probabilities of each of the paths leading to it". Moreover, the official Grade 12 mathematics curriculum states that: "a properly constructed probability tree is a proof", which is no longer conceivable at the university.

CONCLUSION

I have seen through the analysis of students' answers of this exercise that probabilistic modelling is an important issue in the secondary-tertiary transition.

First, it should be noted that students seem to have appropriated the use of probability trees. Indeed, through the analysis of this exercise and the two other exercises of the test that I have not developed here, I saw that the use of trees allowed students to respond better afterwards. The non-use of probability trees at university in appropriate situations could therefore prove to be one of the causes of the difficulty of students in the secondary-tertiary transition.

Through this example of exercise, I have seen that the recognition of the task by students as part of a succession of types of tasks is complex and not always immediate.

Here are the two main results that I can draw from this analysis. First, during the secondary-tertiary transition, the greater the complexity of linking the task to a type of task, the more difficult it is for students.

Second, the probabilistic interpretation of natural language statements poses difficulties for students, in particular when it comes to identifying the events at stake.

In my future research, I will design and evaluate a teaching aiming to overcome these difficulties.

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Appendix: English version of the test given to the students

Test

Exercise 1 :
Let be X and Y two individuals whose lifetimes are independent and are such that
$\mathbb{P}(X \text{ still living 9 years}) = \frac{2}{5}$, $\mathbb{P}(Y \text{ still living 9 years}) = \frac{3}{5}$. Calculate the probabilities that :

1. X and Y are still living 9 years;
2. One of the two at least still lives 9 years;
3. X only lives another 9 years;
4. X lives another 9 years knowing that at least one of the 2 will live another 9 years.

Exercise 2 :
You are the director of staff of the Minister of Health. A disease is present in the population, in the proportion of one sick person out of 10,000. The manager of a major pharmaceutical company comes to you to tell you about his new screening test : if a person is sick, the test is positive at 99%; if a person is not sick, the test is negative at 98%. Do you authorize the marketing of this test?

Exercise 3 :
The marketing department of a telephone store conducted a study of its customers' behaviour. He observed that the latter is composed of 42% of women. 35% of women who enter the store make a purchase while this proportion is 55% for men. A person enters the store. We note the events :

- F : "the person is a woman".
- R : "the person leaves without buying anything".

Throughout the exercise, give values approximating the results to the thousandth.

1. Build a weighted tree illustrating the situation.
2. Calculate the probability that the person who entered the store is a woman and leaves without buying anything.
3. Show that $\mathbb{P}(R) = 0.534$.
4. Are the F and R events independent?

Analysing mathematical and didactic praxeologies in an engineering course: the case of Strength of Materials

Alejandro S. González-Martín and Gisela Hernandes-Gomes

Université de Montréal, Département de Didactique, Canada

a.gonzalez-martin@umontreal.ca; gisela.hernandes.gomes@umontreal.ca

This paper provides an overview of the various stages of our research, which seeks to better understand the use of calculus in university engineering courses. We first illustrate the use of integrals in a classic task (sketching a bending moment diagram) in a Strength of Materials course, showing that although integrals appear in the theoretical block, they are not explicitly used in the practical block. Our analysis of the course's reference book shows that this situation is replicated for all notions defined as integrals. This leads us to seek further information by examining teaching practices and by considering mathematical and didactic praxeologies. Our preliminary results indicate that, although integrals are present in the knowledge block of the course, their presence in the practical block and in the evaluation is significantly weaker.

Keywords: Teaching and learning of mathematics for engineers, teachers' and students' practices at university level, textbooks, Anthropological Theory of the Didactic, integrals.

INTRODUCTION

Recent research in mathematics education and in engineering education has shown that university engineering students encounter a number of difficulties in mathematics courses in their early years of study, resulting in high failure rates and dropouts (Ellis, Kelton, & Rasmussen, 2014; Rooch, Junker, Härterich, & Hackl, 2016). Neubert, Khavanin, Worley, and Kaabouch (2014) state that efforts should be made to increase student retention in engineering courses in the first years of a programme (particularly in first-year non-engineering courses), as this is when most dropouts happen. It is important to note that, in many engineering programmes around the world, mathematics and physics courses are generally taught in the first years, with specific engineering courses appearing in later semesters. This classic structure separates 'basic' and professional disciplines, and can aggravate students' difficulties, reducing their ability to make links between concepts and negatively affecting the teaching-learning process (Perdigones, Gallego, Garcia, Fernandez, Pérez-Martín, & Cerro, 2014). For instance, Loch and Lamborn (2016) report that in engineering programmes, first-year mathematics courses often focus "on mathematical concepts and understanding rather than applications" (p. 30). Authors such as Flegg, Mallet, and Lupton (2011) highlight this lack of connection between mathematics content and engineering content among engineering students, which can lead the latter to view their mathematical courses as irrelevant. We believe this situation may contribute to students' lack of interest and motivation in their mathematics courses. Faced with these problems, the mathematics and engineering education communities have been engaged in research and discussion, not only on which topics engineering students should study, but also on what kind of mathematical knowledge and skills are needed by engineers, with an eye towards improving engineering students' mathematics training (Bingolbali, Monaghan, & Roper, 2007).

Among the pioneering works on this topic, Noss (2002) identified that structural engineers do not "'use mathematics' of any sophistication in their professional careers" (p. 54). Providing testimonies from engineers to support his findings ("an awful lot of the mathematics they were taught, I won't say learnt, doesn't surface again," p. 54), Noss suggests that university mathematics content often goes undetected in real-world engineering practices, although it underlies basic, frequently used operations. For instance, with respect to civil engineers, Kent and Noss (2003) conclude that in "95% of the work [they] do, the mathematics is basic" (p. 18) and that many of them do not even use calculus. In particular, the authors suggest that although calculus can play an important role in engineers' education by helping them grasp basic engineering principles, it may rarely be used explicitly in the workplace. Kent and Noss call for further research on engineers' use of mathematics (calculus in particular) — a pressing concern given the high failure rates in university calculus courses.

In line with the previous, and in order to identify mathematical skills used in engineering, in recent years we have investigated how single-valued integrals are used in engineering courses. We seek to reveal potential ruptures between how notions are first introduced and used in calculus, and how they are later applied in introductory engineering courses. Initially, we analysed how these notions are presented in engineering textbooks, working under the assumption that many university teachers plan their teaching using textbooks as an important resource (e.g., Mesa & Griffiths, 2012). At previous conferences, we presented our results regarding the use of integrals to define first moments of an area (Q), moments of inertia (I), polar moments of inertia (J), bending moments (M), and centroids (C) in a Strength of Materials course for Civil Engineering (González-Martín & Hernandes-Gomes, 2017, 2018, 2019). Our analyses show that, although these notions are defined using integrals, the practices employed either use very basic calculus techniques or eschew them completely (more details are provided in the Data Analysis section below). This echoes Noss' (2002) and Kent's and Noss' (2003) results. Having analysed the entire reference textbook used for this particular Strength of Materials course, in this paper we present a summary of our results, as well as some initial results concerning the teaching of this course and the effective use of integrals, based on interviews with an engineering teacher.

We note that the preliminary results from our analyses of how integrals are used in relation to bending moments, first moments of inertia, and centroids are consistent with Faulkner's (2018) results. Faulkner analysed the entire coursework of a first-year engineering course (Statics), showing that only seven out of the 84 exercises (≈8%) required some explicit knowledge of calculus, with five of these seven exercises appearing in the same chapter. This means that a student with no knowledge of calculus content could still achieve a grade of A- in this course. We are not aware of other research investigating engineering courses in their entirety, and our work aims to fill this gap. Moreover, although some existing research does involve analysis of course material or interviews with engineers, we are not aware of any work of this nature that also examines the classroom practices of engineering teachers or the latter's use of calculus content in their courses. The existing research generally focuses either on the calculus courses that serve as prerequisites for engineering courses or on workplace practices; however, what happens in the middle (the teaching of professional courses) is usually overlooked. Therefore, our research programme seeks to answer the question: how is calculus content used in engineering courses, both in course materials and in teachers' practices?

THEORETICAL FRAMEWORK

As we are interested in variations in practices between mathematics activity and engineering activity, our research uses tools from the anthropological theory of the didactic (ATD – Chevallard, 1999), which considers human activities to be institutionally situated. A key element of ATD is the notion of praxeology, which allows for the modelling of human activity. A praxeology is formed by a quadruplet [T/τ/θ/Θ] consisting of a type of task T to perform, a technique τ which allows the task to be completed, a rationale (technology) θ that explains and justifies the technique, and a theory Θ that includes the discourse. The first two elements [T/τ] are the practical block (or know-how), whereas the second two [θ/Θ] form the knowledge block that describes, explains, and justifies what is done. Although ATD distinguishes between different types of praxeology, due to space limitations we only present our analyses in terms of tasks.

Moreover, teaching practices can also be modelled using praxeologies. In the case of didactic praxeologies, Chevallard (1999) identifies six moments: 1) the first encounter with the content to learn; 2) the exploration of the type of tasks and the elaboration of a technique relative to these tasks; 3) the constitution of the technological/theoretical environment relative to these tasks; 4) the technical work, which at the same time aims to improve the technique making it more powerful and reliable, and to develop the mastery of its use; 5) the institutionalisation; 6) the evaluation of what was learned. We use these moments in our analysis of the interview with the teacher.

METHODOLOGY

Our research project involved the collaboration of an engineering teacher who holds bachelor's and master's degrees in Civil Engineering and who has extensive experience in structural systems and reinforced concrete. This teacher works in a Brazilian university, where Strength of Materials (SM) for Civil Engineering is taught as a second-year course in the engineering programme (part I, SM-I, in the third semester, and part II, SM-II, in the fourth semester). This course is taken once students have completed differential and integral calculus courses in their first year. In Brazilian universities, SM is mandatory in engineering: it is part of basic engineering training and serves as a prerequisite for advanced engineering courses such as Stability of Construction, Concrete Structures and Prestressed concrete. For the data presented in this paper, the methodology was applied in three phases:

- First, we analysed the general structure of the content related to integrals in first-year calculus courses at the engineering teacher's university, using a hard copy and an electronic version of the course reference book (Stewart, 2012). We identified the main tasks concerning integrals proposed to students, the techniques used to solve them and the rationales (technology) employed (see González-Martín & Hernandes-Gomes, 2017, 2018).
- Second, we analysed a classic international SM textbook used at the same university (Beer, Johnston, DeWolf, & Mazurek, 2012), also examining the electronic and hard copy versions. For this book, we identified all notions that are defined as an integral, using keyword searches in the electronic version, and pinpointed all appearances of the symbol " $\int$ " in the hard copy version. For all content defined using a single-valued integral, we identified the tasks involving the latter, as well as the techniques and explanations present. For examples about bending moments and first moments of an area, see González-Martín & Hernandes-Gomes (2017, 2018, 2019).
- Third, we interviewed the engineering teacher on four occasions (I1: March 2016, I2: November 2016, I3: August 2019, I4: September 2019), and had access to his lecture notes. Interviews were conducted in Portuguese; they were audio-recorded and transcribed, with excerpts translated into English. During these interviews, we discussed the specific case of bending moments and how it is presented to students, as well as the tasks and techniques explained. We also discussed the course overall and the use of integrals and calculus content: how frequently this content is used to complete the various tasks presented in the course, and how much this content factors into the students' evaluation.

Due to space constraints, this paper provides a summary of the main results from our textbook analyses on bending moments, followed by an overview of the use of integrals throughout the entire book. We end by providing data from the interviews concerning the use of integrals throughout the entire course.

DATA ANALYSIS AND DISCUSSION

Phases 1 and 2: Calculus and bending moments

At this university, single-variable courses follow the structure of Stewart (2012). The content concerning integrals is organised into two blocks (see González-Martín & Hernandes-Gomes, 2017, 2018). The first block introduces a repertoire of techniques for calculating indefinite integrals (from immediate integration to more complex cases), with theoretical elements mostly absent. The second block introduces Riemann sums to formally define integrals and interpret them as areas, and leads to the Fundamental Theorem of Calculus and the calculation of definite integrals using

Barrow's rule; this leads to some applications of the integral (area, volume…). Many of the techniques used here are derived from the first block.

Regarding bending moments (introduced in SM-I), generally, loads are perpendicular to the axis of a beam (*transverse loading*). These transverse loads can be concentrated, distributed, or both. When beams are subjected to transverse loads, any given section of the beam experiences two internal forces: a shear force (V) and a bending couple (M). In the case of distributed forces, V is defined as the integral of the load (w) and M is defined as the integral of V. The latter creates normal stresses in the cross section, whereas V creates shearing stresses. Therefore, one of the main factors to consider in designing a beam for a given loading condition is the location and maximum value of the normal stress (M) in the beam. For students to determine this location, techniques for sketching bending-moment diagrams are introduced. These techniques produce diagrams such as the one in Figure 1.

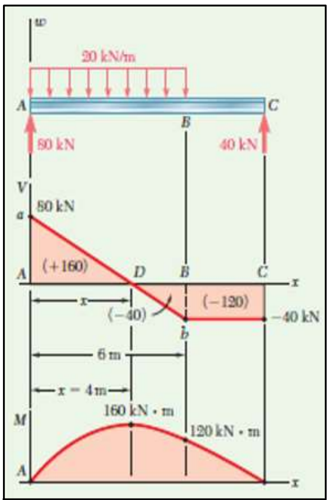

Figure 1: Task concerning bending moment diagrams in Beer et al. (2012, p. 364)

It is important to note that although these notions are defined as integrals, the actual technique does not rely on content or techniques derived from the calculus course (the graph of M in the lower portion of Figure 1 is the graph of the antiderivative of V, which itself is shown in the middle portion). The technique consists of obtaining values for specific points using basic formulae and calculations and then connecting them with a free-hand sketch (see points *A, B, C* at the top of Figure 1—distribution of the load—and how these points determine other points in the two graphs below them). After an initial example, the rationale (technology) for this technique is given: "Note that the load curve is a horizontal straight line, the shear curve an oblique straight line, and the bending-moment curve a parabola. If the load curve had been an oblique straight line (first degree), the shear curve would have been a parabola (second degree), and the bending-moment curve a cubic (third degree). The shear and bending-moment curves are always one and two degrees higher than the load curve, respectively. With this in mind, the shear and bending-moment diagrams can be drawn without actually determining the functions $V(x)$ and $M(x)$" (Beer et al., 2012, p. 362). Although this rationale is based on content from calculus, we note that, as given, this rationale can be used to apply the technique without referring to integrals.

Our results concerning the introduction of bending moments (for more details, see González-Martín & Hernandes-Gomes, 2017) seem to confirm Noss' (2002) and Kent's and Noss' (2003) findings: although calculus underlies the technique used to sketch the above diagrams, the technique itself consists of basic calculations and free-hand sketches. To investigate this phenomenon further, we analysed the entire reference book.

Phases 1 and 2: Calculus and SM

According to the university's curricular guidelines, the content of SM-I, focuses mainly on analysing internal forces, sketching and interpreting their diagrams and studying pieces subjected to flexion and stresses. In SM-II, the content focuses mainly on deepening the study of pieces subjected to flexion (studying different types of flexion), determining tensions and sketching their diagrams and studying torsion, deflexion, and rotation in beams. Both courses follow the structure of Beer et al. (2012), in which each chapter is divided into different sections: theory, concept applications (CA), sample problems (SP), and several homework assignments. Both CAs and SPs appear in the theory sections, focusing on specific topics and helping to illustrate the application of specific content. In our first analyses, we focused on the topics of first moment of an area (Q), moment of inertia (I), polar moment of inertia (J), bending moment (M) and centroid (C). Our results showed that integrals are mostly used in the theoretical sections, to introduce and define these notions, as well as to deduce certain properties (see González-Martín & Hernandes-Gomes, 2019). Figure 2 shows that although these notions are defined as integrals, they are involved in praxeologies where, for the most part, students can use the tables and formulae provided to find the values needed to solve tasks. The actual technique does not rely on using integrals, and it is only if we seek to find the rationale for the technique (technology) that integrals make an appearance. However, as illustrated above with bending moments, explicit justifications of these techniques, when they occur, rely on a professional discourse and are not (at least for the student) explicitly related to explanations and properties as they are taught in calculus courses.

	Theory			Concept Application (CA)			Sample Problem (SP)		
	A	*B*	*C*	*A*	*B*	*C*	*A*	*B*	*C*
First Moment	12	7	0	10	0	1	7	0	0
Moment of Inertia	12	5	0	6	0	3	9	0	1
Polar Moment of Inertia	5	2	1	3	0	1	1	0	0
Bending Moment	33	15	6	15	0	13	24	0	11
Centroid	21	2	0	11	0	0	12	0	0

** Column A: the notion appears without any explicit connection to integrals.*
Column B: the notion appears connected to the integral symbol, with no calculation.
Column C: the notion appears and an antiderivative is calculated.

Figure 2: Frequency of integrals in theory, CA and SP.

The scenario is replicated throughout the book with all notions defined as integrals. Moreover, in the few cases where an integral needs to be calculated in a CA or an SP, the functions involved are constants, x^n, $(x-a)^n$, $1/x$, $\sin(ax)$, or $cos(ax)$. These results, which are coherent with Faulkner's (2018) concerning a Statics course, prompted us to interview our teacher about the actual use of integrals in his course and to study the level of similarity between the mathematical praxeologies in his teaching practices and those in the reference book.

Phase 3: Interview and lecture notes of the engineering teacher

In this section we provide some details about the mathematical and didactic praxeologies that are present in the teaching of SM, based on our interviews with the engineering teacher. The latter agrees that the course employs a number of basic mathematical tools:

T-I1: Thinking about Strength of Materials, we use, for example, proportionality, the Pythagorean Theorem, and basic trigonometry.

Newton's binomial is also used in the course. Although these notions sometimes appear in praxeologies where deductions are necessary to arrive at a needed formula, and while some complexifications are used to study certain phenomena, the teacher adds that "we do not ask these deductions in the exams." Regarding the use of calculus in the course, he confirmed that it appears in the introduction of topics. For instance:

T-I1 At the beginning of Strength of Materials [I], we start studying distributed loads. To deduce the resultant, which we call *mechanically equivalent force*, I will use a concept of Calculus: the infinitesimal. Then you calculate the value in infinitesimal chunks, and the resultant of all this is the integral of [$w(x)dx$]. […] So, I'm using this and all... And again, it's a little bit of Calculus.

The teacher confirms that integrals appear in the course when covering the topic of internal forces in a beam, and when studying the relationships between load (w), shear force (V), and bending moment (M), which are used to sketch bending-moment diagrams. He says he highlights the use of integrals in the theoretical part, as in the textbook. However, he confirms that integrals are set aside during the practical part:

T-I4: This way of doing things [deducing forces using integrals] is set aside when we start sketching. At each point where loads change, we can determine the values of the [shear force and bending moment]. And if we know these values in the extremities of each section [we can perform the task] […] On this beam, you have a uniformly distributed load; if I know the shear force on the left and right [end] of the section, then we know that dV/dx = [w] constant. What thing, when derived, gives a constant? A linear function. Then, if I know that in the extremities [the values] are 40 and -60, how does it vary? Linearly, I know it is linear. So, these two points define a straight line. Then, I can start sketching the diagram directly. I don't need to find the equation [of the straight line]. […] From here, for the [bending] moment, I know its value in extremities and I know the load is constant, the shear [force] is linear, [then] the [bending] moment is parabolic, a quadratic function. […] So, we get to sketch that directly, too.

The teacher confirmed that the calculation of integrals is not necessary throughout this entire section of the course. Moreover, although he makes a connection to derivatives in explaining the technique, the teacher provides students with the rationale from the book (which offers no explicit connection to integrals or derivatives), explaining that this rationale is the one they need to use. Nevertheless, he states that knowledge of integrals is useful "as training, but for many things I don't need to use the integral, although I need to understand it" (I4). We see that, regarding this content, the explicit use of integrals seems to appear in the moments, *exploration of the type of tasks* and *constitution of the technological/theoretical environment*; however, integrals disappear during the *technical work* and its *institutionalisation*. This led us to question the extent to which integrals are used explicitly in the *evaluation* moment.

As taught by this teacher, the SM-I course has two midterm exams (M1 and M2) and one final exam (FE), totalling 10 points each. To pass the course, the condition ((M1+M2)/2 + FE)/2 ≥ 7.5 is necessary, which means that (M1+M2+2FE) ≥ 30. M1 contains a question (two out of 10 points) about bending moments and shear forces, in which students must provide a solution recalling the theoretical explanation using integrals. M2 contains a question (worth approximately six points) on the sketching of bending moment diagrams, which can be solved using the given technique without resorting to integrals. Integrals are not explicitly used in the FE. Therefore, in this case, only 5% of the final mark relates to the explicit use of integrals, which is coherent with Faulkner's (2018) results concerning a Statics course. We therefore see that although integrals are present in some moments of the teacher's didactic praxeologies, their presence in the *technical work* and *evaluation* moments is weak. It seems that this teacher's praxeologies are similar to those present in the book.

FINAL CONSIDERATIONS

As stated in the Introduction, more research is needed to determine the actual mathematical knowledge and skills that are applied in a typical engineering workplace. Pioneering researchers (Kent & Noss, 2003; Noss, 2002) have already suggested that most engineers just need 'basic' mathematics, and that university mathematics content is "transformed into something else" (Noss, 2002, p. 54). Recent research analysing the content of engineering courses seems to confirm this. Faulkner's (2018) analysis of a Statics course revealed that explicit use of calculus is only necessary in 8% of the course. There is a paucity of works analysing and assessing actual (classroom and professional) engineering practices, which would help to clarify how much (advanced) mathematical content should actually be necessary to pass courses.

Using tools from ATD (Chevallard, 1999), we took a holistic approach to analysing an engineering course, Strength of Materials: first, we examined the way the course is organised around specific topics; second, we examined the course reference book in a global way; third, we looked at the teaching practices and mathematical and didactic praxeologies activated during teaching. Due to space limitations, we only provide some data on the general aspects of the course, concerning integrals; however, the results of our three stages of analysis seem consistent. Although integrals are used in the course, primarily in the knowledge block, their explicit use is less necessary in the practical block: most techniques (although implicitly based on the use of integrals) rely on basic calculations, the use of tables or given formulae, and geometric considerations. In addition, knowledge of integrals does not factor much at all in the students' final assessment. We believe that this disconnection between practices in calculus courses and in professional courses may reinforce students' views of their mathematical courses as being irrelevant to their training (Flegg et al., 2011).

We highlight the fact that our teacher states integrals are needed "for anything that goes beyond the trivial", but in the workplace, practices usually follow standardised rules. This seems consistent with the results of Kent and Noss (2003), who suggest that employers look for balanced teams and that "it may be more cost-effective to contract out unusually complex analysis to a specialist design consultancy […] whereas civil engineering consultancies need more, but still only perhaps 10-20% need to have specialist skills in analysis" (p. 18). This is also coherent with recent results from Quéré (2019), who, in a survey of 261 French engineers, found that only 129 (49.9%) used (university) mathematics in their workplace. Of these 129 respondents, only 43% (21.24% of the overall sample) said they used calculus content in their daily practice.

Finally, our results suggest that, regarding the SM course, although content concerning integrals is necessary, it is rather the knowledge block aspects that seem essential for gaining a better understanding of engineering techniques. We intend to deepen our analyses of the interviews with the teacher to gain further insight into this phenomenon. We also intend to pursue analysis of other engineering courses. Both avenues of research will be the source of future publications.

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The professional identity of teacher-researchers in mathematics

Stéphanie Bridoux[1], Martine De Vleeschouwer[2], Nicolas Genier-Boley[3], Rita Khanfour-Armalé[4], Nathalie Lebrun[5], Zoé Mesnil[6], and Céline Nihoul[7]

[1]UMONS (LDAR, EA 4434)[1], Belgique, stephanie.bridoux@umons.ac.be; [2]IRDENa, UNamur, Belgique, martine.devleeschouwer@math.unamur.be; [3]Université Rouen Normandie (LDAR, EA 4434), France, nicolas.grenier-boley@univ-rouen.fr; CY Cergy Paris Université (LDAR, EA 4434), Inspé de Versailles, France, rita.khanfour-armale@u-cergy.fr; [5]Université Lille 1 (LDAR, EA 4434), France, nathalie.lebrun@univ-lille.fr; [6]Université de Paris (LDAR, EA 4434), France, zoe.mesnil@u-paris.fr; [7]UMONS (LDAR, EA 4434), Belgique, celine.nihoul@umons.ac.be

We are interested in the professional identity of teacher-researchers and in particular the influence of their discipline on their teaching practices. In this article, we present a multidisciplinary research based on this issue and report on its results for teacher-researchers in mathematics. First, we introduce the concept of professional identity in order to clarify our research problem. Then, we present our methodology and illustrate our results with excerpts from interviews with teacher-researchers from various institutions in France and Belgium. We compare these results with those obtained with teacher-researchers in physics and suggest some perspectives.

Keywords: teachers' and students' practices at university level, preparation and training of university mathematics teachers, teacher-researchers, professional identity.

INTRODUCTION

Many French universities have created professional development structures for higher-education teachers with a perspective of "pedagogical transformation" in order to respond, in particular, to the diversity of the student population (Endrizzi, 2011). While research has focused primarily on the pedagogy of teaching practices (Annoot & Fave-Bonnet, 2004), this issue has seldom been addressed through an approach based on the discipline of teacher-researchers (Henkel, 2004; Neumann, 2001), even though several authors stress the need for it (Becher, 1994). In this context, we have conducted a research based on three academic disciplines (Mathematics, Physics, Chemistry) (Bridoux et al., 2019): in this contribution, we report on the results of this research for teacher-researchers (TRs) in mathematics. Our objective is to better know this community of teacher-researchers by highlighting several aspects of their professional identity, be it transversal aspects or aspects related to mathematics.

[1] Laboratoire de Didactique André Revuz (LDAR, EA 4434), Université Artois, CY Cergy Paris Université, Université de Paris, Université Paris Est Créteil, Université Rouen Normandie.

This could be a first step for building teacher training courses that take into account the values and qualities they emphasize and specific elements of knowledge they enhance. By this approach, we hope to encourage the transformation of teaching practices and to understand teacher-researchers' expectations about students or the secondary-tertiary transition.

THEORICAL FRAMEWORK AND RESEARCH QUESTIONS

The concept of professional identity is complex and appears in the literature under different approaches. From the standpoint of pedagogical practices, the higher-education teacher is a researcher before being a teacher (Fave-Bonnet, 1999; Musselin, 2008; van Lankveld, Schoonenboom, Volman, Croiset, & Beishuizen, 2017). In mathematics education, the concept of identity of mathematics' teachers is defined in various ways and has been extensively studied (Graven & Heyd-Metzuyanim, 2019). From the point of view of sociology of work, professional identity is defined as "a set of specific elements of professional representations, specifically activated according to the interaction situation and to respond to an aim of identification/differentiation with societal or professional groups" (Blin, 1997). This relation with the community is found in Cattonar (2001) who defines the professional identity as "the characteristics that identify him as a teacher and that the teacher shares with other teachers, which he shares in common with other teachers because he belongs to the same professional group". In this context, de Hosson, Décamp, Morand and Robert (2015) have retained several dimensions of professional identity understood as "the way in which an individual teacher is defined in relation to his or her professional teaching practice" and have highlighted tensions among TRs in physics. The dimensions retained in this work are related to the profession: norms, qualities and skills required, values (Dubar, 1996). Other studies show that teachers identify strongly with their research disciplines (Henkel, 2004). In the latter case, tensions appear, with TRs valuing the research. Drucker-Godard, Fouque, Gollety and Le Flanchec (2013) have shown that French TRs feel "a conflict between values they initially approve (freedom, independence, autonomy, public service) and new values emerging from recent reforms of the university system (scientific productivity, effectiveness, efficiency, individualization of the career, fairness and unequal treatment and esteem" (p. 19). The perception of this conflict seems to be common to many TRs, regardless of their research and teaching disciplines. It is reinforced by the results of de Hosson et al. (2015) who find that the professional identity of the physics TRs [...] interviewed appears to be strongly marked by tensions [...] that sometimes appear under the form *I know that this should be done and yet I do the opposite*. However, we may wonder if there is a discipline imprint on values, and thus on these tensions, whether at the level of public values – the epistemology of each discipline leads to potentially different beliefs and organizations –, or at the level of individual values that can influence each TR in the choice of his/her discipline.

Van Lankveld et al. (2017) also mention that teacher training is perceived as positive in terms of teaching capacity through peer-to-peer exchanges, making them more credible in their institution. This socialization effect is all the more pronounced among teachers who have not received initial teacher education (Goodson & Cole, 1994). However, professional identity can be negatively affected when teacher education is perceived as a supervisory mechanism (van Lankveld et al., 2017).

The aim of our research is to study the following research question for TRs in mathematics: what is the discipline's imprint on teaching practices at the university? We have two main motivations for conducting this study. On the one hand, we would like to understand the way in which the interviewed TRs perceive training courses in "university pedagogy". On the other hand, we would like to know their expectations towards students in terms of difficulties they mention. These motivations are thus taken into account in both the design of the interview protocol and the analysis of the interviews. Our methodology will also allow us to test the relevance of the following assumptions: a TR is a particular teacher because of its multiple missions, the research discipline is a marker of its professional identity, or even of the institution in which he works.

METHODOLOGY

Our data are taken from the anonymous transcripts of 12 semi-directive individual interviews, lasting between 30 and 90 minutes. To see if the institution or the length of service has any influence with respect to our question, the volunteer interviewees come from four universities[2] and have teaching experiences of varying length (3 to 40 years). The interview protocol was initially constructed in such a way that we can identify the norms, qualities and values assigned by the TR to his teaching practices (de Hosson et al., 2015), which give a pertinent access to his teaching professional identity by hypothesis (Cattonar, 2001; Dubar, 1996). We have amended the initial interview protocol to take into account the teaching discipline, in particular mathematics. In the table below, we give examples of questions that are related to each of these dimensions (norms, qualities, values, discipline).

Dimension	Questions
Norms	What is the objective of a course? How do you ensure that the objective is achieved? What is the purpose of an evaluation?
Qualities	What do you find difficult in your job as a teacher? Do you feel the need to be trained as a teacher?
Values	What do you enjoy most about being a TR? What would you be willing to

[2] Université Paris 13, Université Rouen Normandie (France) , UMONS, UNamur (Belgium).

	delegate/not delegate?
Discipline	What are the sources of difficulty for students? What is a good course? What is a good teacher?

Table 1 - Some examples of questions related to study dimensions

During the interviews, we discussed the following topics: organization of teaching, innovative practices, difficulties and evaluation of their students, TRs' training, the profession of TR, didactic questions (objective and content of a course). To analyze the interviews, we have conducted an empirical categorization by identifying verbatim excerpts that seem to be relevant for our research question. We consider a verbatim excerpt as a unit of meaning (Bardin, 1977), which may fall within several significant categories related to the discipline of the TR interviewed and to the attributes of its research activity. We have thus tried to spot regularities and variabilities that are intra or inter-disciplinary. More specifically, we have looked for verbatim statements reflecting the influence of the research profession on the teaching profession, student assessment or teacher training. We have thereby tried to identify elements that refer to a researcher posture in teaching practices (creativity, freedom, collegiality, peer review and peer learning) or to the transposition of research methodological elements into the activities proposed to students (problem solving, reflexivity, student presentations, group work, etc.). We have also identified tension elements (de Hosson et al., 2015) often stemming from contextual factors (institution, relationship with students, teacher training).

RESULTS

Different aspects emerge that allow us to provide various types of answers to our research question.

Mathematical difficulties and transversal competences

During the interviews, TRs were asked to identify students' difficulties when entering university. Whereas the answers to this question are varied, we can spot regularities that concern either mathematical difficulties or mathematical conceptions and skills that are related to their researcher position.

Concerning mathematical difficulties, the majority of TRs point to gaps with specific concepts taught in secondary school: continuity, derivation, tangent to a curve, lack of logical knowledge.... Thus, students have "learned to perform calculations" and do not "know how to pass from an equation to a straight line or from a curve to an equation". The fact that students misunderstand, forget or have compartmentalized knowledge is mentioned by some TRs, with doubts concerning the origin of their difficulties.

M7: And so we cannot know whether students' difficulties are due to the fact that they have not seen or assimilated these notions, or that they have assimilated and forgotten them.

Most TRs also highlight examples of high school mathematical practices that they see as blocking factors at the university level: favoring "recipes" rather than understanding the concepts, having a local understanding of courses rather than a global vision, having problems with formalizing an intuition. Many TRs also speak of a particular difficulty in mathematics related to the production and understanding of proofs. This is both an epistemological (understanding what a proof is) and mathematical issue (understanding the different statements inside a proof and the relations between them).

M5: I think that, at least unconsciously, we think that the students know what a proof is, when to say that something is true or false, etc. and they didn't actually learn that in secondary school.

In particular, TRs deplore a difficulty for students to enter into an abstraction and formalization process, and to conceive the relations between the different mathematical objects. The different excerpts show a tension among TRs between a will to present a rich and complex mathematical universe and a feeling that students are rather in a process of learning in a "fragmented way" without trying to "put the pieces back together".

M7: More and more, a level $n + 1$ course deletes skills from the level n course.

Most TRs agree that the attitudes of many students in courses or in exercise sessions do not allow them to overcome all these difficulties: lack of personal work, difficulties with concentration or attention, lack of autonomy or self-confidence, difficulties to concentrate, be attentive, or to master the language. Some TRs, however, find mitigating circumstances for some students: financial hardship, lack of time, transportation difficulties, having a part-time job.

M8: There are some people who have a job, for example, so they have little time to work or etc. There are others who just don't have a job and do nothing.

However, what seems to be important for the TRs interviewed, is not that students work more, but rather that they really try to confront themselves to exercises or questions, so that they can "become theirs".

M3: The work during exercise sessions cannot replace personal work, for example the course or even trying to do exercises by yourself.

For some TRs, the most important thing in a course is to insist on study processes in order to encourage students to continue the work personally, and to equip them methodologically for such a work.

M2: The subject matter itself is important, but it is somewhat the mechanism that students need to learn in thinking that can be interesting.

For some TRs, this contradicts students' expectations: to have a well-structured course and a clear identification of the elements that will be evaluated.

All TRs point the difficulty for students to adapt to university mathematics. For a large majority of TRs, they "do not do math" in secondary school, in the sense that there is a change in mathematical practices between secondary school and university, "that they do not speak the same language".

M8: I think we have a lot of students who don't know what it's like to do math and who are in this kind of attitude where they're going to try to learn a little bit of algorithmic things, some ready-made methods, that they will reproduce in a very similar context without any thought on the substance…

The following excerpt illustrates a mathematical activity that seems unsuitable for higher education.

M1: In the first year, perhaps, I think that there is a first difficulty in adapting to our requirements. Which is not necessarily related to the matter itself but which is linked to when we define a concept, what does it mean to understand a concept?

This inadequacy is sometimes expressed as a difference between a research practice that requires almost permanent questioning, and students' expectation of a reassuring practice.

M4: As a teacher-researcher, the researcher is not safe because he or she is looking for a way to disrupt everything he or she knows, as soon as there are things he or she doesn't understand, he or she is often faced with difficulties. That's what we'd like to do in our classes.

Assessment practices

The actual evaluation practices of TRs are quite diverse. The assessment may consist of a written and/or oral examination or of a continuous assessment followed by a written examination. It may concern questions about the course, exercises closely related to those presented in exercise session, new problems or a combination of these possibilities. Although the practices differ from one TR to another, we identify a similarity between them: they are generally not satisfied with the way they evaluate students.

M5: Then I'd like to evaluate them on their ability to look a little bit, ask questions that are more open than those I'm actually asking in the exam, where they would look, think, etc. and we don't have time to do that. The exam is too short.

The ideas developed in this excerpt are shared by most of the TRs. In fact, for a large majority of interviewees, the main objective of the evaluation is to determine if students have a thorough understanding of the courses. For this reason, many people want to assess this understanding, particularly through problems in a different context from the exercise sessions. However, they do not do so because of several reasons: lack of time, lack of staff or because it would require too much investment on their part to the detriment of research. In addition, they want to mix various evaluation methods such as continuous monitoring, lectures, homework, mid-semester midterms, etc. The following excerpt clearly shows this trend.

M6: The ideal is still to have a mix of continuous evaluations and evaluations, say at the end of semester or possibly mid-semester.

As a result, very few TRs assess what they really want to test and they seldom vary the forms of assessment. In our view, this reflects a tension between what they value as evaluation methods and what they actually practice. In addition, they are aware that students' success or failure at an examination is not significant in terms of their understanding of the course

M3: It seems horrible to me, but at the same time I can't put the whole system back into play like this, this is the way it works, so I find that sometimes the grades aren't, apart from the very good and the very bad students, but sometimes the grades don't always correspond to the student's understanding of what he's doing.

In the analysis of the interviews, we then identify two tensions related to TRs' assessment practices: one between what is valued and what is evaluated in the exam and another between students' understanding and success.

Teacher-researchers' training

TRs were asked whether they thought it was desirable to have access to continuing education and if so, what form it could take. A first striking aspect is that all TRs agree that, in the case where there is such a training, it must necessarily be non-disciplinary. Indeed, mastery of discipline is taken for granted in the profession of TR.

Second, two major trends emerge from the interview analysis. The first concerns TRs who believe that the training of higher-education teachers is not useful. This concerns about half of the respondents. Some TRs also question what "learning to teach" might mean, as the following excerpt shows:

M9: There is nothing that trains to teach in the strict sense. At the same time, I believe that in a university context, it's not really necessary... when someone knows his field well and... spends time preparing for his course... it's very rare that they actually make a bad course.

The other half of TRs are not opposed to continuing education, provided it has practical aspects that take many forms in their comments. For example, training could cover aspects that are not necessarily disciplinary, such as managing a group of students or how to get them to work more effectively.

M10: I think, for example, of the possibilities to use the ENT[3], computers, this kind of thing, it's a little bit up to each person to go and discover it for himself […], let's say, but nothing very pedagogically precise.

Finally, some TRs mention their lack of knowledge concerning didactics and believe that it would be positive to better understand this field of research through training.

DISCUSSION AND PERSPECTIVES

In this contribution, we have deliberately limited ourselves to presenting results in mathematics. We return here to our general issue concerning the impact of the discipline on teaching practices by pointing out some comparisons between our results and those of Physics.

We have identified that the vision of the discipline is clearly reflected in the statements made by TRs in mathematics, particularly when they point out students' difficulties (difficulties in entering into an abstraction and formalization process, in questioning themselves, etc.). In addition, their posture as researchers is highlighted when they evoke the elements they propose (or would like to propose) to students (research problems, for example). They also raise tensions in the way in which students are evaluated at university.

Thus, the norms, qualities and values mobilized by TRs are strongly marked by the image they have of their own research discipline. They are also marked by the institutional constraint represented by the evaluation standard, particularly at the beginning of university. Similar results have been obtained among TRs in physics that have been interviewed, such as a tension between what they value in terms of forms of assessment (oral, open-ended questions, etc.) and the evaluations they actually propose. Although mathematics are seen as an obstacle for learning physics, TRs in physics say that they still manage to "do physics" with their students: this contrasts with TRs in mathematics who do not manage to "do math" with theirs. For TRs in physics, it is possible to "do physics" because their discipline is at the crossroad of several disciplines and very much related to phenomena that can be observed in everyday life. We also find a strong imprint of their discipline in teaching objectives and practices (group work, reflexivity, creativity, etc.).

These first comparative results between mathematics and physics show some disciplinary disparities but also a strong anchoring of the research profession in teaching practices. In this respect, we find similarities with the theoretical framework presented above, which we need to deepen: the influence of the previous experiences of the TRs interviewed, identification with the research discipline (particularly in relation to learning), the socialization effect,

[3] ENT : digital work environment.

particularly in teacher training. Through a more detailed comparison between the three disciplines, it would be necessary to analyze our results on the basis of the five psychological processes that influence teachers' professional identity (van Lankveld et al., 2017). At least, we observe certain similarities with these processes, thus reinforcing our research hypothesis.

In addition, this research should allow us to identify enough variables to design an online questionnaire (simple or multiple responses, or simple text) to confirm or refute our results and move towards statistical representativeness of the TR population. Another objective is to study *in situ* TRs' practices in order to refine our results (van Lankveld et al., 2017).

Finally, the results presented in this contribution are also encouraging to provide some answers to the problem of university pedagogy mentioned in the motivation for this study. This would enable us to report on the results obtained on TRs in universities where interviews were held, or even during training courses dedicated to university pedagogy. The objective here would be to show TRs a representation of their own practices in order to initiate a discussion on different aspects that could better capture the complexity of practices. Here again, the disciplinary aspect is highlighted, in particular by the desire expressed by some TRs in the three disciplines to deepen their didactic knowledge.

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How applicationism impact modelling in a Belgian school of economy and the viability of an alternative epistemology

Pierre Job

[1]Ichec Brussels Management School, Ladichec, Belgium, pierre.job@ichec.be;

This paper deals with the following issue. Showing students how mathematics can be applied to economy in an applicationist way seems to make them unable to grasp the relevance of using mathematics to study economy. In other words, we expose an example of how a peculiar epistemological standpoint about the relationship between mathematics and economy, namely that of subordinating economy as an application of mathematics, may impact students' views about the interplay between mathematics and economy. We give an example to illustrate this issue and then conclude by giving hints as to an alternative way to articulate mathematics and economy, namely using economy as a semiotic foundation of mathematics.

Keywords: teachers' practices, applicationism, IBME, modelling, economy, ATD.

INTRODUCTION

According to Florensa and al. (2019), Inquiry based Mathematics Education (IBME), has spread, across the world, over the last two decades, being promoted by governments and international organizations through different means. Among these, curricula reforms and specific programs: for instance, PRIMAS and Fibonacci in Europe. Belgium is no exception to this trend and has, over the years, seen its curricula reformed, at the primary and secondary education level, to take into account different aspects of mathematics embodied in IBME, such as mathematical modelling and its relationship to the world. Despite this shared trend, implementations of IBME may put on various clothes, even at the research level, as noted by Artigue and Blomhoj (2013). This variety of approaches justifies that we take a closer look at the Belgian context and thus contribute to the study of IBME. In this paper we will focus on the tertiary level of education and more precisely on the setting of a Belgian high-school[1] in economy, business and management (school of economy in short). We will rely on the following research questions as guidelines. What form does IBME take in a school of economy? In particular, what kind of relationship does a school of economy have with mathematical modelling and economy? What are the factors that might impede or on the contrary facilitate the diffusion of an IBME approach in a school of economy? The aim of this paper is not to answer those questions in a definitive manner but more modestly to provide the following elements of a response. In section 3, we show how difficult it can be for mathematics teachers to engage in genuine modelling activities relevant to economy and how their relationship to mathematics tends to turn modelling into a form of applicationism (Barquero et al., 2013). In section 4, on the other hand, we explain how economy itself might provide a platform to implement a form of

[1] In Belgium, a high-school is university level institution.

IBME, namely a study and research path (Chevallard, 2015). Before getting to these sections we present our theoretical framework as well as relevant literature that puts our research questions in perspective.

THEORETICAL FRAMEWORK

Our theoretical framework is the Anthropological Theory of the Didactic (ATD) developed by Chevallard (1992). The use of the scale of levels of didactic co-determination (Chevallard, 2002) has proved to be a fruitful formalism to study didactic phenomena through the lenses of constraints acting upon institutions and knowledge. In relationship to IBME, Chevallard (2015) puts forward a high-level constraint to the diffusion of IBME. In short, IBME can be considered as an expression of a certain didactic paradigm, that of *questioning the world* (QW). And this rather novel QW paradigm conflicts with a much older one, that of *visiting works* (VW) which is still more spread and rooted in, at least, our western culture. The VW paradigm amounts to approach knowledge as a "monument that stands on its own, that students are expected to admire and enjoy, even when they know next to nothing about its *raisons d'être*" (Chevallard 2015, p.175). On the other hand, the QW paradigm (Chevallard 2015, pp.177-180) starts with a *generating question* tackled by a set of students and a set of guides of the study that together form a *didactic system* whose aim is to generate a final answer to the generating question. This final answer is the culminating point of *moments of study* of available information and *moments of research* that generate intermediate questions and answers. This particular relationship to knowledge delineates what Chevallard calls a *research and study path* (SRP). From these descriptions we can see that these two paradigms are mostly mutually exclusive. Barquero and al. (2013) go further and deal with constraints specific to university level in natural sciences in Spain. Among other things, they show how a certain dominant epistemology called applicationism, which considers that "mathematics has to be introduced by itself, having its own rationale, before being applied to extra-mathematical situations" (Ibid, p.317), tends to greatly restrict how mathematical modelling is understood (Ibid, p.317): "Under its influence [applicationism], modelling activity is understood and identified as a mere application of previously constructed mathematical knowledge or, in the extreme, as a simple exemplification of mathematical tools in some extra-mathematical contexts artificially built in advance to fit these tools". Do similar restrictions apply in our context of a Belgian school of economy? If yes, to what extent?

APPLICATIONISM IN HIGH-SCHOOL: THE BUDGET LINE EXAMPLE

To answer those questions will shall look into the first mathematical course students have to attend in our school, the way it was given between 2012 and 2017. Although it underwent many changes over the years, in terms of teachers, numbers of students, number of dedicated hours and even content, one topic remained the same. It is that of first-degree equations and lines (lines in short). We will focus on this topic as an invariant of the course able to inform us about the relationship to applicationism and modelling *over a substantial period of time*. The content of the chapter devoted to lines summed up to exposing the mapping between equalities of the form $ax + by = c$ and lines in a Cartesian plane. It was structured using the following mould. First the theory was recalled and then some routine exercises were provided for students: an equality was given and they had to draw the corresponding line in a plane and vice versa.

In 2013, an attempt was made for a brief period of time (a few sessions) were teachers decided to reverse the traditional order between theory and exercises. They did so because they had to face the fact that students were losing interest in the theoretical part of the course, in part because students were familiar with lines since their 3rd year in secondary school (14-15 years old) and were recalled for 4 years in a row the same topic. This departure from the norm turned out to be a failure. Students were not able to solve the exercises (no more or no less than when they were first given the theory) but this time they moreover complained they didn't have any theory to rely on and apply. Teachers felt guilty and entrapped because the only perceived way to keep the course going was for them to recall the theory anyway. This episode reinforced most of the teachers in their believe that "theory then exercises" was the only way to go. This shows a first linkage to applicationism. For most teachers in our school, the only possible way to teach mathematics is to expose the abstract theory and then apply it to some routine exercises, because their students don't have the required mathematical autonomy to learn outside the framework of a well-established theory.

A second linkage to applicationism is the following. It was no conceivable for teachers to leave aside the chapter on lines because lines were to be used in a subsequent chapter were elementary linear programming problems were solved using a geometrical presentation based on lines. Thus, students had to master the theory of lines before they could possibly encounter linear programming problems in a fruitful way. In other words, these teachers did not conceive that it could have been possible to teach mathematics in a manner that does not mimic its deductive architecture. The meaning of concepts is not driven by problems but rather by their logical embedding which is considered as the quintessential level of rationality of mathematics.

This reduction of teaching mathematics to architectural aspects had deep consequences on how modelling was treated by teachers as well. Being in a school of economy, teachers felt important to deliver a course that would be closely linked to economy. This desire to relate the two subjects was implemented by inserting "economic applications" into the course. They felt that doing so they contributed to introduce students to mathematical modelling, thus showing the relevance and importance of mathematics to economy. In the case of lines, the mathematical application considered was that of a budget line.

It was presented to students in the following manner. First a numeric example was given to them: "If two goods can be bought at respective prices of 2 and 5 euros, and if we have to buy a quantity x of the first good and a quantity y of the second in order to spend exactly 100 euros, then the equality $2x + 5y = 100$ must hold true. Thus, 100 euros outlays can be represented by a line in the plane of all possible outlays. This line is called a budget line".

Following this numeric example, a "general" version with letters was given in exactly the same way leading to $p_1x + p_2y = B$ with p_i standing for the unit price of good i, x and y being the respective quantity and B the budget at disposal. $p_1x + p_2y = B$ could also be represented by a line according to the theory that had been recalled earlier in the course. The budget line application was then over and students were given exercises like "Draw the budget line that represents the outlays related to two goods that can be bought at respective prices of 7 and 14 euros with a given budget of 1400 euros".

The way budget lines were introduced is very informative as to the impact of applicationism on modelling activities. Let us turn to students' relationship to budget lines to better understand the underlying mechanisms.

If we start with the exercises on budget lines like the one mentioned above, students behaved in a similar fashion as with exercises on abstract lines as soon as they had understood that they had to take $p_1 = 7$ and $p_2 = 14$ and $B = 1400$ to solve the exercise (and they did because teachers told them when they were stuck). They were then able to comply to the teachers' expectations by relying on the didactic contract (Brousseau, 2002) when they were not able to understand on their own what was required from them to solve the exercise.

But when we interacted with students, asking them what was the point of these budget lines according to them, many of them told us that they didn't feel at ease. For them, it was like they couldn't grasp the difference between budget lines and abstract lines, only in one case they had to use some economy related terminology (budget line, goods, …) but not in the other case. They could not figure out was budget lines were really useful for but they didn't bother too much with these concerns, because they could do the exercises and convince themselves that it is natural in a school of economy to have some economic terminology percolate through mathematical courses.

With this example, we can measure the gap that sets up between students and teachers, gap hidden under the appearance of the ordinary functioning of a regular course. Essentially, for students, budget lines don't make much sense and are definitely not the end product of a modelling activity as would be the case in a genuine study and research path. Indeed, there is no economic problem to which budget lines are an appropriate answer, the way the course was given, despite the fact that such problems seem pretty obvious and a priori within the reach of students. The following questions for instance might contribute to design an SRP. Is this possible to buy that amount of these two goods given that budget? What budget would be required to be able to buy that amount of these two goods? In other words, presenting budget lines the way they were lacks some fundamental character in the sense of Brousseau (2002). How did we get to that point?

The prevalence and naturality of the VW epistemology among teachers, of which applicationism is an offspring, tends to hide in the back *open* questions and problems in favour of a theory whose power to solve *closed* questions and problems (questions and problems designed to be solved by that theory) will justify its prominence. As a consequence, the idea to develop mathematics from the need that arise to solve an initially open problem is mostly absent. Instead teachers tend to reduce and focus teaching on the design of the best outfitting in which a theory should be dressed to minimize students' reluctance. The more energy they put in the design of such outfitting, the more they are unable to get in touch with their students' epistemic concerns because, from their teachers' point of view they made everything possible for students to understand the theory. This outfitting can take on the form mentioned above of using concrete numeric examples before using letters for the theory. This way, teachers feel they are really engaging in mathematical modelling and making it accessible to students, whereas students don't understand what is the point of budget lines besides learning some economic related terminology. It shows the mechanism by which the applicationist point of view deprives itself from the ability to design an economic problem where a mathematical model would be relevant to solve a genuine economic problem.

Chevallard (2015) relates the VW epistemology's long prevalence to the "social structure of formerly undemocratic countries" (p. 175) among other aspects. In the case of the interplay between economy and mathematics we may invoke another reason. The famous title of Wigner (1960) "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" is symptomatic of a train of thought that dates back to at least Galileo stating that mathematics is the natural language of nature and thus by extension the signature of any approach that would qualify as scientific. Although complex, the penetration of mathematics within economy can be partly related to this trend. Economy had its proponents to turn it into a hard science and not a "mere" social and human science and thus mathematize it: "« Avec sa théorie de la valeur, Debreu développe une approche résolument axiomatique dont le critère exclusif est la cohérence logique et non le rapport à la réalité » (de Vroey, 2002). In this context, being able to subordinate economy as an application of formal mathematics may be considered an achievement that equals the historic refoundation of physics and geometry based on "pure" mathematics. Thus, the way mathematics and economy interact at a pedagogic level is tainted by the means through which economy established itself as a hard science. From the perspective of economy that wants to establish itself as a hard science, the ability to be subordinated to mathematics is considered as a mark scientificity and this translates to mathematical courses given to economy students. These courses tend to be display economic application the way it is illustrated above with the budget line e.g. "pure" mathematics are developed with no connexion with economy and then "applied" to economy.

USING ECONOMY AS A POSSIBLE SEMIOTIC PLATFORM TO DEVELOP A STUDY AND RESEARCH PATH

In this section, we provide empirical data showing that economy itself might be used to develop a modelling activity where the meaning of mathematical concepts relies on the semantic of economic ground and thus showing a possible way out of strict applicationism. During the period 2012-2017, part of a chapter dealing with linear programming problems was devoted to the teach students that inequalities of the form $ax + by \leq c$ could be represented by half-planes and vice versa. This result was then used to give a geometric representation of linear programming problems that would allow to solve them geometrically. The argument used by the teachers was purely mathematical with no reference to economy and relied on the decomposition $\{(x,y)|ax + by \leq c\} = \bigcup_{k \leq c} \{(x,y)|ax + by = k\}$. The idea was to show students that a half-plane can be seen as a stacking of lines and thus reduce the study of the geometric representation of inequalities to that of lines (which had been recalled in a previous chapter). It turned out that students agreed on the geometric idea of a half-plane being a "stacking of lines" but irrespective of the above decomposition. They didn't not understand why the decomposition was used to assert that a half-plane is a "stacking of lines" as for them it was self-evident. As consequence they didn't understand either how to use the decomposition to draw the half-plane representing a given inequality. Teachers themselves had much trouble understanding what students couldn't understand in their argument. All in all, teachers felt pushed to leave aside the theory of inequalities and fall back on teaching what algorithm to apply to draw half-planes from inequalities. In 2017-2018, we had the opportunity to depart from the way the course was taught during the period 2012-2017 and were able to experiment on a small scale a different approach to inequalities and half-plane. This experiment is rather modest but nevertheless meaningful in our context because being in charge of hundreds of students does not allow much room for ideas that would be considered as "failures", by the institution. So, we had to make adjustments in the course very cautiously, in a step by step fashion, that would make changes not appear as dramatic modifications. The economic context used is the following. We have a budget of 400€ that allows to buy two types of tea. The first type T_1costs 5€/100g and the other one T_2 costs 4€/100g. The experiment can be divided in steps. We will fly over the first steps as we do not have enough space to details all of the experiment and focus on the steps directly related to the mapping between inequalities and half-planes. Step1. Through a set of questions like the following, students are led to calculate numerical expenditures: can you give expenditures spending the entire budget, can you give give expenditures spending more than the budget, given two expenditures which one is more costly, etc. This step allows students to feel at ease in the chosen context and relate the required calculation to an economic context that makes sense to them. Step 2. It also prepares them for the next step which seeks to make them move from the numerical register to algebra with the use of questions asking them to reflect upon expenditures like the following: what calculation do you have to make to determine if an expenditure spends the entire budget? This step leads to ostensive objects like $5q_1 + 4q_2 = 400$ where (q_1, q_2) denotes a certain expenditure. Students are then asked to give a geometrical representation of $5q_1 + 4q_2 = 400$ in a plane so as to be able to visualize all the expenditures spending the entire budget. Given their previous acquaintance with the topic, students are well aware that it gives rise to a line, even if they may have trouble drawing that line. The next step is what interests us most for our purpose. Step 3. Based on the following economic we lead students towards inequalities. In one company, the workers which to spend all the allotted budget for tea because at the end of the year, the amount that hasn't been spent returns to the company. In another company, budget is handled in a different way. If the budget is not spent entirely the remaining is left to the workers. The only condition imposed by the company is to not spend more than the initial budget. This second context leads to the expression $5q_1 + 4q_2 \leq 400$ which models whether or not an expenditure (q_1, q_2) will exceed the allotted budget. Student are then asked to also give a geometric representation of $5q_1 + 4q_2 \leq 400$ and contrast it to the previous context of $5q_1 + 4q_2 = 400$. At this point students are not accustomed to expressions like $5q_1 + 4q_2 \leq 400$ even though they have already met inequalities in secondary school. What is interesting for our purpose is that some students were able to give a geometric meaning to $5q_1 + 4q_2 \leq 400$ based on an economic reasoning. The trail of their reasoning is the following.

- Some students note, for instance, that $q_1 = 40$ and $q_2 = 50$ exhausts the 400 € budget.
- It thus means that any increase of q_1 or q_2 will exceed the budget. And any decrease will no exhaust the budget.
- The geometric consequence is that starting from a point on the line representing $5q_1 + 4q_2 = 400$ like $q_1 = 40$ and $q_2 = 50$, increasing wether q_1 or q_2 or both at the same time, will give birth to a point in the plane (q_1,q_2) that will be located "above" the line. A similar conclusion can be drawn while decreasing those quantities. Such points will all be located "below" the line.
- From these considerations, students are able to give an economic meaning to the interplay between algebra and economy. Expenditures spending less than the budget verify $5q_1 + 4q_2 < 400$ and are geometrically located "below" the line represented by $5q_1 + 4q_2 = 400$. Points exceeding the budget verify $5q_1 + 4q_2 > 400$ and are geometrically located "above" that same line.
- Thus geometrically, $5q_1 + 4q_2 \leq 400$ can be divided into points on the line $5q_1 + 4q_2 = 400$ and points "below" it $5q_1 + 4q_2 < 400$.

The way these students reasoned about the geometric meaning of $5q_1 + 4q_2 \leq 400$ is remarkable in our institutional context for several reasons. First it shows that it is possible for students to take responsibility of a fragment of interplay between algebra and geometry. To the best of our knowledge, it never happened in our institution in the framework of the course we are studying in this paper: all theoretical aspects have always been taken in charge by teachers. It means that students can be made much more responsible than thought and have the ability to contribute to the development of a theory. Second it shows that applicationism is not the only “possible” way to teach mathematics in a school of economy. Third, the mapping between algebra and geometry performed by students relies on economy i.e. economy is used to give a geometric meaning to algebra. We would like to stress this aspect because it shows not only that something different than applicationism is feasible but that the subordination of economy to algebra and geometry is not inescapable. Economy can be envisioned as a stepping stone on which mathematics can be build whose objects’ semiotic relies on economy like for instance inequalities.

CONCLUSION

We showed how much applicationism is rooted in our school of economy. It means that it underlies the whole course that has been studied but more than this, that teachers feel difficult to teach another way because attempts to modify the “theory then exercise” model turned out to be failures. One reason for this failure worth exploring in another paper would be the idea that teachers lack other levels of rationality than the deductive one which deprives them from envisioning the teaching of mathematics according to other organizing principles. As a consequence of their failed attempts, it seems to comfort them with the idea that teaching mathematics mostly consists in thinking about which outfitting should be used to wrap the theory they want to teach in a way that minimizes frictions with students. From this perspective, the “theory then exercise” model appears to be a generic outfitting that allows to use exercises as a mean to discharge students from taking responsibility and making sense of the theory: when students are successful at exercises it is considered as a mark of understanding.

We also showed of much applicationism impacts modelling. It reduces mathematical modelling of economy to applying “pure” mathematics to economy. This has a potential impact on students’ perception of economic applications. The case of budget lines suggests that presenting them as a mere application of mathematics deprive students for the ability to consider this application as meaningful. This leads to a vicious circle. Teachers feel that economic applications give more credit and substance to the usefulness of their course by tightly interacting with economy, when in fact, the very way it is presented to students has the reverse effect on them as it deprives them from the possibility to understand which relevant economic problem has been tackled.

These results around applicationism and modelling are in line with those found in Barquero and al. (2005). It would be interesting to study the extent to which such phenomena apply in other institutions in Belgium and around the world but also within our institution in other courses which we so far had no access to.

Lastly we showed that it is possible, even if it was experimented on a small scale, to develop some mathematics starting from economy where economic can act as a milieu which students can interact with to construct a semiosis that connects first order inequalities to half-planes thereby showing the possibility to deconstruct the applicationist paradigm and opening to a tighter integration between mathematics and economy.

The ability to make mathematics rely on economy is, as noted before, an important result, at least in the context of our institution. It nevertheless raises a question that might at first seem to downplay the relevance of this result on which we will end this paper. To what extent the use of economy as the semiotic foundation of mathematics might contribute to create epistemological obstacles? Indeed, if we imagine a course entirely built on economy, it might lead students to not be able to grasp the meaning of mathematical concepts in any other way than being rooted in economy. We think for instance of mathematical structures that emerge from needs internal to mathematics.

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The Role of Learning Strategies for Performance in Mathematics Courses for Engineers

Michael Liebendörfer[1], Lara Gildehaus[1], Robin Göller[2], Jörg Kortemeyer[3], Rolf Biehler[1], Reinhard Hochmuth[4], Laura Ostsieker[5], Jana Rode[6], Niclas Schaper[7]

[1]Universität Paderborn, Institut für Mathematik, Germany, michael.liebendoerfer@math.upb.de; [2]Leuphana Universität Lüneburg, Institut für Mathematik und ihre Didaktik, Germany, [3]Technische Universität Clausthal, Institut für Mathematik, Germany, [4]Leibniz Universität Hannover, Institut für Didaktik der Mathematik und Physik, [5]Technische Hochschule Köln, Institut für Produktentwicklung und Konstruktionstechnik, [6]Universität Kassel, Institut für Mathematik, [4]Universität Paderborn, Institut für Humanwissenschaften

We analyse the impact of learning strategies on engineering students' performance in mathematics. Learning strategies play an important role in self-regulated learning and are a possible predictor of student performance. Especially for mathematics-related learning strategies, the question arises how such strategies can be measured and how they relate to mathematics performance. Therefore, we present a new learning strategy questionnaire that takes into account the specifics of mathematical learning at universities. We then present correlational data of a longitudinal study with n = 403 engineering students. We further regress their performance on students' use of their learning strategies as well as their prior performance. The results indicate which learning strategies help students succeed.

Keywords: Teachers' and students' practices at university level, Teaching and learning of mathematics for engineers, Learning strategies, Students' performance.

INTRODUCTION

Mathematics is still a big hurdle for many students entering university across different study programs. Heublein (2014) reports that at German universities, 36 % of all bachelor-students in engineering drop out and the most prominent reason for drop-out is their problematic performance. Improving students' performance is not only important with regard to drop out but can rather be seen as the major goal of university teaching.

One variable to explain students' performance is their use of strategies. The learning of mathematics at universities usually involves many self-study phases in which students have to self-regulate their learning. However, we lack a clear understanding of what strategies should be recommended and what strategies explain performance, especially when it comes to mathematics courses for engineering students. Only few studies have used instruments that take the characteristics of university mathematics into account and many results are based on cross-sectional but not longitudinal data.

Liebendörfer et al. (submitted) have developed the LimSt questionnaire (Learning strategies in mathematical studies) to measure students' learning strategies specifically in higher mathematics. They showed that several strategies could be empirically discerned. The question of how these strategies explain students' performance is still open. In this paper, we use these strategies to predict performance.

THEORETICAL BACKGROUND

Learning strategies

Students' use of learning strategies is usually framed within self-regulated learning (Pintrich, 1999) and examined with questionnaires like the "Motivated Strategies for Learning Questionnaire" (MSLQ; Pintrich, Smith, Duncan, & McKeachie, 1993) or the German adaption "Inventar zur Erfassung von Lernstrategien" (LIST; Schiefele & Wild, 1994). These questionnaires operationalize cognitive and resource management strategies. They also include metacognitive strategies, which we do not focus in this paper. Cognitive strategies are strategies for the processing of information. The MSLQ distinguishes rehearsal strategies (such as repeating words or other items to remember them) elaboration strategies (such as paraphrasing or summarizing to build internal connections between items), organization (such as outlining or clustering to select appropriate information) and critical thinking (Pintrich et al., 1993). Resource management strategies regulate the use of internal resources, such as time and effort management, and external resources, such as peer learning and help seeking.

University mathematics, however, has some specialties that lead some researchers to either use only parts of the general instruments (e.g., Griese, 2017 dropped the scale for critical checks from the LIST) or completely design new scales (e.g., Kaspersen, 2015 developed a new scale on working conceptionally with mathematics).

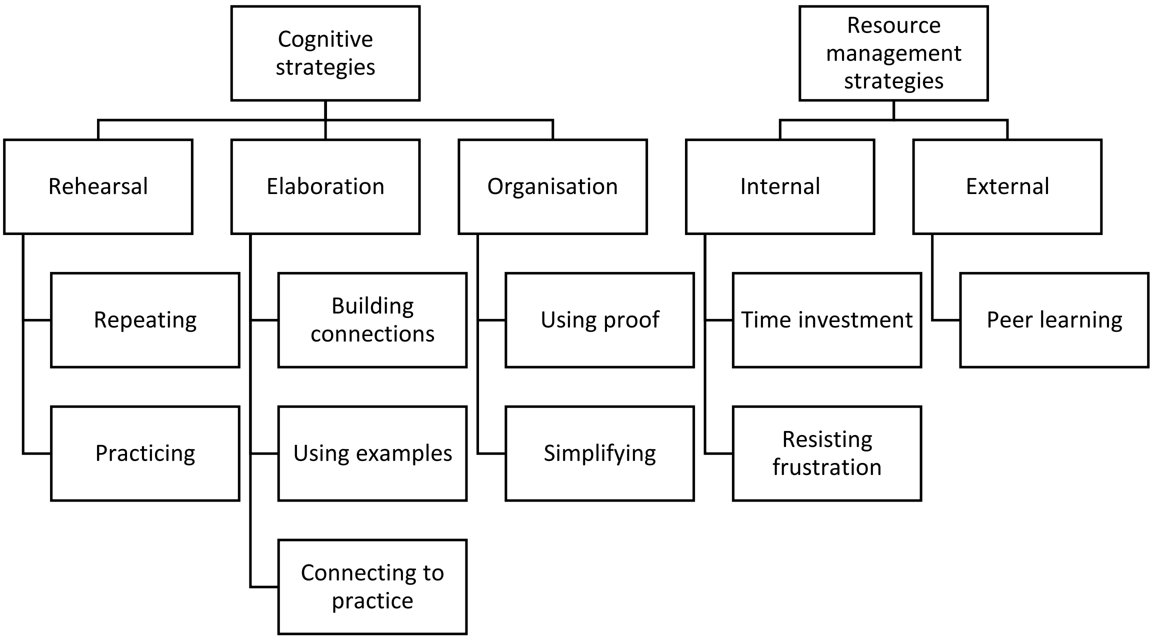

Figure 1: Structure of the LimSt scales used for this research

Important specifics of university mathematics are the multifaceted role of proof (Auslander, 2008; Jones, 2000; Weber, 2014) and the role of procedural knowledge, e.g. in performing calculations (Bergsten, Engelbrecht, & Kågesten, 2017; Hiebert, 2013). To address the specialties of university mathematics, Liebendörfer et al. (submitted) developed an instrument similar to the LIST and MSLQ that adds and differentiates more forms of learning strategies to cover these specifics of mathematics.

The LimSt questionnaire

The LimSt questionnaire (Fragebogen zur Erhebung von Lernstrategien im mathematikhaltigen Studium; Liebendörfer et al., submitted) maintains the distinction between cognitive and resource management strategies, as well as the subdivision in rehearsal, elaboration, organisation strategies, internal and external resources respectively, see Figure 1. However, these strategies have been refined with regard to the specifics of mathematics at the tertiary level. Item examples are given in Table 1.

Rehearsal strategies may refer to the repeated reading, writing or saying aloud of content to be learned. For the learning of mathematics, the rehearsal strategy of practicing is also relevant, which refers to carrying out procedures and algorithms in various examples in order to learn how to perform them. The difference between repeating and practicing strategies is not necessarily due to the content to be learned, since one could also learn about procedures by repeating, e.g. saying aloud the steps in their order. However, practicing is considered necessary for the acquisition of procedural knowledge.

With regard to the elaboration strategies, building connections includes comparing content, relating it to content already learned and finding analogies. For mathematics, two specific forms of connections are particularly relevant. The first form is the use of mathematical examples to illustrate general rules and phenomena or constructions and procedures. The second form refers to the establishing of real-world connections, e.g. via mathematical modelling.

Organization strategies were subdivided in the use of proofs and the simplifying of contents. Using proof refers to any activity that includes the proofs given in lectures or learning materials. Although proof is the central organizing principle of academic mathematics, students often focus on facts and procedures only (Göller, in press). The strategy of simplifying refers to transformations of complex content into less complex forms, even if they are not perfectly correct, like essential ideas that can be memorized more easily.

Resource management strategies include the management of inner resources like students' effort. Whereas effort is often described in terms of time investment, we discern pure time investment from resisting frustration during the learning, which refers to different inner resources like volition or self-control. Finally, peer learning makes use of peers as external resources, like seeking help or collaborating in solving tasks.

Learning strategies and performance

The driving motive for the development of theories of learning strategies is the assumption, that different ways of learning may explain different results, in particular differences in students' performance. Students who tend to use some but not other strategies may thus tend learn the content to more effectively. In the literature, some studies on learning strategies and their connection to student performance in service mathematics can be found. We complement our review by the meta-analysis of Schneider and Preckel (2017).

Correlational data show positive connections of working on exercises (focusing procedural knowledge) with performance in exams (Eley & Meyer, 2004). Since working on exercises can be seen as following a surface approach, this may explain, why although surface learning is generally related to minor success in higher education (Schneider & Preckel, 2017), this is only sometimes the case for mathematics (Griese & Kallweit, 2017), but sometimes not (Laging & Voßkamp, 2017; Liston & O'Donoghue, 2009).

Theoretically, elaboration strategies are expected to improve study performance because they lead to deep processing of the information, which should lead to deep and stable knowledge. Correlational data show connections of gaining an overview (i.e., using elaboration strategies) with performance in one study on mathematics (Eley & Meyer, 2004); however, this connection could not be confirmed in several other studies (Griese, 2017; Griese & Kallweit, 2017; Laging & Voßkamp, 2017; Liston & O'Donoghue, 2009; see also Schneider & Preckel, 2017 for results across different domains). Similarly, organization strategies do not correlate with students' performance (Griese, 2017).

Students' management of internal resources (effort) is an important predictor of academic performance both across different domains (Schneider & Preckel, 2017) and in university mathematics (Griese, 2017). In contrast, peer learning as a form of managing external resources has proven helpful in various domains (Schneider & Preckel, 2017) but not mathematics (Griese, 2017).

In sum, these findings from studies of service mathematics show that students' effort is the only strategy having a consistent connection to their performance. We should note, however, that except for the study by Laging und Voßkamp (2017), the presented findings were not based on longitudinal data that include a measure of prior performance, which is generally known to explain much of the future performance (Schneider & Preckel, 2017). Since there are no earlier studies on the relation of the LimSt scales and students' performance, it is an open question, whether the more specific scales may reveal that specific strategies predict students' performance.

Research Questions

Given the new LimSt scales and the few results from longitudinal studies that take students' prior performance into account, we want to explore the connections of students' learning strategy use to their performance. We have two research questions:

RQ1: Which learning strategies correlate with students' performance?

RQ2: Which learning strategies predict students' performance?

METHOD

We draw on data gathered in summer 2015 in a second-semester course on mathematics for engineers at the University of Hanover (Germany) that follows a first-semester mathematics course. The cohort consists of students from electrical engineering, civil engineering, mechanical engineering and similar programs. The topics of the first-semester course included analytic geometry, complex numbers, linear algebra (as far as eigenvalues) and univariate analysis (sequences and series, differentiation and integration). In the second semester, multivariate analysis up to integral theorems and ordinary differential equations followed.

In both the course on mathematics in semester 2 and its preceding course in semester 1, students were offered to take four short exams spread over the semester that replace the final exam at the end of the semester (that still was offered). In each short exam, students could reach up to 10 points, so the possible maximum score is 40. The pass mark was 15 points and higher results yielded better grades. We use the sum of the four short exams in semester 1 as indicator of students' prior performance and the sum of the four short exams in semester 2 as their performance in the second-semester course.

The tasks focused mainly on calculations. Examples from the four short exams in the second semester are to investigate the convergence of power series, to give Taylor polynomials for given functions in one and two dimensions, to find extreme values, to calculate line integrals or to solve differential equations. In contrast, no task required proof. Students were not allowed to bring their notes or calculators.

Students were further asked to answer a paper and pencil questionnaire during lecture time. We measured their learning strategies on Likert scales from the LimSt questionnaire described above ranging from 1 (strongly disagree) to 6 (strongly agree). Most of them were newly developed, only the time investment scale consists of four items of the LIST scale for effort (Schiefele & Wild, 1994) that focuses on time investment, supplemented by one more item, see (Liebendörfer et al., submitted) for details to all scales. All scales showed a high internal consistency, see Table 1.

The learning strategies were assessed at the beginning of the course, so the students answered the questionnaire after having completed all short exams that measure their prior performance and prior to the short exam measuring their future performance. We analyse the data of the subgroup of all engineering students (more than 1000) who had taken the short exams and agreed on sharing their results (n = 403; 93 % were in their second semester, 77 % were male).

Both the questionnaire data and the short exam results were treated as metric data in the analysis; i.e. we give means and standard deviations and use Cronbach alpha and Pearson correlations as well as a linear regression analysis. This treatment may not perfectly match the ordinal data given, but is simple and seems to yield appropriate results. The methods are well known in the field and questionnaire data is often handled similarly.

RESULTS

Before answering the research questions, we give mean values and standard deviations in Table 1. The mean values show that students strongly report the use of the rehearsal strategies practicing and repeating as well as peer learning. In contrast, using proof is the strategy with the lowest mean, but highest standard deviation.

Correlations

To answer RQ1, we report the correlations of prior performance, performance and learning strategies in Table 1. The two rehearsal strategies repeating and practicing have positive correlations with performance. Of the elaboration strategies, only building connections has positive correlations to performance. Of the organization strategies, only using proof has a small positive correlation to prior performance. The two forms of effort, time investment and resisting frustration both show positive correlations to performance and peer learning has a small correlation to future performance. Note that generally, the correlations with prior performance and future performance are almost equal.

Regression analysis

To answer RQ2, we conducted a linear regression using all learning strategies and the prior performance as predictors of future performance. Together, these variables could explain 57 % of the variance of future performance ($R^2 = .57$). The non-standardized regression coefficients are displayed in Table 1. Prior performance is a clear predicator of future performance. Of the rehearsal strategies, repeating is a negative predictor, whereas practicing is a positive predictor. None of the elaboration strategies predicted performance. Of the organization strategies, simplifying is a negative predicator. Resisting frustration but not time investment predicts performance and finally peer learning does not predict performance.

	Item example	Items	α	M	SD	r_{PP}	r_P	b
Prior Performance				19.02	8.18			**0.71**
Performance				15.81	8.71	**.73**		--
Repeating	I repeatedly go through important content so that I will not forget it.	3	.74	4.49	0.84	**.21**	**.14**	**-1.35**
Practicing	I learn algorithms by repeatedly performing the procedure.	3	.77	4.43	1.16	**.24**	**.23**	*0.66*
Building connections	I try to understand how new content relates to what I have learned before.	4	.82	3.99	0.98	**.24**	**.21**	0.26
Using examples	I search for application examples for formulas.	4	.77	3.89	0.97	.06	.07	-0.02
Connecting to practice	I think about what one can practically do with new content.	3	.74	3.20	1.30	-.08	-.02	0.29
Using proof	I try to understand the proofs of the theorems.	3	.85	3.14	1.33	*.12*	.10	-0.11
Simplifying	I try to simplify difficult content.	3	.75	4.36	0.93	.03	-.04	*-0.76*
Time investment	I take more time for learning than most of my peers.	5	.79	4.15	1.01	**.28**	**.29**	0.21
Resisting frustration	I do not give up, even if the content is very difficult or complicated.	3	.79	4.15	1.13	**.34**	**.40**	**1.47**
Peer learning	I meet with fellow students to develop ideas for solutions together.	3	.78	4.48	1.23	.09	**.13**	0.35

Table 1: Example item, Cronbach's Alpha, mean (M), and standard deviation (SD) for students' performance and the learning strategies measured, as wells as correlation coefficients for prior performance (r_{PP}) and performance (r_P) and the non-standardized regression coefficient (b) for performance regressed on prior performance and learning strategies. Coefficients significant at p < .05 are in italics, they are bold if p < .01.

DISCUSSION

Based on a refined scale on learning strategies for mathematics and a longitudinal sample of engineering students, we investigated the relationship between performance and learning strategy use.

Besides the significant and high correlation of performance and prior performance, the correlation analysis showed that the rehearsal strategies of repeating and practicing, the strategy of building connections, and the scales for time investment and resisting frustration showed high correlations to performance. It may seem surprising, however, that stronger students put much of their effort into rehearsal strategies that are often labelled surface strategies but not most of the elaboration and organization strategies. This result can be understood if we consider the kind of mathematics that was requested in the short exams, which is mainly procedural knowledge.

The regression analysis showed that performance could mainly be explained by prior performance confirming the literature (Schneider & Preckel, 2017). Yet, some of the learning strategies can explain further parts of students' performance. The rehearsal strategies practicing and repeating both are significant predictors. Surprisingly, while practicing is a positive predictor, repeating is a negative predictor. Of course, a negative coefficient does not mean here that a specific form of learning does not help the individual but that students who used this strategy learned less than the average of the student cohort, so the strategy may be effective but not efficient. This finding highlights the constructivist view that mathematics is an activity and learning mathematics means doing mathematics. From the elaboration strategies, only simplifying, which was highly used by students, is a significant (negative) predictor for performance. Whereas simplifying could help students to get a rough overview of a topic, it seems as if they do not get deeper into the content. From the internal resource management strategies, only resisting frustration is a (positive) predictor of performance.

Comparing correlations and regression results, we see that repeating is something that rather good students do but does not help them getting better. Similarly, stronger students use proofs more often but that does not explain future performance. This fits the general consideration that the knowledge required in written exams can be mostly achieved through practicing strategies. In addition, stronger students invest more time, but that does not explain their performance. The quality of students' learning may thus be more important than the quantity of their time invested.

Our conclusion is that performance is raised by practicing but not repeating, and by resisting frustration but not simplifying. Doing the hard and frustrating work pays off.

Strengths and limitations

The strengths of our study encompass using a validated instrument that was specifically designed for higher mathematics, relying on longitudinal data in a large cohort taking into account prior performance, and a high ecological validity by using exam scores. This allowed revealing differences in related variables like repeating and practicing or time investment and resisting frustration that can be clearly linked to engineering students' performance in exams.

Limitations include that we only analysed a subset of students that may have their specialties. Further, questionnaire data do not perfectly represent real behaviour and testing at the beginning of the semester does not cover later changes in students' learning behaviour. This may have blurred some results. In addition, the exams we used as a measure of students' performance focus on procedural knowledge. We should therefore limit our findings to the learning of procedural mathematics.

Implications for theory and practice

Besides the identification of relevant learning strategies, our theoretical differentiation of the individual mathematics-related learning strategies built a useful frame. In particular, rehearsal and effort were split into forms with different roles as predictors. Future research could explore the role of these refined strategies for the learning of other forms of mathematics, e.g. in teacher education. The longitudinal design further revealed that correlational patterns do not need longitudinal patterns so we should not take correlations of learning strategies as indicator of causality (see repeating or using proof).

Our recommendation for students' learning is to practice mathematics and work hard but not simplify and repeat (as many do according to the mean values).

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First-year university students making sense of symbols in integration

Hans Kristian Nilsen

University of Agder, Norway, hans.k.nilsen@uia.no

This paper focuses on first year university engineering students and their sensemaking of integration and its symbolism. Through a semiotic approach, attention is given to two students and their attempt to verbally express their reflections on integration and the related meaning of symbols. Findings suggest that students tend to interpret the symbols mainly as operations, in terms of calculations to be carried out. They express uncertainty concerning what the symbols stand for, and the mathematical objects they represent. For example, the symbols ∫ and dx are respectively conceived of as "finding the integral with respect to x" and students are unclear on how Riemann sums connect to integration as a concept.

Keywords: Teaching and learning of analysis and calculus, Teaching and learning of mathematics for engineers, Integration, Riemann sums, symbols.

INTRODUCTION AND RESREACH QUESTIONS

This paper is based on a study conducted as a part of MatRIC's (*Centre for Research, Innovation and Coordination of Mathematics Teaching*) research activity, with the overarching goal of improving mathematics teaching and students' learning in higher education. Integration constitutes a major part of first-year university engineering students' calculus curricula, and students from computer and mechatronic engineering programmes constituted the target group of this study. The university teacher explicitly emphasized the importance of obtaining in-depth understanding of fundamental concepts and ideas of integration. A relevant object of study could therefore be to investigate if this could be traced in students' reflections. At the same time, to investigate the challenges that students encounter while working with this topic in general, could be of great value for further developing teaching and student activities. Sofronas et al. (2011) point to three significant "sub-goals of the integral" as essential for students understanding: 1) the integral as net change or accumulated total change, 2) the integral as area and 3) techniques of integration (p. 138). In this paper, the focus is a combination of 1) and 2), as students are asked to reflect on the Fundamental Theorem of Calculus (hereby denoted as FTC) and the symbols involved. Several aspects of conceptualizing integration have been dealt with in research, for example the importance that students conceive of integration as an *accumulation function* (Thompson & Silverman, 2008; Bressoud, 2011) and the use of *Riemann sums* towards obtaining this goal (Wagner, 2018). In line with these aspects and through a semiotic approach, this study aims to investigate on students' perceptions of integrals and how their interpretations of the symbols ∫ and dx contribute to their perception on integration in general. The following research question guides the focus of this paper:

What characterizes students' interpretations of the symbols involved in integration, and how do these interpretations affect upon their perception of integration as a mathematical concept?

The term "characterizes", indicates a focus on students' own subjective interpretations of symbols and the analysis will be based on Steinbring's (2005) epistemological triangle.

THEORETICAL BACKGROUND

Rooted within the socio-cultural perspective, I take the position that learning, and the use of language are inseparable phenomena (Vygotsky, 1978). Within this perspective, *mediation* describes the communicative interplay between *tools* and *signs* (Vygotsky, 1978), where tools can be understood as physical utilities, like a piece of chalk or a pencil. Several approaches and models have been offered to explain the meaning of signs, like dyadic and triadic models. In dyadic models, the object (signified) is represented through a certain symbol (signifier), and the sign is constituted of both, taken together (Walkerdine, 1988). For this study, I will adapt a triadic model, as suggested by Steinbring (2005) in terms of the so-called *epistemological triangle*. Steinbring's main idea is that mathematical signs do not have a meaning of their own, and therefore meaning should be "produced by students or teacher by establishing mediation between signs/symbols and a suitable reference context" (Steinbring, 2005, p. 22). In this sense, two functions can be associated with mathematical signs:

1) A semiotic function: the role of the mathematical sign as "something which stands for something else".
2) An epistemological function: the role of the mathematical sign in the context of the epistemological interpretation of mathematical knowledge.

(Steinbring, 2005, p. 21)

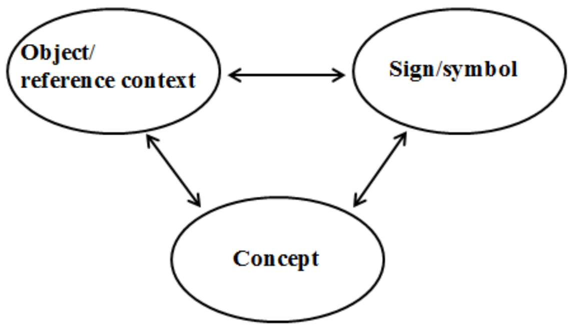

Figure 1: The epistemological triangle (adapted from Steinbring, 2005, p. 22).

The "object/reference context" in the epistemological triangle represents what the sign/symbol may refer to. In this model the epistemologically grounded mediation between the object/reference context and the sign/symbol is emphasized. At the same time, this mediation, with its epistemological possibilities and constraints, also allows for the construction of "new and more general mathematical knowledge" (Steinbring, 2005, p. 22). In the context of this study, the upper left corner is labeled "reference context" as I find this term more clearly to open for different interpretations among the students. The upper right corner, I label "signs". Signs of course include symbols, but also open for other possibilities, like the use of technical terms, sketches or gestures. Within this framework, it is also possible to construct *semiotic chains*.

> Furthermore, one can accordingly draw up a sequence of epistemological triangles for the interaction, or a sequence of learning steps to reflect the development of interpretations made by the subject (Steinbring, 2005, p. 23).

Examples of signs could be the symbols $\int$ and dx and possible reference contexts for these symbols among the students might be "area under a curve", "summation of small magnitudes" or a series of operations like "finding the anti-derivative". The lower corner labeled "concept" is the mediated mathematical meaning gained from the interplay between the reference context and the signs. This could be subjective, and not necessarily in line with mathematically correct definitions. If a student for example link the concept "differential", represented by the symbol dx, solely to a reference context consisting of the convention "with respect to x", it is not very likely that this student's interpretation of dx fully covers the mathematically definition of differentials. Hence, in that example, the students' "concept corner" of the epistemological triangle probably would differ from the mathematicians.

Historically, in the context of symbols and integration, it should be emphasized that meaning of symbols in integration has changed. Ely (2017) points out that in most textbooks, dx and $\int$ are still used, but without the meanings Leibniz assigned to these, rooted in the idea of infinitesimals. Instead, integrals are now often presented and accounted for in terms of limits. The original meaning of symbols in some sense then become vestiges and no longer directly represent quantities that students can manipulate. To overcome this dilemma, teaching projects like DIRACC (*Developing and Investigating a Rigorous Approach to Conceptual Calculus)* (Thompson, 2018) and "an informal approach to infinitesimals" (Ely, 2017) have been carried out. In DIRACC one emphasizes that variables vary smoothly, and that differentials are variables. Differentials are conceptualized by letting x vary "by dx through intervals of length Δx" and be letting dy vary "at a constant rate with respect to dx" (Thompson, 2018, p. 2). Ely (2017) approaches differentials in similar terms, by letting the differentials refer to actual, small quantities that varies. The underpinning theory of hyperreal numbers, that mathematically justifies this approach, is not made explicit to the students and the approach is therefore labeled as an "informal infinitesimal approach" (p. 155). Common to both Thompson's (2018) and Ely's (2017) ideas for teaching, is that differentials are treated more like the historical origins of infinitesimals rather than limits, as opposed to what commonly is preferred in modern calculus textbooks.

METHODOLOGY

By focusing on "how individuals make sense of the world" this research takes a phenomenological approach (Bryman, 2016). A qualitative study was carried out with the aim of gaining insight into several individual students' reasoning and experiences with integration and could in this sense be regarded as a multiple case study. 15 students were individually interviewed for about 40 minutes through semi-structured interviews. These were the students from four randomly selected "working groups" from an earlier research project, which in turn was selected based on voluntariness. During the interview, the students were asked to explain what an integral is, followed by the challenge of describing, in their own words, the content of the FTC as displayed in their textbook (figure 2).

The Fundamental Theorem of Calculus

Suppose that the function f is continuous on an interval I containing the point a.

PART I. Let the function F be defined on I by

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is differentiable on I, and $F'(x) = f(x)$ there. Thus, F is an antiderivative of f on I:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

PART II. If $G(x)$ is *any* antiderivative of $f(x)$ on I, so that $G'(x) = f(x)$ on I, then for any b in I we have

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Figure 2: The Fundamental Theorem of Calculus as presented to the students (adapted from Adam & Essex, 2018, pp. 313-314).

Follow-up questions were based on the students' own statements. Subsequent to these rather "open" questions, the students were asked to interpret the meaning of the symbols in the expressions, respectively $\int$ and dt. Towards the end of the interview, if the students did not bring it up themselves, I asked them to reflect on Riemann sums and how these might relate to integrals.

The interviews were coded and transcribed in several stages, focusing on students' interpretations of meaning related to symbols and their accounts of Riemann sums and integration. In the analysis, these two parts of the interview are treated rather holistically, aiming to collectively account for the students' interpretations of meaning. In this paper, I focus on two students, "Eric" and "Matt". These two students are selected since they expressed multifaceted symbol interpretations, involving both conventions and conceptual aspects. Further, I found their conceptual challenges arising from the mediation between the symbols and the different reference contexts to be shared by several of the students involved in the study. It is therefore my aim that the insights gained from an analysis of Eric and Matt's accounts, could contribute to illuminate some core issues, relevant for several of the cases involved in this study. In turn, this might also contribute to raise some discussions relevant to students in introductory calculus courses, in general.

ANALYSIS

Based on the research question, the subsequent analysis is focusing on students' reasoning and reflections, rather than teaching, but to provide some context, a small summary of the observed teaching is still offered. The introduction to integration took place during two lectures, each lasting for two hours. Briefly, Riemann sums were introduced, and examples of areas under curves, in terms of polynomial functions, were dealt with. These areas were calculated from concrete numbers, by using limits of areas of sums of bars, to be obtained from summation formulas already known to the students. Subsequently, the general Riemann integrals were introduced by taking the limit of such sums. Finally, a visual proof of the FTC was offered in terms constructing the accumulating area function, $A(x)$ for $f(x)$ and it was showed that the derivative of $A(x)$ resulted in $f(x)$.

The analysis to follow is based on interviews with two students from the computer engineering programme. As accounted for in the previous section, only a part of the interview is relevant for the subsequent analysis. The students were asked to explain in their own words what is meant by integration, followed by reflections on the Fundamental Theorem of Calculus, as displayed in the textbook (figure 2). They were asked to give a general comment in terms of verbally interpreting what they saw, followed by some specific questions giving attention to the symbols $\int$ and dt from the first expression (figure 2). Drawings and sketching along with their verbal explanations were encouraged during the conversation.

The case of Eric

Eric expressed that he was uncomfortable with the definition of an integral, and to a large extent interpreted the symbols only as mathematical operations or conventions. Being presented with the FTC the focus in the following excerpt is on the first expression (figure 2).

Interviewer: I wondered if you could take me through the use of symbols here [points to the expression] and tell me a bit about what they mean, and how you interpret them?

Eric: […] Basically everything between the s [integral sign] and the dt that is what should be integrated. In this case that is ft

Interviewer: Why does dt –

Eric: -since it is dt instead of normally dx, this means with respect to t

Interviewer: Is everything that could be said about dt, that the function should be integrated with respect to dt, the way you see it?

Eric: Yes

This excerpt illustrates several students approach to the question of interpreting the symbols in this expression. From a semiotic perspective, in this case the signs primarily stand for a series of operations in terms of $\int$ representing the process of integration through anti-derivates and dt is interpreted to mean "with respect to t". The positive response to the last question combined with a holistic analysis of the interview, do not indicate that these symbols, in this setting, had any additional meaning to Eric. Later in the interview an attempt was made to approach the definition of Riemann integrals:

Interviewer: Why is it, do you think, that the integral between two points, gives an area as an answer? Have you thought about that?

Eric: I do not know why, I have only learned that that is the way it is, but I do not know the reason.

Since it was known to me that Riemann sums and estimation of areas under curves had been dealt with in the lectures, I mention the word "Riemann sums" and showed a picture from one of the tasks they had been working with where the point was to estimate the area under a curve by drawing suitable bars and add the areas.

Interviewer: What do you do when you divide into different bars?

Eric: You estimate.

Interviewer: What happens to the bars when you calculate the integral?

Eric: One takes the area of the bars.

By using the word "estimate" and by mentioning that "one takes the area of the bars", Eric to some extent demonstrates that he manages to estimate the total area by adding the areas of the bars. Still, he is not explicit on limits and hence, the connection to integration. For potential follow-up comments from Eric, I continued by offering an explanation, to hear his reaction.

Interviewer: If we let the width approach zero, then it would be more exact.

Eric: Yes […] this is something that always have stressed me. Ok, here we have an estimate, but if we say that it approaches zero it suddenly becomes accurate.

From Eric's reflection it seems like he at some point have heard about limits, but when it was brought up, he expresses confusion towards the idea that limits lead to accurate answers. Since at no point in the interview, from Eric's side, any explicit connections between Riemann sums and integration are made, one can assume that he treats these as rather separate mathematical ideas. For Eric, the mediated meaning of integrals only seems to involve mathematical operations in terms of the anti-derivative and the calculation of areas. Riemann sums act more like a stand-alone activity, involving estimation of areas, almost like an alternative to integrals, for example when the formula is unknown.

The case of Matt

Matt was a talkative student, and through his interpretations of symbols he demonstrated some important connections, but he still struggled with some underlying conceptual issues. With the FTC presented, Matt quickly takes the word before any specific questions have been posed:

Matt: Before I watch this too much, and without cheating, I can say that to integrate something is the opposite of finding the derivatives of something, and this is what this whole theorem is about.

From Matt's statement it is apparent that integration is associated with anti-derivatives, and that this view seems to constitute the main part of the mathematical meaning he associates to the concept. Next, an attempt of approaching Riemann sums and the definition of an integral was made, based on some discussions concerning the integral being an area under a curve.

Interviewer: Why does the antiderivative result in an area under a curve?

Matt: […] Like I want to think about this is that you have these heights and bars, and then you just multiply the areas of these bars […] And one thing that I have a problem with is when you have this [points at the upper right corner of a bar under a curve on a drawing] and this suddenly becomes a point. How did we get there?

From the context, I interpret Matt to mean "add" instead of "multiply", and that he connects the sum of these areas of bars to the concept of integration. Without explicitly mentioning limits, or that the width of the bars approaches zero, I still interpret Matt's statement, combined with his gestures, to point to the same dilemma as Eric. As for Eric, Matt describes a seemingly paradox that arises from approaching integrals through sums of areas of bars with a certain width. By the statement "I have a problem when […] this suddenly becomes a point" Matt expresses the difficulty of adding up the areas of these bars when the width becomes zero, and one are left with an infinite number of vertical "lines". Going back to the symbols in the presented expressions, Matt elaborates on the symbol dt.

Matt: One thing I find problematic is this d. It happens something special each time you put this d in front of [from the context, Matt probably means "after" or "next to"] a variable. I like to think about this as an infinite small number, like a constant, and this is multiplied with for example x, if we want the width of our bar. But I am not allowed to do calculation with it, like I could if it was a constant.

Interviewer: What makes you say that you are not allowed to do calculations?

Matt: Well, for example if it says dy dx [means dy/dx] then it looks like a fraction, and sometimes I can do calculations on this like with fractions, but not always. That is the impression that I have got. Sometimes I can regard this as

> a fraction, but it is not really a fraction, it is something else which share the same symbol.

In his first statement, Matt conceptualizes d as an "infinite small number", which is in line with how one could define this in classical infinitesimal calculus, historically rooted in Newton and Leibniz. Further, the difficulties arise when Matt equates this "infinite small number" with "a constant". In turn, this causes the essence of Matt's confusion as small constants, no matter how small, could be treated exactly like "normal" fractions. In Matt's second statement it seems like his confusion is enforced by experiences he had involving such differentials, as they in some case can be regarded and treated like fractions (for example in linear differential equations or in integration tasks involving substitution). On the other hand, he is aware that this is not "really a fraction".

DISCUSSIONS AND CONCLUSIONS

Although 15 students were interviewed in total, the examples illustrated through the cases of Eric and Matt exemplify the main issues that several of the students struggled with. The connection between integrals and Riemann sums were vague, and among those students who made such connections, the limit of these sums was hard to conceptualize. These difficulties were expressed in terms like "how can one take the sum of bars with no width" or, as in Eric's case, who are "stressed" by the view that "we have an estimate, but if we say that it approaches zero it suddenly becomes accurate". The mathematical meaning in this sense strongly relates to an underlying idea of limits which, historically speaking, is a newer idea than the original ideas of infinitesimals. Ely (2017) points out that in most textbooks, dx and $\int$ are still used, but without the meanings Leibniz assigned to these. Instead, modern calculus textbooks often reformulate integrals in terms of limits. The notations in some sense then become vestiges and no longer directly represent quantities that students can manipulate. In both Eric's and Matt's case a challenge seemingly appeared in the reference contexts from first interpreting an estimated area as the sums of bars with a certain width, followed by the reference context of accurate area as the "sum of bars with no width". According to Ely (2017), this ambiguity is not easy to solve, unless one introduces hyperreal numbers to substantiate the algebraic sense-making of infinitesimals. Semiotically, and phrased in the language of Steinbring's (2005) epistemological triangle, one can model the mediated meaning as the idea that bars estimate the area, and that the area becomes more accurate when the width of these bars becomes smaller. The confusing next step for the students was to make sense of this sum when the width *becomes* zero, which to some implied a sum consisting of an infinite number of vertical lines with no width.

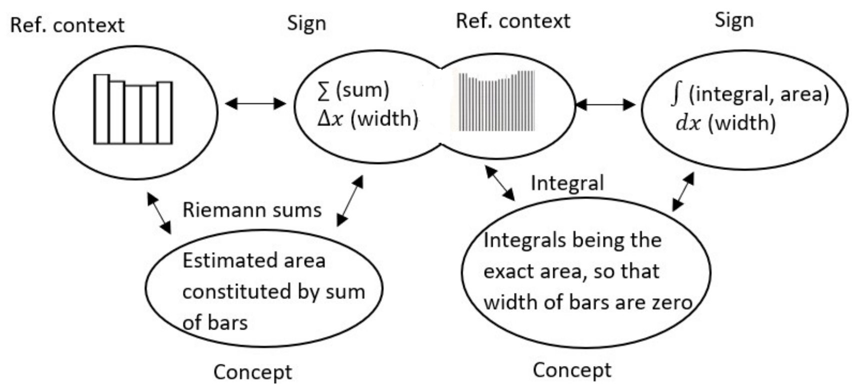

Figure 3: Semiotic chain of epistemological triangles to illustrate students' mediated meaning of Riemann sums and integrals

From a semiotic perspective, figure 3 illustrates students' mediated meaning related to Riemann sums and integrals. Their concepts arose from verbally interpreting the symbols and what the symbols represent. In the first part of the chain, most students, like Eric and Matt, somehow associated the Riemann sums with estimation of the area under a curve through a sum of area of bars with width Δx. For many students, the confusion arose when they should evaluate the exact area in this manner, which eventually led to the interpretation of the curve area as the sum of area of bars having width equal to zero. This observation could find its explanations in earlier studies, like Oehrtman (2009), where students turned out to have inappropriate metaphors for limits in terms of focusing on points in sequences rather than for example a continuous motion approaching something. In Eric and Matt's cases, this way of reasoning was evident as they quantified Δx as a constant, instead of for example regarding this as something that varies. This also involved treating the limit itself (zero) as if it was one of many possible values of Δx. According to Thompson (2018), key elements in understanding differentials is the ability to regard these as variables, and through the emphasis on differentials as something that vary, such confusions might be avoided.

Another aspect that appeared throughout the interviews was the tendency that students viewed integrals and limits of Riemann sums as rather separate phenomena. Students' reference context for the symbols dx and $\int$ was mainly interpreted to be mathematical conventions and operations in terms of "with respect to x" and "finding the anti-derivative". Only when directly asked, some students offered attempts of expressing more conceptual aspects. One can hypothesize that this phenomenon is influenced by how Riemann sums are presented in teaching and in student activities. In this sense, findings support Wagner's (2018) claim that "too many students dismiss Riemann sums as an unpleasant stepping-stone to be endured in a curriculum whose goal was really to get to the FTC" (p. 354). As pointed out by Thompson and Silverman (2008), Riemann sums bear the potential of playing a major role for the students' perception of integrals as accumulation functions, which in turn could contribute to students' understanding of the FTC. For the 15 students in this study, the neglection of integrals as an "accumulation function", enforces the suspicion that the potential of Riemann sums is not sufficiently utilized in teaching. In this respect, Bressoud (2011) suggests that if we want students to see the need for evaluating limits of Riemann sums, we ought to provide students with good unfamiliar problems involving accumulation.

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Exploring undergraduate engineering students' competencies and attitudes towards mathematical problem posing in integral calculus

Farzad Radmehr[1,2], Mahboubeh Nedaei[2], and Michael Drake[3]

[1]University of Agder, farzad.radmehr@uia.no, [2]Ferdowsi University of Mashhad
[3]Victoria University of Wellington

The present study explores engineering students' mathematical problem posing competencies in relation to integral calculus, and their attitudes towards mathematical problem posing. The sample comprised of 135 undergraduate engineering students from a public university in Iran. Students' problem posing competencies were explored using a test including eight problem posing tasks related to the fundamental theorem of calculus and integral-area relationships. Furthermore, students completed a questionnaire that explored their attitudes towards mathematical problem posing. Nine students also participated in a semi-structured interview. The findings show that many students could improve their problem posing abilities further, and around 60 percent of students had positive attitudes towards mathematical problem posing activities.

Keywords: calculus, mathematics for engineers, mathematical problem posing, attitudes towards problem posing, integral calculus.

INTRODUCTION

Engineering relies heavily on using mathematics for building and designing projects for the use of society, however, attracting and retaining students in engineering degrees are sometimes problematic because of the role of mathematics in engineering (Flegg, Mallet, & Lupton, 2012). Mathematical problem posing can be considered as one of the central activities in mathematics and is a useful tool in mathematical teaching and learning (NCTM, 2000). Problem posing referred to "the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems" (Stoyanova & Ellerton, 1996, p. 1). To pose mathematical problems, several skills are required such as the abilities to formulate mathematical situations, recognizing relationships between different mathematics concepts, and choosing an appropriate approach for each situation (Abu-Elwan, 1999). Problem posing has potential benefits for improving the quality of teaching and learning of mathematics. For instance, it could help students to develop their mathematical understanding (e.g., Cai & Hwang, 2002) and problem-solving skills (Cai & Hwang, 2002). Also, it can help teachers to identify students' mathematical misconceptions and difficulties (Chen, Van Dooren, & Verschaffel, 2015).

Calculus is an important topic in advanced mathematics, and has various applications in other disciplines such as physics and engineering (Jones, 2015). It is essential for students to fully understand calculus concepts and be able to apply them in different situations (Mahir, 2009). Integral calculus is a valuable topic in calculus, and is a prerequisite for further coursework (Sealey & Oehrtman, 2005; Thompson &

Silverman, 2008). It consists of important concepts such as the fundamental theorem of calculus and integral-area relationships.

Many studies have explored students' attitudes towards mathematics and mathematical problem solving (e.g., Lim & Chapman, 2013; OECD, 2013). Most of these studies have shown a close relationship between various domains of attitudes towards mathematics and mathematics achievement (Lavy & Bershadsky, 2003; OECD, 2013; Samuelsson & Granstrom, 2007). However, a few studies have explored students' attitudes towards mathematical problem posing (e.g., Chen et al., 2015). Considering the potential value of problem posing activities in the teaching and learning of mathematics, this study explores undergraduate engineering students' problem posing competencies in relation to integral calculus, and their attitudes towards mathematical problem posing. Therefore, the research questions of this study are: What are undergraduate engineering students' competencies in posing problems related to integral? And what are their attitudes towards mathematical problem posing activities?

LITERATURE REVIEW

The literature review section reviews the previous studies related to problem posing, integral calculus, and attitudes towards mathematics.

Problem posing

Problem posing activities offer potential benefits to develop students' mathematical understanding. Problem posing activities could have a positive influence on students' creativity (e.g., Bonotto & Dal Santo, 2015), attitudes toward mathematics (e.g., Chen et al., 2015), and critical thinking skills (Nixon-Ponder, 1995). Furthermore, several studies have reported that there is a close relationship between students' problem posing and problem-solving competencies (Cai & Hwang, 2002; Silver & Cai, 1996; Xie & Masingila, 2017). For instance, Silver and Cai (1998) analysed middle school students' responses to problem posing and problem solving tasks. They found that problem solving and problem posing performance are closely related, and successful problem solvers can pose more complex mathematical problems compared to unsuccessful problem solvers. Several frameworks have been proposed to design problem posing tasks (e.g., Christou, Mousoulides, Pittalis, Pitta-Pantazi and Sriraman, 2005; Stoyanova & Ellerton, 1996). For instance, Christou et al. (2005) have designed a taxonomy for designing problem posing tasks that has four categories: *Editing quantitative information*- posing problems without restriction, *selecting quantitative information*- posing problems based on a given answer, *comprehending quantitative information*-posing problems based on a given calculation/equation, and *translating quantitative information*- posing problems based on a given graph, diagram or table (Christou et al. 2005). In relation to analysing students' posed problem, different frameworks have been proposed (e.g., Leung, 2013). Recently, Cankoy and Özder (2017) have proposed a framework that can be used to analyse students' posed problems across five dimensions: solvability; reasonability; mathematical structure; context; and language.

Integral calculus

Many studies have reported that students have various misconceptions in learning integral calculus (Jones, 2013; Kouropatov & Dreyfus, 2013; Radmehr & Drake, 2017, 2019; Sealey, 2014). Integral calculus includes important topics such as the Fundamental Theorem of Calculus (FTC) and integral-area relationship. FTC links definite and indefinite integrals and is often used to solve definite integral problems (Radmehr & Drake, 2017). Several studies have highlighted that many students rely on learning routine procedures and integral techniques, and do not develop a conceptual understanding of integral calculus (e.g., Radmehr & Drake, 2019). Sealey (2014) explored students' understanding of the definite integral, and suggested a framework to characterize students' understanding of Riemann sums and the definite integral. The results indicated that "conceptualizing the product of $f(x)$ and Δx proves to be the most complex part" (p. 230) for students. Radmehr and Drake (2017) have explored university students' mathematical performance, and metacognitive experiences and skills in relation to FTC. The results showed that several students had difficulties in solving problems related to the FTC. For example, in relation to $F(x) = \int f(x)dx$, many students did not understand that $f(x)$ is the rate of change of the accumulated area function $F(x)$.

Attitude

Attitude could be defined as "a predisposition to respond to a certain object either in a positive or in a negative way" (Zan & DiMartino, 2007, p. 28), consequently, students' attitudes towards mathematics underlie their tendency to engage in mathematical activities. Students' attitudes towards mathematics can impact directly on students' mathematical learning, problem solving, and achievement (Ngurah & Lynch, 2013; Sarouphim & Chartouny, 2017). Positive attitudes towards mathematics can encourage students to engage more in mathematical learning activities (Singh Granville, & Dika, 2002) while negative attitudes towards mathematics can increase students' mathematics anxiety (Trujillo & Hadfield, 1999). Several studies have reported that there is a strong relationship between different attitude domains (e.g., enjoyment of mathematics; motivation to do mathematics) and mathematics achievement (e.g., Ngurah & Lynch, 2013; OECD, 2013; Sarouphim & Chartouny, 2017). Though, a literature search exposed only one study which explores students' attitudes toward problem posing (Chen et al., 2015). Chen et al. (2015) investigated students' problem posing and problem solving competencies, as well as their attitudes towards mathematical problem posing and problem solving. Their findings showed that problem posing activities had a positive impact on students' problem-solving abilities, and attitudes towards problem posing and problem solving also improved.

RESEARCH METHODS

The present study takes a sequential explanatory mixed method approach. To form a comprehensive understanding of students' problem posing abilities and their attitudes towards problem posing in mathematics, first, students participated in a problem posing test, and completed a questionnaire about their attitudes towards problem posing. Then, nine students were invited to participate in semi-structured interviews. The sample comprised of 135 undergraduate students from different engineering majors of a public university in Iran. For the problem posing test, eight problem posing tasks were designed based on Christou's problem posing taxonomy (2005) related to two topics in integral calculus: the FTC and integral-area relationships. The attitude questionnaire consisted of twelve items on a five-point Likert-style scale and two open-ended questions. To illustrate, two items of the questionnaire were "I get a great deal of satisfaction from posing a mathematical problem" and "By practicing mathematical problem posing, I become a better mathematical problem solver". The problem posing test and the attitude questionnaire were piloted with nine students from an engineering calculus 1 course. After piloting and refining, 135 students participated in the problem posing test and completed the attitude questionnaire. Students' problem posing abilities were analysed using an adapted version of Cankoy and Özder's (2017) rubric. Using purposeful sampling, nine students with different levels of performance on the problem posing test were selected to participate in a semi-structured interview. To explore the validity of the attitude questionnaire, two senior lecturers in mathematics education examined the readability of the questionnaire items, and factor analysis was also conducted to examine the relationships among the questionnaire items. To explore the reliability of the questionnaire, Cronbach's alpha was calculated, the value 0.89, indicates that the questionnaire items had good internal consistency. To explore the validity of the problem posing test, two senior lecturers in mathematics examined the problem posing tasks and then it was piloted with nine students.

RESULTS

This section comprises the results of analysing responses to the problem posing tasks, the attitude questionnaire, and students' responses to the interview questions. The results of two tasks are described in this paper because of the page limits, one related to the integral-area relationship (Figure 1) and one related to the FTC (Figure 3).

Students' responses to the problem posing tasks

Task 1 is classified as *translating quantitative information* based on Christou's (2005) problem posing taxonomy because students are asked to pose a problem based on the given graph. Ninety-eight out of 135 (72.6%) students posed a problem for this task, however, the remaining 37 (27.4%) did not pose any problem. Students' posed problems were classified into three categories (Table 1).

Task 1. Please write a problem based on the given graph which its solution would require using area under curves (The red graph is $y = (x-1)^3 + 1$ and the green is $y = x$).

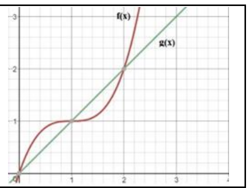

Figure 1. Task 1

Categories	A sample response	N
Finding the enclosed area between two curves	Find the area between $f(x) = (x-1)^3 + 1$ and $g(x) = x$ in [0,2].	77 (79%)
Calculating integral	Calculate the following integral: $$\int_0^2 ((x-1)^3 + 1 - x)dx$$	11 (11%)
Real-world context	Two runners are in a running competition. The first runner runs with the speed of $v(t) = (t-1)^3 + 1$. The speed equation of the second runner is $v(t) = t$. Calculate the displacements of these two runners after one minute?	10 (10%)

Table 1. Posed problems for the integral area-relationship task (N=98)

Furthermore, the results showed that 90 out of 98 (92%) problems were solvable and only 8 (8 %) problems were unsolvable. Ten (10%) problems were based on real-world context while 88 (90%) problems were 'bare tasks without contexts' (Vos, 2020). Ninety out of 98 (92%) posed problems had clear language and only 8 (8%) problems were not clear. Furthermore, analysing students' posed problems showed that many students had several difficulties when posing problems. The interviewed students were asked to pose a new problem for each task during the interviews, and also solved their posed problems. During this process, also some difficulties were identified. Students' difficulties in relation to posing a problem for Task 1 are summarized in Table 2. For example, 30 students in the problem posing test and three interviewed students did not understand that the enclosed area between curves should be always positive as some of them calculated the enclosed area between the two curves zero or negative (Figure 2).

$$\int_0^2 ((x-1)^3 + 1 - x)\,dx = \int_0^2 (x-1)^3\,dx + \int_0^2 dx - \int_0^2 x\,dx = \left[\frac{(x-1)^4}{4} + x - \frac{x^2}{2}\right]_0^2$$

$$= \frac{1}{4} + 2 \rightarrow -\frac{1}{4} = 0$$

Figure 2. A sample response to Task 1

Types of difficulty	Test	Interview
Difficulties with the concept of signed areas	30 (22.2%)	3 (33%)
Not checking whether the upper and lower functions change within the enclosed area	15 (11.1%)	7 (78%)
Difficulty with identifying applications of enclosed area between curves in the real world	-	6 (67%)

Table 2. Students' difficulties in Task1

Task 2. “Please can you write a problem based on the following graph whose solution would require using the FTC?” (Radmehr & Drake, 2017, p. 1052).

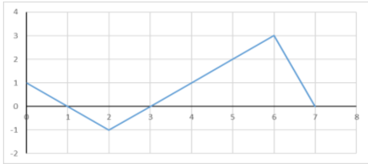

Figure 3. Task 2

Task 2 is also classified as *translating quantitative information* based on Christou’s (2005) problem posing taxonomy as students are required to pose a problem based on the given graph. Forty-two (31.1%) students posed a problem for this task, however, the remaining 93 (68.8%) did not. The posed problems were classified into two categories (Table 3). Forty out of 42 (95.2%) problems were solvable and two (4.8%) were unsolvable. Twenty (47.6%) problems were based on a real-world context while 22 (52.3%) problems were bare tasks without contexts. Forty-one (97.6%) posed problems had clear language, and only one problem was not clear.

Categories	A sample response	N
Find the area under curves	Find the enclosed area between x-axis and the given graph in [0,7].	22 (52%)
Real-world context	The given graph shows the speed of a car between t=0 and t=7 minutes. Calculate the distance travelled by the car.	20 (48%)

Table 3. Posed problems for to the FTC task (N=42)

Students also had several difficulties when posing problems for Task 2 in the test and during the interviews (Table 4). The results showed that ten students in the problem posing test and six out of nine interviewed students had difficulties in identifying how FTC could be used in the real-world. For instance, a posed problem was “the given graph shows the distance a man ran in a running competition. Calculate the acceleration of the man between t=0 and t=5”. In this problem, it seems the student incorrectly thought integrating the displacement equation, that could be obtained from the graph, results acceleration function. However, integrating an acceleration equation results the velocity function, and integrating the velocity equation results the displacement function.

Types of difficulty	Test	Interview
Difficulties in understanding the role of $F(x)$ in the FTC	13 (30%)	7 (78%)
Difficulties in understanding the applications of the FTC in the real-world	10 (23.8)	6 (67%)
Difficulties in calculating antiderivatives	5 (11.9%)	4 (44%)

Table 4. students’ difficulties in Task 2

Students’ attitudes towards mathematical problem posing

Students’ responses to the attitude questionnaire showed that over 50 percent of students enjoyed the problem posing activities, and more than 60% of students believed that problem posing and problem solving are closely related. Students’ responses to open-ended questions showed that they believed engaging in problem posing activities help them to develop their mathematical understanding. For instance, one student said “practicing problem posing activities might increase our creativity in mathematics and also helps us to solve more complicated problems which need more creativity”. The results of the interviews showed that eight out of nine students believed problem posing tasks are enjoyable activities and could be included in the teaching of mathematics. An examples was: “After I posed problems, I finally understood the applications of the mathematics we learned in the school and university. In fact, problem posing activities make mathematics more practical and bring it to our real life”. These eight students also expressed that problem posing tasks could be used in the mathematical exams.

DISCUSSION

The present study explored undergraduate engineering students’ competencies and attitudes towards mathematical problem posing in integral calculus. The findings showed that many engineering students could develop their problem posing skills. Of the 1080 problems that potentially could have been posed for the eight tasks, only 501 (46%) problems were posed. Of these 501 problems, 411 (81%) were solvable which was consistent with previous studies which have reported most of the students’ posed problems are solvable (Bonotto & Dal Santo, 2015). One possible reason for the high percentage of solvability in the present study is that many of the posed problems were bare tasks without contexts. Moreover, only 157 problems (31.3 %) were based on the applications of integrals in the real world and 344 problems (68.6 %) were bare tasks without contexts which could be an indication of students’ lack of knowledge about the applications of integrals in the real world. The language used in the 440 of the posed problems (88%) was clear and understandable which might indicate that students at university level could pose clear and understandable problems. Furthermore, the study findings suggest that problem posing tasks could be used by teachers and lecturers to explore students’ mathematical understanding. In this study, using problem posing tasks, several students’ difficulties in relation to integral calculus were identified. The difficulties that have been identified are in line with previous studies that have explored students’ understanding of integral calculus (e.g., Mahir, 2009; Radmehr & Drake, 2017, 2019). In relation to students’ attitudes towards mathematical problem posing, the findings showed that more than 50% of the engineering students believed problem posing is an enjoyable activity. This is consistent with previous studies (Arikan & Ünal, 2015) which have reported that students enjoyed practicing problem posing tasks. Students also expressed that problem posing tasks could improve their mathematical learning, and they brought several reasons for their responses. For examples, they mentioned problem posing activities help them to foster their creativity, and identify their mathematical misunderstandings. To conclude, this study suggests that problem posing activities could be used to improve the teaching and learning of integral calculus in engineering mathematical courses as the problem posing tasks could identify students' difficulties in integral calculus and motivate them to improve their understanding of applications of integral calculus in real life. Moreover, since many students believed problem posing activities are enjoyable and help them to improve their mathematical learning, using such tasks could encourage students to be more active in mathematical classrooms, and might motivate them to learn mathematical concepts meaningfully.

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Impact of attitude on approaches to learning mathematics: a structural equation modelling approach

Yusuf F. Zakariya[1], H. K. Nilsen[1], Kirsten Bjørkestøl[1], and Simon Goodchild[1]

[1]University of Agder in Kristiansand, Faculty of Engineering and Science, Norway, yusuf.zakariya@uia.no

This study aims at validating an attitude subscale of a national mathematics test that has been repeatedly used for over two decades and to expose the relations between students' attitude towards mathematics and their approaches to learning mathematics. A sample of 196 year-one engineering students completed two survey instruments used for the study. Using a structural equation modelling approach, empirical evidence of construct validity, discriminant validity and reliability were found for the attitude subscale. Further, it was also found that students' attitude towards mathematics had a substantial positive impact on deep approaches to learning and a substantial negative impact on surface approaches to learning. These findings could be of help to university teachers and other stakeholders in designing appropriate interventions to support the students.

Keywords: students' practices, deep approach, surface approach, attitude, structural equation modelling.

INTRODUCTION

Approaches to learning in higher education are part of students' practices that have a considerable effect on learning outcomes. Students who care for every detail in their course content with the intent to achieve conceptual understanding (deep approach) are more likely to perform better than others who only rely on memorization of key points (surface approach). Approaches to learning have been conceptualized to include "predispositions adopted by an individual when presented with learning materials and strategies used to process the learning contents" (Zakariya, Bjørkestøl, Nilsen, Goodchild, & Lorås, 2020). It is an essential factor in students' practices that has received increased attention in recent times. Perhaps, as a result of international campaigns on aligning university education towards developing learners' deep approaches that will enable them to navigate easily through an increasingly changing society.

Several empirical studies have been reported on the factors that encourage or discourage the adoption of either deep or surface approaches to learning. One of these studies is a critical review by Baeten, Kyndt, Struyven, and Dochy (2010). Therein, a total of 118 empirical studies were reviewed, and the results can be summarized as follows: satisfaction with course quality, big five personality traits except for neuroticism, and emotional stability are some of the factors that stimulate adoption of the deep approaches to learning. It was also found that students that experience intrinsic motivation, and who are self-efficacious and self-confident are most likely to adopt deep approaches to learning. In a follow-up quasi-experimental study Baeten, Struyven, and Dochy (2013) investigated the contribution of some teaching methods on students' approaches to learning. They found that adoption of deep approaches to learning decreases among the participants in a lecture-based group while they remain stable in a student-centred learning environment over a period.

More so, Von Stumm and Furnham (2012) conducted an empirical study involving 579 psychology and computer science undergraduate students on relations between approaches to learning, personality, intelligence and intellectual engagement. It was found that deep approaches to learning strongly related to intellectual engagement while personality and intelligence explained 25% variability in surface approaches to learning among the subjects of their study. In an attempt to unravel the interwoven bond between critical thinking, self-efficacy and learning approaches, Hyytinen, Toom, and Postareff (2018) conducted an empirical study involving 92 science education undergraduate students in Finland. Their results showed that students with high self-efficacy also adopt deep approaches to learning. Some researchers have studied relations between approaches to learning and other factors in domain-specific contexts. For instance, Mji (2000) found that there was a strong relationship between students' different conceptions of mathematics and their approaches to learning the subject.

Despite the importance of approaches to learning and its relations with some affective constructs, e.g. self-efficacy there are few studies on its relations with attitudes of students. One of the relatively recent studies on this topic is the report by Alkhateeb and Hammoudi (2006) on the relations between attitude towards mathematics and students' approaches to learning. In their study, students with a positive attitude towards mathematics were identified with deep approaches to learning while those with a negative attitude towards mathematics were identified with surface approaches to learning approaches. However, their study had some methodological issues such as the use of regression analysis to examine the relations between these constructs, given that the regression analysis does not account for measurement errors in the predictor variable(s). Another methodological issue in their study involved the use of mean scores derived from item parcelling of ordinal variables which could lead to biased results because of violation of multiple assumptions, e.g. tau-equivalent, and normal distribution (Zakariya, 2020).

Thus, the present study was motivated by the sparsity of studies on the relationship between attitude towards mathematics and approaches to learning coupled with some methodological issues observed in available studies (e.g., Alkhateeb & Hammoudi, 2006). Further, to the best of our knowledge, there was no validation study on the attitude subscale of the Norwegian national mathematics test for the past fifteen years. The national mathematics test is a test that is conducted every two years and designed to assess pre-university knowledge of mathematics of year-one undergraduate students across universities in Norway. The validity of this test is essential to ensure the test measures what is purported to measure, which will facilitate more accurate interpretations of its ensuing results. Therefore, the primary purposes of the present study are to use a structural equation modelling approach to (a) validate the attitude subscale of the Norwegian national mathematics test; (b) expose the impact of attitude towards mathematics on students' learning approaches. The use of the structural equation modelling approach will avail us an opportunity of taking of care of the two methodological issues involved in the use regression analysis that is typically used in the literature (e.g., Alkhateeb & Hammoudi, 2006). In the next section, a conceptual framework coupled with a theoretical perspective that justifies the rationale for finding the relations between these constructs is discussed.

CONCEPTUAL FRAMEWORK

A theoretical structure that could be used to justify the relations between attitude towards mathematics and approaches to learning is social cognitive theory. This theory sees an individual's behavioural changes as consistently being regulated and modified by interacting with social factors in the environment whose feedback influences the next actions and outcomes (Bandura, 2001). Central to this theory is the concept of reciprocal determinism that postulates a dynamic relationship between personal, behavioural, and environmental determinants (Bandura, 2012). Even though both the attitude towards mathematics and approaches to learning are personal factors, it is presumed that the dynamic relationship between the determinants (personal, behavioural, and environmental) can be extrapolated to within the personal determinants (cognitive, affective and biological factors). As such, a causal relationship between attitude towards mathematics and approaches to learning can be theoretically postulated. Empirical evidence has shown that students' attitude towards learning mathematics is greatly influenced by consistent interactions with teachers, peer groups and parents (e.g., Davadas & Lay, 2017). In other words, students whose teachers are efficacious, motivate them to learn, give positive feedback, maintain good teacher-student relations are more likely to develop a positive attitude towards mathematics. This, in turn, influences their approaches to learning the subject.

Several attempts have been made to conceptualize and operationalize both attitude towards mathematics and approaches to learning. Attitude towards mathematics has been conceptualized to include appraisal, valuation and enjoyment of mathematics (Zakariya, 2017). It is a construct whose multifaceted nature has influenced, to a great extent, the development of its measuring instruments (e.g., Palacios, Arias, & Arias, 2013; Zakariya, 2017). Some of these instruments have contributed significantly to the measurement of this construct as well as in relating it to other constructs from quantitative research perspectives. However, for the purpose of this study, a 5-item unidimensional attitude scale which is part of the national mathematics test in Norway, was selected. Our choice of this scale was prompted by two factors: (a) availability in the Norwegian language; (b) our quest to provide construct and discriminant validity which is lacking in the literature.

In addition, the "revised two-factor study process questionnaire" (R-SPQ-2F) has been identified as one of the best instruments for measuring students' approaches to learning

(López-Aguado & Gutiérrez-Provecho, 2018). R-SPQ-2F was chosen for the present study because of its high psychometric properties, a small number of items and ease of score interpretations. Further, Norwegian validations of this instrument have been undertaken (e.g., Zakariya, 2019; Zakariya et al., 2020). The Norwegian version has ten items on deep subscale and nine items on surface subscale with evidence of construct validity, discriminant validity, and internal consistency of its items.

Based on the postulates of the social cognitive theory coupled with previous literature, the two hypotheses of the present study are stated as follows, while Figure 1 depicts these hypothesized relations:

> (H01) There are substantial positive impacts of attitude towards mathematics on deep approaches to learning.
>
> (H02) There are substantial negative impacts of attitude towards mathematics on surface approaches to learning.

Figure 1 shows hypothesized impact of attitude measured by five items (att01 – att05) on both deep and surface approaches each measured by ten items and nine items respectively with an error correlation (indicated by the double-headed arrow) between deep and surface approaches. The plus (+) and minus (-) signs indicate the hypothesized positive and negative impacts of attitude on deep and surface approaches, respectively.

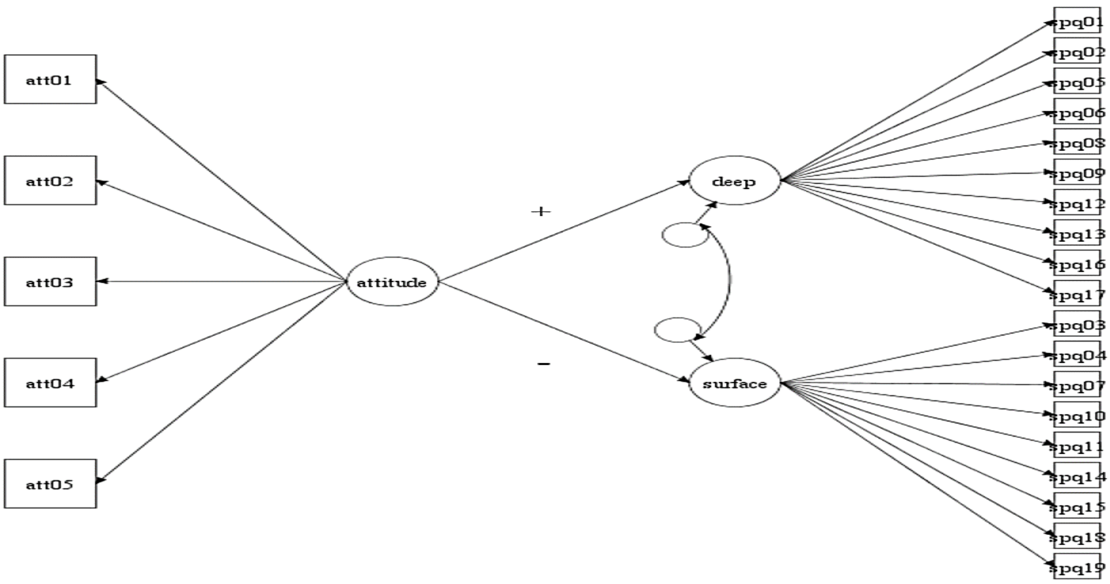

Figure 1: A hypothesized model of the relations between attitude towards mathematics and approaches to learning mathematics

METHOD

SAMPLE AND MEASURES

The sample for this study was made up of 196 year-one engineering students, including 34 females and 162 males with an average age of 24.64 years. Two online survey instruments were completed by the students, including R-SPQ-2F (Norwegian version) and attitude towards mathematics scale (AtMS). R-SPQ-2F is a 19-item questionnaire in which respondents rated their level of agreement from (1) '*never* or *only rarely* true of me' to (5) '*always* or *almost always* true of me' to statements like "I test myself on important topics until I understand them completely" (deep approach), "I see no point in learning material which is not likely to be in the examination" (surface approach), etc. On the other hand, AtMS is a 5-item scale in which respondents rated their level of agreements from (1) '*strongly disagree*' to (4) '*strongly agree*' to statements like "I work with mathematics because I like it", and "I'm interested in what I learn in math".

DATA ANALYSIS

The analyses proceeded in two stages. Stage one involved fitting a measurement model to examine the construct validity and unidimensionality of the AtMS. In this stage, AtMS data were screened for outliers, normality assumption, skewness, and kurtosis. It was found that AtMS contained excess kurtosis (absolute value > 2) and both Kolmogorov-Smirnov's and Shapiro-Wilk's tests were significant for each item which showed that the data were not normally distributed. Thus, weighted least square mean and variance adjusted (WLSMV) estimator was used for the confirmatory factor analysis as it is robust enough to perform well under violation of multiple assumptions (Suh, 2015; Zakariya, Goodchild, Bjørkestøl, & Nilsen, 2019). Further, both the item and scale reliability indices of AtMS were investigated using latent factor approach as opposed to the Cronbach alpha coefficient.

Analyses in stage two involved validating a structural model that explains the relations between attitude towards mathematics and approaches to learning. It consisted of evaluating the model and conducting exploratory post hoc analysis for its improvement. The structural equation modelling approach was used to either confirm or falsify the causal hypothesized relations between attitude towards mathematics and approaches to learning without claiming outright causation between the constructs. Model fits were assessed based on a combination of criteria as proposed in literature which includes: χ^2 ratio to the degree of freedom (df) less than 3, significant estimated parameters, comparative fit index (CFI), Tucker-Lewis index (TLI) close to or ≥ .95, root mean square error of approximation (RMSEA) ≤ .06, and standardized root mean square residual (SRMR) ≤ .08 (Chen, 2007; Hu & Bentler, 1999). All the analyses were performed using Mplus 8.3 software, and the results are presented in the next section.

RESULTS

STAGE ONE: MEASUREMENT MODEL AND RELIABILITY

A one-factor model was evaluated for the measure of attitude towards mathematics, and the results are presented in Table 1.

GOF indices	Model 1	Model 2
χ^2-value	84.078	3.562

df	5	3
p-value	< .001	.313
χ^2/df	16.816	1.187
CFI	.927	.999
TLI	.854	.998
RMSEA (90% CI)	.284 (.233 - .339)	.031 (< .001 - .128)
CFit	< .001	.510
SRMR value	.076	.038

Table 1: Selected goodness of fit indices (GOF) for measurement 1-factor model of AtMS
Note. CFit: close fit (i.e. probability of RMSEA <= .05)

The results of the selected goodness of fit (GOF) indices, presented in Table 1 (Model 1) showed an appropriate fit of the one-factor model AtMS. However, the significant chi-square value, its ratio to df > 3, and the low value of TLI, suggest that the model can be improved. Thus, a post hoc analysis was conducted using suggestions from modification indices. On this basis, two error covariances were included in the model between item 02 and item 04 as well as between item 01 and item 05. These resulted in a significant improvement in the model (Model 2) as indicated by the significant chi-square difference test statistics with Satorra-Bentler correction $\Delta\chi^2_{[2]} = 80.516, p < .001$. The model chi-square value is no longer significant, which is expected and its ratio to df < 3, CFI, TLI, RMSEA and SRMR, all now within the recommended ranges. These suggest an excellent fit of Model 2. All the factor loadings are found to be significant, and all the items are reliable with an ordinal coefficient alpha of .78 on the whole measuring instrument.

STAGE TWO: STRUCTURAL MODEL

In an attempt to test hypotheses one and two, we evaluated two structural models. The first model (Model 3) concerns the impact of attitude towards mathematics on the two dimensions of approaches to learning the subject. The second model (Model 4) concerns the final improvement of Model 3 through post hoc analyses. Selected GOF indices of these models are presented in Table 2, while Figure 2 displays standardized estimates of factor loadings, regression weights, variance explained, etc.

GOF indices	Model 3	Model 4
χ^2-value	425.784	301.876
df	247	204
p-value	< .001	< .001
χ^2/df	1.724	1.480

CFI	.913	.953
TLI	.902	.947
RMSEA (90% CI)	.061 (.051 - .070)	.049 (.037 - .061)
CFit	.037	.518
SRMR value	.070	.059

Table 2: Selected goodness of fit indices for measurement 1-factor model of AtMS

The results in Table 2 (Model 3) showed an appropriate fit of the model except that both CFI and TLI are relatively low and close fit probability assessment of RMSEA was significant which implies the model is not close enough to the data. As the first step in post hoc analysis of the structural equation modelling approach, we scanned through the estimates and discovered that item 4 and item 10 of the surface approach subscale of R-SPQ-2F had non-significant factor loadings. These items were deleted from the model, and the resulting model improved significantly as indicated by the significant chi-square difference test statistics with Satorra-Bentler correction $\Delta\chi^2_{[43]} = 123.908, p < .001$. The results, as presented in Table 2 (Model 4) suggest an excellent fit of the model. Figure 2 gives more detail on the parameter estimates of Model 4.

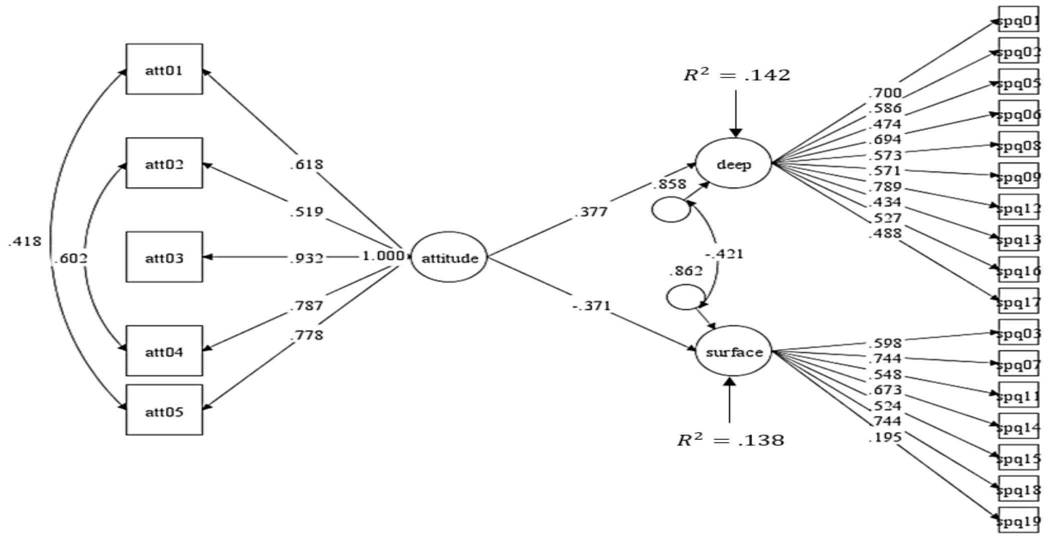

Figure 2: Validated structural model of the relations between attitude towards mathematics and approaches to learning mathematics

The illustrated results by Figure 2 show that there is a significant positive impact of attitude towards mathematics on deep approaches to learning (β= .377, p < .05) and a significant negative impact on the surface approaches to learning (β= -.371, p < .05) which confirm hypothesis one (H01) and hypothesis two (H02) respectively. These findings could be interpreted to mean that students who have a high (positive) attitude towards learning mathematics are more likely to adopt deep approaches to learning the subject. On the other hand, students who have a low (negative) attitude towards mathematics are more likely to adopt surface approaches to learning the subject. It is also revealed in Figure 2 that attitude towards mathematics explained 14.2% and 13.8% variances in predicting deep and surface approaches, respectively. These percentages of explained variances appear low. However, they are statistically significant. The low percentages of explained variances in deep and surface approaches are suggestive of the presence of other factors that are not captured in the present study and yet influence the adoption of students' learning approaches. In the next section, we present a brief discussion of the significant findings.

DISCUSSION

Attempts are made in the present study to provide empirical evidence for construct and discriminant validity of a 5-item attitude subscale of the Norwegian national mathematics test and to expose the impact of attitude towards mathematics on students' learning approaches. The attitude subscale was found to be unidimensional, and possesses construct validity, it discriminates cleanly between two approaches to learning and it has high internal item consistency with an ordinal coefficient of .78. However, this validity evidence was achieved after accounting for two error covariances between item 02: "I work with math because I like it" and item 04: "I'm interested in what I learn in math" as well as between item 01: "making an effort in math is important because it will help me in work I will be doing later" and item 05: "mathematics is an important subject for me because I need it when I want to study further".

It is important to remark that the error covariances appear to make sense conceptually since both item 02 and item 04 seem to capture intrinsic motivation part of attitude and item 01 and item 05 seem to capture usefulness of mathematics part of attitude. This finding corroborates other studies that have reported multidimensional attitude scales (Palacios et al., 2013). Further, the reliability coefficient of the AtMS ($\alpha = .78$) is higher than that of the perception of utility subscale ($\alpha = .679$) reported in (Palacios et al., 2013) and that of the usefulness of mathematics subscale ($\alpha = .75$) reported in (Zakariya, 2017) even though the final reliability coefficients of the whole scales reported in the two previous studies are higher than $\alpha = .78$ that was found for the AtMS.

Another important finding of the present study is the substantial positive impact of attitude towards mathematics on the deep approaches to learning as well as the substantial negative impact on the surface approaches to learning. These findings, on the one hand, suggest that year-one engineering students who enjoy mathematics, who are interested in the subject and recognize the utility of mathematics to their future studies are more likely to adopt deep approaches to learning the subject. On the other hand, the findings suggest that year-one engineering students who find mathematics less enjoyable and struggle to discover its relevance to their future studies may tend to adopt surface approaches to learning the subject. These findings agree with the report by Alkhateeb and Hammoudi (2006) and partly support some reported results by García, Rodríguez, Betts, Areces, and González-Castro (2016). More importantly, we do not claim outright causal relations between these constructs. However, our results have only provided tentative empirical evidence that confirms our hypothesized causal relations between engineering students' attitude towards mathematics and approaches to learning. Future replication studies are recommended to confirm these findings in independent samples. Finally, it is hoped that the findings of this study have shed some light on a general understanding of the causal relations between the attitude of students towards mathematics and their approaches to learning the subject. This could be of help to university teachers and other stakeholders in designing appropriate interventions to support the students.

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Car Race - an interdisciplinary Calculus' project using Python

Sandra Gaspar Martins[1,2]

[1]Instituto Superior de Engenharia de Lisboa, Instituto Politécnico de Lisboa, Portugal, sandra.martins@adm.isel.pt;

[2]Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Portugal

Students performed a CarRace in Python programming language using Calculus' parametrizations. Students get a sample Python program where a car performs a race in a speedway and adapt that program to a speedway chosen by them. Every student chooses a speedway, so every student has a different problem and must adjust his parametrizations (and some more details) to his speedway.

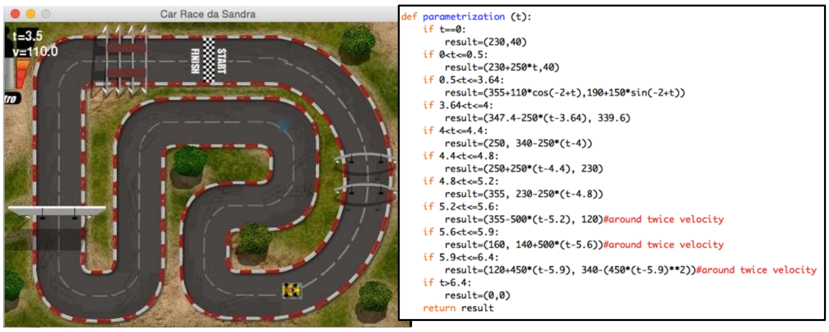
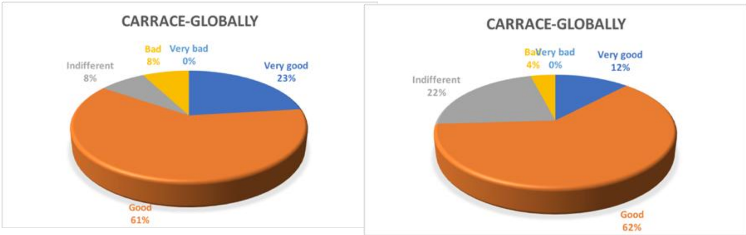


Figure 1: Screenshot of a car performing a trajectory on a speedway. And the parametrizations to run the spreedway. The example given to students.

Research question: the CarRace project is a positive project to propose to computing engineering students? The course had 113 and 114 students respectively, in 2017 and 2018. Respectively, 95 and 86 accomplished the project. The respondents to a survey were 26 and 51, respectively.



Graphic 7: Answers to the question: "Globally I classify the existence of CarRace project as:" in 2017 and 2018.

As shown in Graphic 7, nearly 80% of respondents globally classify the existence of CarRace as positive.

In the answers to the survey around 75% of students state that were pleased or highly pleased making the project. And 80% were also pleased or highly pleased when concluded it.

To about 50% of respondents, the project made them to better understand the subject. And only 15% says that using Python increased difficulty to the project. Nearly all students agree that is important to use Python transversally in graduation. CarRace also shows an immediate application of Mathematics and around 80% agreed that it is interesting.

Globally students classified CarRace project as positive, 80% of respondents. Teachers refer that students show excitement about the project, were attentive, and posed many questions. The teachers and the graduation commission consider important that students work with Python transversally in graduation, connecting Python with others subjects and also to work with an immediate application of mathematics.

The global assessment of this project is positive thus we will keep this project in the following years and recommend its use by other teachers.

Bridging the gap between mathematics courses and mathematics in the workplace: the example of a study and research path for future engineers

Pierre-Vincent Quéré[1]

[1]Centre de Recherche sur l'Éducation, les Apprentissages et la Didactique (CREAD), Rennes, France, pierre-vincent.quere@ac-rennes.fr

Keywords: Teaching and learning of mathematics for engineers, Novel approaches to teaching, Study and Research Path

INTRODUCTION

The work presented here refers to the Anthropological Theory of the Didactic (ATD). I consider engineering schools and the workplace for engineers as two different institutions, where mathematics is present in different components of the praxeologies (Chevallard, 2017).

In a recent research (Quéré, 2017) I have evidenced that French engineers encounter different types of mathematical praxeologies in the workplace. I have classified them in two main categories: "proper" praxeologies (Basic, Statistic and Specific) or "transversal" praxeologies (Modelling, Reasoning, and Communication and Documentation). Whereas some of them seem to be taught in the engineering studies, the Modelling and the Communication and Documentation praxeologies are declared as lacking by the engineers I've met through the research. This suggests the need for specific innovations in the preparation of future engineers.

In this poster I describe and analyse the design and implementation of a Study and Research Path (SRP, Barquero, Bosch, & Gascón, 2008) in Chemistry and Statistics, in an engineer school in France. I try to show how this kind of innovative teaching can be a way to bridge the gap between mathematics courses for future engineers and mathematics in the workplace.

THEORETICAL FRAMEWORK AND RESEARCH QUESTION

An SRP can play several roles. Here I focus on its role as a teaching model for an inquiry-centred education based on a generating question Q_0. It is aimed to make the teaching getting out from the usual *monumentalism* of the mathematical teaching (Barquero et al., 2008). The ATD framework provides efficient tools to analyse the global and mathematical praxeologies of the students during the progress of a SRP, with in particular 9 dialectics (for example, between questions and answers, Parra & Otero, 2018). The main research question I shall try to answer here is "Can a SRP be an answer to the concerns of motivation, mobilisation of mathematical knowledge, lack of connection between diverse disciplines, satisfaction of both teachers and engineering students?"

METHODOLOGY

The SRP described in this poster starts with the following generating question Q_0 written by the chemistry teacher supervising the SRP with me: *"In the pharmaceutical industry, how do you make sure that the product (medicine) meets the dosage on the package?"*

To answer this question the aim for the 14 participating students is to provide a website that will be accessible for the future students of the next year. At the beginning of the experiment, we have all together planned a number of 6 working sessions for the two coming months before the students have to present their work to the other students of the class.

To monitor the global and mathematical activity of the students, I've firstly filled a personal diary with my observation notes during the working sessions. After each of these sessions, designated students leaders had to fill an online diary detailing their actions (meetings, research, topics, resources, etc.). At the end, each student had to fill a personal questionnaire about her or his own experience. Moreover, I have recorded and transcribed a final interview with the chemistry colleague.

RESULTS

An important result is that this SRP has allowed students to develop some praxeologies usually lacking in the training of engineers and useful in the professional context (for example, documentation and communication). Moreover, the students and the teachers have emphasized codisciplinarity as a real asset.

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TWG3: Number Theory, Algebra, Discrete Mathematics, Logic

TWG3 report: University Mathematics Didactic Research on Number Theory, Algebra, Discrete Mathematics, Logic

Rolf Biehler[1] and Viviane Durand-Guerrier[2]

[1]University Paderborn, Germany, biehler@math.upb.de

[2]University of Montpellier, France, viviane.durand-guerrier@umontpellier.fr

INTRODUCTION

Ten oral communications were presented, reflecting the variety of the themes of the group: Number Theory, Algebra, Discrete Mathematics, Logic. In detail, two papers on Linear Algebra (Span, Linear transformation) were presented; 2 papers on Abstract Algebra (Group theory, the concept of ideal); 4 papers on Logic, Reasoning and Proof (syntax and semantic, Mathematical induction and recursion, Backward reasoning, personal meaning of proof); 2 papers on innovative teaching (first-year university students, Geometry capstone course). They were presented during sessions 1 and 3, being followed in each case by a discussion session nourished by the issues raised in the communications. Thirty-eight participants were registered for the sessions in this thematic working group. The number of attendees varied between 29 and 19, from Tunisia, Europe, North and South America and Japan; this was a challenge, due to the differences in local time zones.

SYNTHESIS OF THE COMMUNICATIONS

The two papers on Linear Algebra were presented respectively by Mitsuru Kawazoe (Japan) and Asuman Oktaç (Mexico). Mitsuri Kawazoe's paper is entitled *Relation between understandings of linear algebra concepts in the embodied world and in the symbolic world*. In this study, linear (in)dependence and basis were focused on, and the relation between understandings of them in the embodied world and the symbolic world. I includes a study of the effectiveness of an instruction emphasizing geometric images of them. The main results of the study were the following: 1/ conceptual understanding of linear dependence of four spatial vectors such that any three of them do not lie on the same plane was positively associated with the understanding of the basis in the symbolic world. 2/ A geometrical instruction had not improved understanding of linear dependence of such vectors; indeed, in both pre-test and post-test, this task showed to be problematic for nearly half of the students.

Asuman Oktaç presented a paper written with by Diana Villabona, Gisela Camacho, Rita Vasquez and Osiel Ramirez on *Process conception of linear transformation from a functional perspective*. The paper discusses student conceptions involved in the construction of conceptions about a domain, image and inverse image of a linear transformation from IR^2 to IR^2 as well as the relations between these notions. The authors present the design of a set of tasks that allow exploring different facets of the above concepts, evidenced by the analysis of the production of a student. Thanks to the design of the instrument, it was possible to highlight some conceptions that may not be evident in typical teaching situations.

The two papers on Abstract Algebra were presented respectively by Koji Otaki (Japan) and Julie Candy (Switzerland and France). Koji Otaki presented a paper written with Hiroaki Hamanaka and Ryoto Hakamata entitled *Introducing group theory with its raison d'être for students*. This paper reports results of a sequence of didactic situations for teaching fundamental concepts in group theory, e.g., symmetric group, generator, subgroup, and co-set decomposition. Students in a pre-service teacher-training course dealt with such concepts, together with card-puzzle problems the analysis of which provide students with the raisons d'être of these concepts.

Julie Candy presented a paper entitled *Etude de l'enseignement du concept d'idéal dans les p*remières années postsecondaires: élaboration de modèles praxéologiques de référence. The paper presents the construction and interpretation of a praxeological reference model for teaching the concept of ideal in the first two post-secondary years in France, in two different institutions, before this concept is taught systematically in Ring Theory. The model allows a comparison of the choices made by the two institutions and a first discussion of the implementation of structuralist thinking, in the perspective of the teaching of abstract algebra in the third year of university.

The four papers on Logic, Reasoning and Proof were presented respectively by Zoé Mesnil (France), Nicolas Leon (France), Ines Gómez-Chacón (Spain) and Sandra Krämer (Germany). Zoé Mesnil presented a paper written with Virginie Deloustal-Jorrand, Michèle Gandit, and Mickael Da Ronch, entitled *Utilisation de l'articulation entre les points de vue syntaxique et sémantique dans l'analyse d'un cours sur le raisonnement*.The authors highlight the relevance of the articulation between syntax and semantics in proof and proving activities. With this lens, they present a logical and didactical analysis of a university course entitled "Mathematical Reasoning", relying on interviews with teachers, worksheets and an assessment test. The case study presented here is the first step for a comparative study aiming at characterizing the teachers' views on proof and proving, as a preliminary before studying students' appropriation of the various aspects of proof and proving.

Nicolas Leon presented a paper, written with Simon Modeste and Viviane Durand-Guerrier, entitled *Récurrence et récursivité: analyses de preuves de chercheurs dans une perspective didactique à l'interface mathématiques*. The authors present the analysis of researchers' proofs of the equivalence of two definitions of the concept of tree in graph theory, one of the two definitions being recursive and the other not. The analysis aims to shed light on the relationship between the notions of recurrence and recursion, as perceived by experts. The authors will rely on the results of this study when designing didactic sequences aiming to work with students on recurrence and recursion and their interactions.

Ines Gómez-Chacón presented a paper written with Marta Barbero and Ferdinando Arzarello entitled *Backward reasoning and epistemic actions in discovery processes of strategic games problems*. The authors focus on the epistemic and cognitive characterization of backward reasoning in strategy games problems with PhD students in a Spanish and an Italian university. They report a case study showing the process of discovery that a PhD student carries out to formulate a general recursive formula. They propose a unified framework that allows focusing on both short-term and long-term processes in students' activities. Sandra Krämer presented a paper written with Leander Kempen and Rolf Biehler entitled *Investigating high school graduates' personal meaning of the notion of "mathematical proof*". In this paper, the authors report on the results of a pilot study to investigate high-school graduates' personal meaning of mathematical proof. By using proof tasks and a following interview phase with meta-cognitive questions, they describe students' personal meaning of the notion of mathematical proof and show, among others, that some students hold different meanings of the word "proof" simultaneously.

Each of the last two papers presents innovative courses in teaching mathematics. They were presented respectively by Patrick Gibel (France) and Max Hoffmann (Germany). Patrick Gibel presented a paper written with Isabelle Bloch entitled *Analyse des effets d'un dispositif innovant sur l'évolution des représentations des étudiants en première année de licence de mathématiques*. The authors present an innovative course set up at the University of Pau in order to help undergraduate students to overcome difficulties in the secondary-tertiary transition. A main mean is to involve students in research into mathematical problems. An example situation is described and analysed.

Max Hoffmann presented a paper written with Rolf Biehler entitled *Designing a Geometry Capstone Course for Student Teachers: Bridging the gap between academic mathematics and school mathematics in the case of congruence*. The authors present a geometry course for upper secondary student teachers aiming to show links between academic mathematics and school mathematics. In the paper, they focus on the concept of congruence, illustrating how specific aspects of the course are used to systematize the mathematical background of the topic, thus enabling future mathematics teachers to diagnose and react in fictitious teaching situations professionally based on subject matter knowledge in mathematics. Finally, they provide examples of learning activities in the course and first results of analysing students' work.

The paper by Khalid Bouhjar, Christine Andrews-Larson, and Muhammed Haider, *On students' reasoning about span in the context of Inquiry-Oriented Instruction*, has not been presented, but is available in the proceedings. The authors analyse differences in reasoning about span by comparing the written work of 126 linear algebra students who learned through a particular inquiry-oriented (IO) instructional approach compared to 129 students whose instructors used other instructional approaches. Their analysis of students' responses to open-ended questions indicated that IO students' concept images of the span were more aligned with the corresponding concept definition than the concept images of non-IO students. Additionally, IO students exhibited richer conceptual understanding and greater use of deductive reasoning than Non-IO students.

MAIN ISSUES DISCUSSED DURING THE SESSIONS

The main theoretical and methodological issues discussed were 1/ the role of ATD (Anthropological Theory of the Didactic) for analysing, designing, giving access to the raisons d'être of a mathematical topic; 2/ the means to address the complexity of mathematical notions with students (e.g. Cayley diagram in group theory, recursion, strategic games in 3D); 3/ the relevance of analysing data through the lens of concept images versus concept definition, and of considering the impact of the choice of definition in students' activities (e.g. definition of the image starting from the domain or codomain; 4/ what can we infer from case studies depending on the two following cases: 4.1: a significant amount of data have been analysed – a representative case; 4.2 a small number of interviewees, but a diversity of profile providing a great richness in the data. In both cases, it is not possible to generalize, but such a case study might contribute to enrich a priori analysis and identify candidates for operational invariants. Issues on proof and reasoning prevailed in the four papers focusing on this topic, but also in other papers, and were widely discussed. Several questions on proof classification were raised: what counts as an empirical argument? What is the difference between generic proof, narrative proof, symbolic proof? What links exist with the classification of the type of proofs by Balacheff? How to distinguish between correct and incorrect proof, considering the audience of a proof? Some participants wonder if there is a consensus among university teachers on what is a mathematical proof; more precisely, in a didactical transposition perspective, is there a common reference on proof that would make easier its teaching and learning. The answer is that this is not obvious because there might be dependence on the educational context or personal views of teachers. In some cases, a local consensus may exist among a pedagogical team.

Different and related (necessary) aspects of teaching proof have been considered in the discussions: 1/ showing proofs to students seems necessary but is clearly not sufficient; 2/ solving problems with not too obvious solution to motivate the need for proof; 3/ teaching what is a proof and its role in mathematics to provide students with meta-knowledge on proof in mathematics. 4/ having students experience how to construct and analyse proofs, in their mathematical and logical dimensions; 5/ considering not only proof but also proving as a practice; 5/ teaching proof as a separate topic or integrated into teaching mathematical topics (with reflections on proof)? 6/ considering the role of proof on conceptualization and the reverse.

FURTHER RESEARCH AVENUES

Finally, we have identified main open questions and research areas deserving more attention for the years to come. A promising avenue of research is addressing the second transition of Klein (in programs for mathematics teacher education), by developing innovative teaching modules to allow students in a teacher training program to deeply understand the relationships between university mathematics and school mathematics in a professional perspective. There are convincing examples but also several challenges: 1/ finding relevant topics with strong epistemological foundation (e.g. congruence, symmetry, integration, proof); 2/ developing collaborations between university teachers and researchers in didactics of mathematics (some might be both) for implementation and analysis; 3/ managing to implement it, depending on the context: department of mathematics *versus* faculty of education; 4/ finding a way of dissemination of research results toward mathematics university teachers. Exploration of paths of collaboration between mathematicians and researchers in mathematics education, considering various institutional contexts: 1/ having researchers in didactics of mathematics in a department of mathematics; 2/ having professional mathematicians in a faculty of education; 3/ developing collaboration in a doctoral programme - co-supervision of PhD students; 4/ designing training modules for mathematics university teachers (mandatory in many countries – should also be specific to the domain of mathematics, not just general pedagogy); 5/ organizing workshops aiming at participants to get acquainted with didactic aspects of the teaching and learning of university mathematics.

Address proof and proving issues at all level of university mathematics. 1/ going on investigating the possibility of a common background (*versus* specificity) on proof and proving for developing university mathematics (both undergraduate and graduate) studies; 2/ deepening studies on the role of proof and proving in conceptualization on advanced topics (e.g. number theory, linear and abstract algebra, algebraic topology, algorithms, discrete mathematics, recursion); 3/ developing research on proving as a practice linked to solving problem: epistemological and didactical issues; 4/ developing students' meta-knowledge on proof as a topic of its own on top of students' experiences of proof and proving as part of problem-solving processes; 5/ working in a given axiomatic *versus* exemplary participation in an axiomatization of a domain; going on addressing logical issues in mathematics and establishing links with mathematics and computer science. These issues are in line with than some of those identified and discussed in Chellougui et al. (2021).

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ANALYSE DES EFFETS D'UN DISPOSITIF INNOVANT SUR L'EVOLUTION DES REPRESENTATIONS DES ETUDIANTS EN PREMIERE ANNEE DE LICENCE DE MATHEMATIQUES

Isabelle Bloch[1] et Patrick Gibel[2]

[1]Université de Bordeaux, Lab-E3D, isabelle.bloch@u-bordeaux.fr; [2]Université de Bordeaux, Lab-E3D, Patrick.Gibel@u-bordeaux.fr

Résumé : Cet article présente un dispositif mis en place à l'université de Pau afin d'aider les étudiants de Licence 1 à surmonter les difficultés d'adaptation aux mathématiques du niveau supérieur, et à s'impliquer dans la recherche de problèmes mathématiques. Une situation est précisée et les productions des étudiants analysées.

Mots-clés : situations de recherche à l'Université, concepts mathématiques, outils sémiotiques, disponibilité des savoirs et des signes.

INTRODUCTION

La volonté politique de l'UPPA[1] d'expérimenter des dispositifs innovants pour lutter contre l'échec en première année de Licence (L1) de mathématiques et de licence MIASHS[2] a conduit des enseignants intervenant en L1 et les didacticiens de l'ESPE[3] à se réunir pour envisager la mise en œuvre d'un projet pédagogique innovant. Ceci a été rendu possible grâce aux interactions engagées, depuis plusieurs années, entre l'ESPE d'Aquitaine (site de Pau) et le département mathématique de l'université dans le cadre des mémoires liés à l'enseignement des mathématiques (Master MEEF) et au co-encadrement de thèses en didactique des mathématiques. Ces échanges ont contribué à expliciter les principaux enjeux du champ de la didactique des mathématiques et à montrer l'efficacité des concepts de didactique pour questionner et étudier les problématiques liées à l'enseignement des mathématiques dans le secondaire et le supérieur. Ces interactions ont conduit certains collègues de mathématiques de l'UPPA à vouloir diversifier leurs méthodes d'enseignement, suite au constat du manque manifeste d'investissement des étudiants de L1 dans l'étude des cours dispensés en CM[4] et dans la recherche des activités proposées en TD[5], et donc de leur échec – potentiel ou avéré.

CARACTERISATION DU DISPOSITIF PEDAGOGIQUE

L'élaboration du projet

Le souhait des enseignants de l'UPPA était donc de mettre en place un dispositif pédagogique spécifique ciblant les difficultés des étudiants et visant à augmenter leur

[1] Université de Pau et des Pays de l'Adour
[2] MIASHS : Mathématiques et Informatique appliquées aux Sciences Humaines et Sociales
[3] Ecole Supérieure du Professorat et de l'Education, devenue en 2019 INSPE : Institut National Supérieur du Professorat et de l'Education
[4] CM : cours magistraux
[5] TD : séances de travaux dirigés

compréhension des savoirs mathématiques en jeu, en vue de permettre une plus grande responsabilisation de ces étudiants de L1 et de favoriser leur implication dans les activités de résolution de problèmes, notamment en les faisant travailler en groupes. La volonté de privilégier les échanges au sein de groupes d'étudiants et de permettre des travaux en trinômes a reposé sur une méthode : il a été décidé d'aménager un effectif réduit pour chacun des groupes, soit 24 étudiants. Puis ont été constitués huit trinômes d'étudiants par groupe afin que les enseignants présents puissent suivre et encadrer les travaux engagés par chacun des trinômes, et étudier les principales difficultés rencontrées. Les étudiants ont été laissés libres de constituer un trinôme, notamment par affinité. Le but du projet est donc de motiver et responsabiliser chaque trinôme d'étudiants, en leur dévoluant une situation mathématique proche d'une situation adidactique ou à dimension adidactique, c'est-à-dire une situation comportant une dimension de recherche. Les situations présentées ont pour but de faire approfondir les connaissances présentées notamment durant les CM et les TD classiques.

Nature des difficultés des étudiants constatées en L1

Le décalage de connaissances et de pratiques mathématiques observé lors de la transition secondaire/supérieur s'avère particulièrement difficile à gérer pour les étudiants ; en effet le contrat didactique évolue, conduisant les étudiants à une plus grande responsabilisation liée aux choix :

- des connaissances et des savoirs qu'il convient de mobiliser pour répondre à la situation de recherche dévolue par l'enseignant ;
- du cadre (numérique, géométrique, algébrique, graphique…) qui apparait le plus adéquat pour répondre à la question posée ;
- du mode de raisonnement qu'il faut mobiliser (inductif, déductif, abductif) et la forme de raisonnement associée (raisonnement par l'absurde, raisonnement par récurrence, raisonnement 'direct' ou par contraposée) pour élaborer la solution ;
- de l'interprétation et de l'usage des signes mathématiques, signes dont le niveau de complexité est élevé, par exemple les quantificateurs.

Les nombreuses études menées en didactique sur la transition secondaire-supérieur mettent en évidence un attendu spécifique en L1, à savoir une pratique maîtrisée du raisonnement au travers de ses différentes fonctions (Bloch et Gibel, 2016), (Gibel, 2018). Parmi les attendus du raisonnement on peut citer : organiser sa recherche, chercher, conjecturer, expliquer, justifier, prouver, démontrer, valider, invalider, réfuter.

Les précédentes recherches en DDM montrent des déficiences importantes des étudiants dans la prise d'initiative quant aux connaissances et savoirs à mobiliser pour résoudre un problème et élaborer une procédure de résolution. Au lycée les connaissances mobilisées dans le cadre des activités de résolution de problèmes sont le plus souvent en lien direct avec la notion mathématique étudiée précédemment, et ne nécessitent que la mise en œuvre de procédures répertoriées directement (et enseignées dans le cours) en lien direct avec la notion étudiée, voire explicitement indiquées dans l'énoncé du problème. Il en est de même dans l'épreuve du baccalauréat. De plus les signes utilisés relèvent peu du formalisme mathématique.

A l'université, les étudiants doivent être capables d'élaborer, puis de rédiger et communiquer une preuve mathématique de façon autonome, mais également de débattre de la validité et de la pertinence d'une preuve complète, ceci en intégrant un niveau de justification adéquat, et des outils nouveaux, comme les quantificateurs : il y a donc un basculement du contrat didactique. Le décalage entre les deux niveaux a été largement illustré dans la recherche, par exemple dans Bloch (2016) :

> (…) ce décalage fait que les étudiants ne savent manipuler que des fonctions définies par une formule algébrique, et […] ne prennent que peu en compte le fait que l'étude des fonctions implique des calculs et raisonnements à différents niveaux, soit ponctuel, global, local, ce dernier point de vue étant celui qui se trouve le moins investi. Le global n'est pas non plus bien maîtrisé, les graphiques, par exemple, n'étant parfois vus que comme des icônes de fonctions, et non comme des outils de travail sur ces fonctions. Benitez & Drouhard (2015) mettent aussi en évidence que les étudiants testés dans leur étude […] ont d'abord à surmonter des difficultés de calcul algébrique et de raisonnement, et que les étudiants qui ne manifestent plus ces difficultés algébriques sont confrontés à des obstacles venant de leur conception inachevée des objets mathématiques, et des liens entre les différents objets.

Rogalski (2008) pointe également la difficulté qu'ont les étudiants à passer du niveau global au local, ou réciproquement, notamment dans l'étude des fonctions. C. Winslow signale aussi que :

> Parmi les enseignants universitaires, il y a un sentiment répandu que l'étudiant doit, effectivement, accomplir des « sauts cognitifs » dans le parcours […] vers l'analyse abstraite enseignée à l'université. (Winslow, 2007, p.189)

Les grandes lignes du dispositif et le contrat didactique

La mise en œuvre du dispositif décrit précédemment repose donc sur un contrat didactique universitaire *et* spécifique qu'il nous semble important de définir. Ainsi les principales responsabilités dévolues aux étudiants sont :

- L'implication dans la recherche, l'élaboration de raisonnements en réponses aux questions de l'énoncé, la formulation de questions en vue de surmonter certains obstacles ainsi que la rédaction d'une solution intégrant un niveau de justification adéquat et l'usage conforme des signes mathématiques.
- La présentation par le trinôme, tiré au sort, du raisonnement mathématique produit en réponse à chacune des questions de la situation de recherche.
- Le questionnement, par les trois étudiants constituant le jury, des raisonnements et des réponses produites par le binôme exposant son travail.
- L'analyse critique du raisonnement produit par des étudiants ayant ou non travaillé cette situation

- La rédaction[6] d'une solution intégrant les commentaires et les remarques effectués par les étudiants et l'enseignant suite à la présentation.

Les principales responsabilités de l'enseignant lors de la mise en œuvre de la séance et à l'issue de celle-ci sont :

- Observer les interactions au sein de chaque trinôme pour identifier leurs difficultés, les accompagner dans la résolution en apportant des réponses au questionnement des étudiants en vue de favoriser l'appropriation de la situation de recherche et d'éventuels changements de cadres ou de registres. Il s'agit de conserver la composante recherche de la situation, en s'inscrivant dans l'optique d'une 'guidance faible' (Bartolini-Bussi, 2009) ne dénaturant pas la dimension heuristique.
- Identifier précisément la nature et l'origine des difficultés des étudiants en analysant les différents types d'erreurs produites lors de l'exposé de leurs travaux.
- En fin de questionnement par le jury, choisir de revenir sur certains éléments de la solution en vue de lever le doute sur certaines interrogations quant aux savoirs en jeu, à l'adéquation des signes mobilisés et à la pertinence des réponses proposées.
- A l'issue des questions, décider des connaissances et des savoirs qu'il convient d'institutionnaliser : identification des objets mathématiques qui définissent la situation objective, nature et forme des signes et des raisonnements mobilisés lors de l'étude, liens entre les procédures distinctes mises en œuvre par différents groupes ; principaux enjeux didactiques et mathématiques de la situation, retour sur les registres sémiotiques mobilisés.
- Commenter et questionner l'écrit de synthèse produit par le trinôme – en charge de produire la mémoire de la situation de recherche – en vue d'une réécriture valide sur le plan sémantique et syntaxique.

Ce contrat spécifique est exposé en début d'année aux étudiants inscrits dans cette unité d'enseignement, et une première situation de recherche leur est proposée, situation qui est ensuite présentée et corrigée par l'enseignant afin de mettre en évidence les attendus de la restitution des situations.

METHODOLOGIE UTILISEE POUR ANALYSER L'EVOLUTION DES REPRESENTATIONS ET DES CONCEPTIONS DES ETUDIANTS

Contexte du dispositif et de la recherche

L'expérimentation a été menée dans cinq groupes de travaux dirigés au premier semestre de la première année de Licence de Mathématiques et de licence MIASHS[7]. Les étudiants ont choisi de suivre cette Unité d'Enseignement intitulée « Outils de méthodologie pour comprendre les mathématiques ». Les cinq enseignants qui ont

[6] Par un trinôme n'ayant pas présenté son travail.
[7] Mathématiques et Informatique appliquées aux Sciences Humaines et Sociales

mené ces expérimentations sont ceux à l'origine du projet pédagogique innovant. Ils ont interagi avec les chercheurs en didactique, en vue de définir précisément le déroulement de chaque séance, de rédiger les énoncés des situations de recherche et de déterminer pour chacune des situations le contrat didactique correspondant.

Mise en œuvre

Afin de déterminer en quoi la confrontation des élèves à des situations de recherche favorise non seulement l'identification des objets mathématiques et l'appropriation des concepts mathématiques, mais aussi la pratique du raisonnement, nous avons procédé à l'analyse des différentes versions produites successivement par un trinôme en charge de la restitution de la solution construite à partir de la situation 1.

La situation 1 objet d'étude : Suite de carrés[8]

On construit une « suite » de carrés juxtaposés de la manière suivante : le côté du premier carré est de longueur 1 (en référence à une unité donnée), puis chaque carré a pour mesure de côté $\frac{3}{4}$ de la mesure du côté du carré précédent.

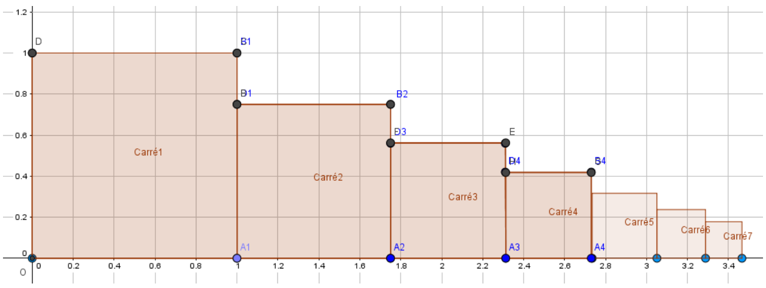

Figure 1 Les sept premiers carrés obtenus par le procédé de construction

Dans cette situation, les élèves doivent déterminer s'il est ou non possible de construire un « énième » carré, dont a_n, l'abscisse du point A_n – correspondant à la mesure OA_n, où O désigne l'origine du repère – est strictement supérieure à 4. Puis ils doivent montrer que les points B_i sont alignés, et calculer l'aire totale de la figure. Il s'agit d'une situation à dimension adidactique, visant à confronter les étudiants à la notion de limite. Cette dernière est obtenue ici comme le résultat du processus de construction des carrés, itéré à l'infini. Cette situation offre la possibilité d'étudier la notion de limite finie d'une suite ; son intérêt principal est qu'elle est issue de connaissances du secondaire, et qu'elle offre la possibilité d'articuler différents cadres (algébrique, graphique et géométrique) pour une meilleure appréhension du concept.

Déroulement de la séance et analyse a priori de la situation « Suite de carrés »

Place de la situation dans le dispositif

La situation « Suite de carrés » est la première situation de recherche dévolue aux étudiants et dont la résolution leur incombe pleinement. Auparavant l'enseignant a proposé une situation sur les fonctions (variation d'une aire de triangle dont un sommet

[8] La situation complète est en annexe.

varie selon deux variables possibles : l'abscisse x du point variable ou la mesure α de l'angle opposé du triangle) à l'ensemble des trinômes de son groupe ; après la phase de recherche, c'est l'enseignant qui a effectué la présentation au tableau de la solution attendue. Puis il a répondu aux questions posées par un trinôme d'étudiants tiré au sort, constituant le jury. Il a enfin institutionnalisé les connaissances et les savoirs en jeu dans la situation étudiée : distinction entre fonction de x ou α et variation d'aire étudiée, étude des fonctions et représentation graphique, etc.

Analyse a priori de la situation suite de carrés

Cette situation est fondée sur la notion de limite ; il s'agit de différencier la limite d'une suite de carrés, d'une suite de points, d'une aire… et de calculer l'aire totale d'une figure constituée d'une infinité de carrés dont l'aire tend vers zéro. La situation permet de s'appuyer sur le registre graphique ; la raison de la suite des aires de carrés est un rationnel ; les abscisses des points $(B_i)_{i\geq 1}$ sont à identifier, et à distinguer des points mêmes, et il faut montrer l'alignement des (B_i). Ensuite un calcul de somme de série géométrique est à effectuer pour calculer l'aire.

Les étudiants peuvent confondre les points, et leurs coordonnées ; ils peuvent avoir des difficultés à montrer l'alignement demandé, et à calculer aisément la somme de la série. Les savoirs relèvent a priori tous du secondaire, mais l'usage des signes peut être problématique. De même, le choix des raisonnements peut poser problème : les élèves du secondaire ne sont pas habitués à choisir un raisonnement par l'absurde, par récurrence, par contraposée… de leur propre chef.

Questions sur les restitutions et rédactions des étudiants

On focalisera donc l'analyse sur les points suivants :

- L'identification correcte du problème par les étudiants ;
- Leur capacité à poser des conjectures ;
- La capacité à distinguer les termes d'une suite et la limite, une fonction et un ensemble de fonctions ;
- La capacité à mener des calculs puis à en tirer des conclusions ;
- La réponse donnée, ou non, au problème ;
- Comment utilisent-ils des savoirs : lesquels, de façon adéquate, ou non, avec les signes et notations ;
- Réussissent-ils à généraliser un calcul, puis à identifier et utiliser un mode adéquat de validation ? Ceci concerne le niveau d'utilisation des signes atteint par les étudiants, notamment du point de vue de leur capacité à généraliser les calculs et donc à atteindre le niveau des arguments génériques (cf. tableau signes et milieux, Bloch & Gibel 2011).

Du point de vue de l'enseignant, quelle institutionnalisation est pertinente, ainsi :

- Quel problème a-t-on résolu, et quels étaient les objets mathématiques en jeu ?
- Comment fonctionnent les signes utilisés, quelles sont les règles d'écriture ?

PREMIERS RESULTATS EXPERIMENTAUX : EFFETS DU DISPOSITIF SUR LES REPRESENTATIONS ET LES CONCEPTIONS DES ETUDIANTS

Dans ce paragraphe nous rendons compte de l'évolution de la rédaction de la solution produite par un trinôme d'étudiants issus de section scientifique ; nous examinons la première version rédigée par le trinôme, puis les suivantes, ainsi que les commentaires rédigés par l'enseignant (en police Calibri gras) et, dans la colonne de droite, nos analyses de ces productions.

La suite de carrés représente la suite C_n	Analyses
Puisque $C_1=1=\left(\frac{3}{4}\right)^0$; $C_2=\left(\frac{3}{4}\right)^1$; $C_3=\left(\frac{3}{4}\right)^2$; $C_4=\left(\frac{3}{4}\right)^3$ *et* $C_5=\left(\frac{3}{4}\right)^4$ *On a donc* $C_{n+1}=\left(\frac{3}{4}\right)^n$ "**Pb C_1 est un carré et (3/4) est un nombre ! Ceci n'a pas de sens !**" 1) *On conjecture que les points $(B_n)_{n\epsilon N}$ sont alignés* *Pour démontrer cette conjecture nous utiliserons la fonction affine tq f(x)=ax+b dont la droite passe par les points B_n.* *On calcule le coefficient directeur* *f(1)=1 f(1,75)=0,75* $$a=\frac{y_B-y_A}{x_B-x_A}=\frac{0{,}75-1}{1{,}75-1}=\frac{-0{,}25}{0{,}75}=-\frac{1}{3}$$ *On résout l'équation f(1)=1* $\Leftrightarrow \frac{1}{3}.1+b=1$ $$\Leftrightarrow b=1+\frac{1}{3}=\frac{4}{3}$$ *Donc* $\forall x \in R\,, f(x)=-\frac{1}{3}x+\frac{4}{3}$ "**Ceci est la droite (B_1B_2) Pourquoi $B_n \epsilon (B_1B_2)$ si n>2 ? Expliquer !**" 2) $A_n>4 \Leftrightarrow x>4$ $\Leftrightarrow -\frac{1}{3}x<-\frac{4}{3}$ $\Leftrightarrow -\frac{1}{3}x+\frac{4}{3}<-\frac{4}{3}+\frac{4}{3}$ $\Leftrightarrow f(x)<0$ *A partir de x=4, la fonction f est négative. Donc il ne peut pas exister de carré C_n tel que l'abscisse du sommet A_n>4*	Le professeur relève la confusion entre les points et les abscisses, et la non conclusion pour B_n Les étudiants tentent de trouver un critère pour affirmer que A_n a une ordonnée positive

Dans cet épisode, les étudiants montrent leur difficulté à identifier et transcrire en signes mathématiques le caractère générique de l'alignement, qui doit être valable pour tout *n*. Dans la dernière ligne de conclusion, ils semblent se baser sur le registre graphique sans parvenir à traduire dans le registre algébrique afin de produire une preuve recevable. Selon le tableau des signes (Bloch & Gibel 2011), ils restent au niveau des calculs et conjectures ponctuelles, et non au niveau des calculs génériques.

Voici à présent la version 2, rédigée par le trinôme d'étudiants après retour du jury :

1) *On conjecture que les points $(B_n)_{n\epsilon N}$ sont alignés*	Version 2

<table>
<tr>
<td>

Pour démontrer cette conjecture nous utiliserons la fonction affine tq f(x)=ax+b dont la droite passe par les points B_n.
On calcule d'abord le coefficient directeur de (B_1B_2) : B_1(1 ;1), B_2(1,75 ;0,75)
D'où f(1)=1 f(1,75)=0,75

$$a=\frac{y_{B2}-y_{B1}}{x_{B2}-x_{B1}}=\frac{0{,}75-1}{1{,}75-1}=\frac{-0{,}25}{0{,}75}=-\frac{1}{3}$$

Puis celui de (B_1B_3) : B_1(1 ;1) et $B_3\left(\frac{37}{16};\frac{9}{16}\right)$
D'où f(1)=1 $f(\frac{37}{16})=\frac{9}{16}$

$$a=\frac{y_{B3}-y_{B1}}{x_{B3}-x_{B1}}=\frac{\frac{9}{16}-1}{\frac{37}{16}-1}=\frac{-\frac{7}{16}}{\frac{21}{16}}=-\frac{7}{21}-\frac{1}{3}$$

(B_1B_2) et (B_1B_3) ont le même coefficient directeur, donc les points sont alignés.
On résout l'équation "**Pourquoi ??? Expliquer !**"
f(1)=1$\Leftrightarrow \frac{1}{3}.1+b=1 \Leftrightarrow b=1+\frac{1}{3}=\frac{4}{3}$
Donc $\forall x \in R\ , f(x)=-\frac{1}{3}x+\frac{4}{3}$

</td>
<td>Les étudiants rencontrent la même difficulté de généralisation à $(B)_n$, $\forall n$</td>
</tr>
</table>

Découvrons à présent la troisième et dernière version rédigée par les étudiants en intégrant les remarques faites par l'enseignant :

<table>
<tr>
<td rowspan="2">

1) *On conjecture que les points $(B_n)_{n\epsilon N}$ sont alignés*

Pour démontrer cette conjecture nous utiliserons la fonction affine tq f(x)=ax+b dont la droite passe par les points B_n.
On calcule d'abord le coefficient directeur de (B_1B_2) : B_1(1 ;1), B_2(1,75 ;0,75)
D'où f(1)=1 f(1,75)=0,75

$$a=\frac{y_{B2}-y_{B1}}{x_{B2}-x_{B1}}=\frac{0{,}75-1}{1{,}75-1}=\frac{-0{,}25}{0{,}75}=-\frac{1}{3}$$

On effectue maintenant une récurrence pour montrer que peu importe (B_n) $a=-\frac{1}{3}$ *, donc ils appartiennent tous à la droite.*
<u>*Initialisation*</u> *: On calcule le coefficient directeur de (B_1B_3) :On a B_1(1 ;1) et* $B_3\left(\frac{37}{16};\frac{9}{16}\right)$

$$a=\frac{y_{B3}-y_{B1}}{x_{B3}-x_{B1}}=\frac{\frac{9}{16}-1}{\frac{37}{16}-1}=\frac{-\frac{7}{16}}{\frac{21}{16}}=-\frac{7}{21}=-\frac{1}{3}$$

<u>*Hérédité*</u> *: On suppose que (B_n) appartient à la droite. On souhaite démontrer que B_{n+1} aussi. Pour cela on souhaite démontrer que* $a_{BnB1}=-\frac{1}{3}$
Et $B_{n+1}(x_{n+1} ;y_{n+1})$ avec x_{n+1}= $1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\cdots+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^n+\left(\frac{3}{4}\right)^{n+1}$ *et* $y_n=\left(\frac{3}{4}\right)^{n+1}$
$A_{BnBn+1}=\frac{y_{n+1}-y_n}{x_{n+1}-x_n}=\frac{(\frac{3}{4})^{n+1}-(\frac{3}{4})^n}{\frac{3}{4})^{n+1}}=\cdots=\frac{-\frac{1}{4}}{\frac{3}{4}}=-\frac{1}{3}$

</td>
<td>Version 3</td>
</tr>
<tr>
<td>Dans cette version, les étudiants mettent en forme une récurrence claire avec ses trois étapes</td>
</tr>
</table>

<u>Conclusion</u> $\forall n \in N^*$, *on a* $a = -\frac{1}{3}$, *donc la droite passe par tous les points (B_n) donc ils sont alignés.* *2) On cherche si un carré C_n existe pour $A_n>4$, on a $A(x_n ;0)$. Si $A_n(4 ;0)$, $B_n(4 ;y_n)$. Pour savoir si un carré C_n existe, nous devons trouver si y_n existe pour $x_n>4$.* *x_n correspond à x dans la fonction affine, on a alors f(4)=* $-\frac{1}{3} \times 4 + \frac{4}{3} = 0$ *Donc pour $A_n(4 ;0)$ on a $B_n(4 ;0)$ soit $A_n=B_n$.* *Donc pour $A_n>4$, soit $a_n>4$, il n'existe pas de carré C_n.* **"Car alors $y_n<0$ d'après l'équation de droite."**	Il manquait l'argument principal : *y_n* doit être strictement positif car l'ordonnée de B_n est une puissance de 3/4

CONCLUSION

Le dispositif expérimenté a confirmé que les étudiants ne sont pas habitués à vérifier si un calcul conduit à un argument recevable puis à une preuve : pour eux, le but serait juste de "faire un calcul". Il a permis une véritable implication des étudiants dans la recherche et la formulation des situations proposées. La problématique a mis en lumière la nature des obstacles des étudiants concernant l'identification des objets mathématiques, et leur formulation par des signes adaptés. On a aussi constaté leur difficulté quant au choix raisonné des procédures de preuve, et leur manque de discernement quant à la forme adéquate du raisonnement adapté au problème posé. Les étapes rédactionnelles prévues dans le dispositif les aident à surmonter ces obstacles et à adopter une posture réflexive quant aux exigences d'une preuve argumentée. L'étude didactique menée montre donc que ce dispositif s'avère complémentaire par rapport au cursus classique de licence, et peut aider les étudiants à comprendre et à réussir ce cursus.

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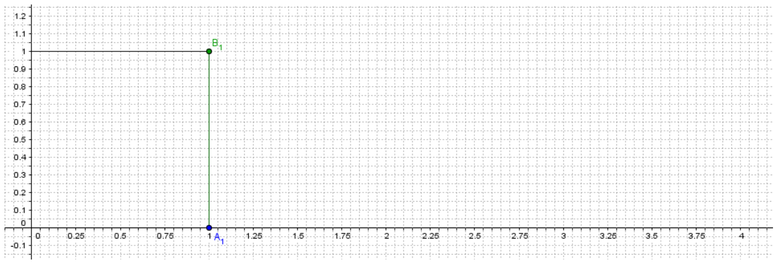
Annexe : La situation Suite de carrés

Le côté du premier carré a pour mesure 1. Le côté du deuxième carré mesure $\frac{3}{4}$ du premier, le côté du carré suivant mesure $\frac{3}{4}$ du précédent et ainsi de suite (on itère ce procédé de construction).



1. Construire les cinq premiers carrés dans le repère ci-dessous.

2. On note B_1, B_2, B_3, … les sommets « en haut à droite » de chaque carré (la suite $(B_n)_{n\geq 1}$ est une suite de points). Quelle conjecture peut-on émettre quel que soit n sur les points B_1, B_2, B_3, …, B_n ? Justifier ou invalider la conjecture.

3. On note A_1, A_2, A_3, … les sommets « en bas à droite » de chaque carré (la suite $(A_n)_{n\geq 1}$ est une suite de points).

<u>Question :</u> on se demande si, en itérant le processus de construction un nombre *n* de fois suffisant, on peut obtenir un n-ième carré dont l'abscisse du sommet A_n est strictement supérieure à 4.

4. Déduire de la propriété des sommets B_1, B_2…,,B_n, la réponse à la question ci-dessus.

5. On note a_n l'abscisse du point A_n ; la suite $(a_n)_{n\geq 1}$ est une suite numérique.

$a_1 = 1$, $a_2 = 1{,}75$, etc. $\forall n \geq 1$, exprimer a_{n+2} en fonction de a_{n+1} et a_n.

6. En mettant en œuvre un outil numérique, faites une conjecture sur la limite de la suite (a_n)

Soit un nombre $s \in [1;4[$. Peut-on déterminer le rang n_0 à partir duquel on ait $a_n \geq s$?

On pose s=4-ε, avec $\varepsilon=10^{-6}$. Par la mise en œuvre d'un outil numérique, effectuez une approximation de la valeur du rang n_0 correspondant.

7**.** Déterminer l'expression algébrique de a_n en fonction de *n*. Démontrer la convergence de la suite (a_n).

On students' reasoning about span in the context of Inquiry-Oriented Instruction

Khalid Bouhjar[1], Christine Andrews-Larson[2], and Muhammed Haider[3]

[1]American University of Malta, Malta, khalid.bouhjar@aum.edu.mt; [2]Florida State University, Tallahassee, FL, USA; [3]Southern Methodist University, Dallas, TX, USA

In this report we analyze differences in reasoning about span by comparing written work of 126 linear algebra students who learned through a particular inquiry-oriented (IO) instructional approach compared to 129 students whose instructors used other instructional approaches. Our coding of students' responses to open-ended questions indicated that IO students' concept images of span were more aligned with the corresponding concept definition than the concept images of Non-IO students. Additionally, IO students exhibited richer conceptual understanding and greater use of deductive reasoning than Non-IO students. Importantly, we argue that in order to reason about span in conceptually rich ways, students had to make use of ideas about linear independence and dimension.

Keywords: teaching and learning of linear and abstract algebra, Teaching and learning of specific topics in university mathematics, inquiry-oriented instruction, student reasoning, span.

INTRODUCTION

Active approaches to learning have been linked to improved student learning in undergraduate science, technology, engineering and mathematics courses (Freeman et al., 2014). Though, there is limited research that documented the differences in students reasoning about particular disciplinary ideas under particular instructional approaches. The purpose of this paper is to reveal these differences in reasoning about span of students whose instructors received instructional supports to teach linear algebra in an inquiry-oriented way (IO students) from those who did not (Non-IO students). Inquiry-oriented (IO) instructional approaches feature student inquiry into mathematics through problem-solving and instructor inquiry into student reasoning, and foreground the importance of leveraging student ideas to move forward the mathematical agenda of the class (Rasmussen & Kwon, 2007).

In this analysis we draw on a data from an assessment developed to assess student performance and reasoning around core concepts in linear algebra (Haider et al., 2016; Haider, 2109). This report will focus on students' responses to two multi-part questions that offer insights into students' understanding of span. The central research question for this study is: How did IO and Non-IO students, reason differently about span?

LITERATURE & THEORETICAL FRAMING

Algebraic and geometric interpretations were salient in research on students reasoning about span. Several studies found that students were more likely to approach problems about span algebraically rather than using the geometric intuition (Bogomolny, 2007;

Aydin, 2014; Ertekin, Erhan, Solak, & Yazici, 2010; Stewart & Thomas 2010). Bogomolny (2007) found that for some students, geometric and algebraic representations were not well-coordinated; students gave a geometric representation of the solution set of the homogeneous system $Ax = 0$ instead of providing a geometric representation of the span of the columns of the matrix A. By definition, span does not require linear independence, but by involving this concept students successfully interpreted span as a subspace of certain dimension (Wawro, Sweeney, & Rabin, 2011).

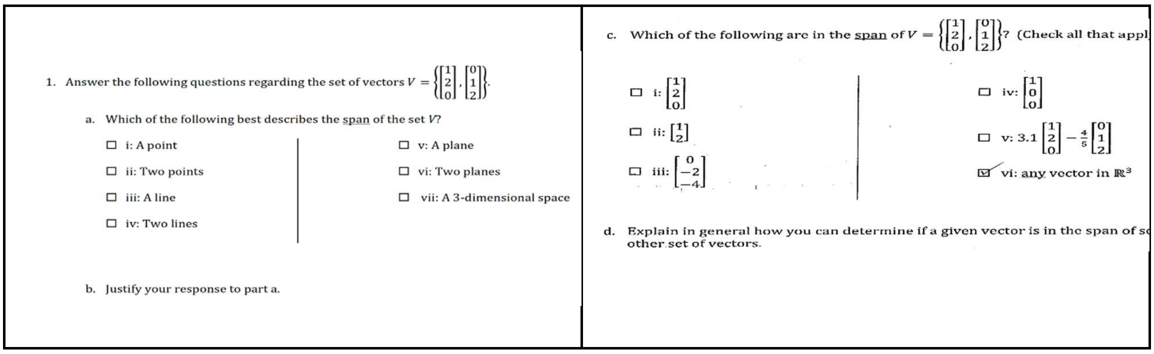
In this paper we coordinate three theoretical constructs to gain insight into systematic differences in student reasoning under different instructional approaches. The first construct we leverage is Tall and Vinner's (1981) notion of concept image which refers to the ways in which particular mathematical ideas are engaged by individuals, and concept definition which refers to formal definitions generally accepted by the broader community of mathematicians. The second construct we leverage is Hiebert and Lefevre's (1986) definition of conceptual knowledge as "knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relatonships are as prominent as the discrete pieces of information" (pp. 3-4). We methodoligically operationalize conceptual understanding by examining connections between particular ideas related to span in the context of our assessment items. The third construct we use considers the use of deductive reasoning. In this work we also paid attention to proof-like arguments in which deductive reasoning, leveraging appropriate concepts and linking them with appropriate logical connections take place. Ayalon and Even (2008) described deductive reasoning as: "…unique in that it is the process of inferring conclusions from known information (called premises) based on formal logic rules, where conclusions are necessarily derived from the given information and there is no need to validate them by experiments" (pp. 235). Johnson Laird (1999) argued that "deduction yields valid conclusions, which must be true given that their premises are true (pp. 110)." Jean Dieudonne (1969) considered logical deduction as the one and only true powerhouse of mathematical thinking.

DATA SOURCES AND STUDY CONTEXT

Our data comes from a study in which instructors received three instructional supports for inquiry-oriented mathematics instruction: instructional materials, a summer workshop focused on the intended implementation of the instructional materials, and weekly online meetings with other instructors during the term when materials were implemented. IO instructors received these instructional supports. For this analysis, we have analysed the work of a total of 255 students where 126 IO and 129 Non-IO students; to collect assessment data for comparison between performance and reasoning of IO and Non-IO students, six IO instructors were involved in these instructional supports and three Non-IO instructors from different institutions in the US. Non-IO linear algebra instructors were recruited either at the same institutions as IO instructors or at other similar institutions (e.g. similar size of student population, similar acceptance rate at institution, similar geographic area). The linear algebra assessment was administered in IO and Non-IO classes as a paper-pencil based test at the end of the semester. There were 9 assessment questions that include combinations of multiple-choice and open-ended items. Students were given one hour to complete the test. All questions were designed such that a calculator was not required. In this analysis, our focus will be on an in-depth analysis of students' reasoning on the assessment questions related to span.

Assessment items analyzed

The assessment questions analyzed for this analysis are shown below in Figure 1. Questions Q1a and Q1b offer insights into how students conceive span geometrically and Q1c and Q1d offer insights into how students interpret the elements of span. The choices in Q1a and Q1c will provide systematic insights on these students' concept images of span, whereas their open-ended responses Q1b and Q1d will provide information about nuances of students' reasoning and justification.



1. Answer the following questions regarding the set of vectors $V = \left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix} \right\}$.

a. Which of the following best describes the span of the set V?

☐ i: A point
☐ ii: Two points
☐ iii: A line
☐ iv: Two lines
☐ v: A plane
☐ vi: Two planes
☐ vii: A 3-dimensional space

b. Justify your response to part a.

c. Which of the following are in the span of $V = \left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix} \right\}$? (Check all that appl

☐ i: $\begin{bmatrix}1\\2\\0\end{bmatrix}$
☐ ii: $\begin{bmatrix}1\\2\end{bmatrix}$
☐ iii: $\begin{bmatrix}0\\-2\\-4\end{bmatrix}$
☐ iv: $\begin{bmatrix}1\\0\\0\end{bmatrix}$
☐ v: $3.1\begin{bmatrix}1\\2\\0\end{bmatrix} - \frac{4}{5}\begin{bmatrix}0\\1\\2\end{bmatrix}$
☑ vi: any vector in $\mathbb{R}^3$

d. Explain in general how you can determine if a given vector is in the span of s
other set of vectors.

Figure 1: Assessment items related to span

IO instructional approach

Since the focus of this analysis is on items relating to span, we characterize the instructional sequence implemented in the IO approach aimed at supporting students understanding of these ideas. The approach draws on the instructional design heuristics of Realistic Mathematics Education (RME) that task sequences begin by engaging students in problem solving in an experientially real setting, that the sequence of tasks follow a trajectory that anticipates students' construction of understanding of important mathematical ideas, and that the sequence supports a shift in which models-of students' mathematical activity in one phase serve as models-for students' subsequent mathematical activity (Wawro, Rasmussen, Zandieh, Sweeney & Larson, 2012).

In the context of span, students begin with an experientially real setting in which they have two modes of transportation: a hover board and a magic carpet. Each can move only in a certain direction, with movement that is symbolized by a vector so that journeys can be described using linear combinations of vectors. The task sequence has 4 core tasks. In task 1, students work to determine if it is possible to use the two modes of transportation to take a journey that starts at home (the origin) and ends at a particular location. In the second task, students work to determine if there is any location they cannot reach with the two modes of transportation. This provides students with an intuitive way of exploring the set of all possible linear combinations of two vectors – or the span of two vectors in R^2; the instructor formalizes this definition after students have worked on task 2 and provides typical examples for them to practice applying the definition in R^2 and R^3. Tasks 3 and 4 were designed to approach the linear dependence and independence concepts (see Wawro et al., 2012).

METHODS OF ANALYSIS

To answer our broad research question about students' reasoning about span, we will deal with three sub-questions. (1) How did IO students' reasoning compare to that of Non-IO students? (2) How did IO- students compare to Non-IO students connect between ideas as evidence of conceptual understanding? (3) How did IO- students compared to Non-IO students with regard to deductive reasoning?

To identify differences between IO and Non-IO students' reasoning about span, we first look quantitatively at response patterns to multiple choice questions to Q1a and Q1c, and then look qualitatively at open ended responses to Q1b and Q1d to better understand the nature of student reasoning and differences between IO and Non-IO students. To qualitatively see how IO and Non-IO students reasoned, we engaged in open coding by first examining a subset of student responses to identify the variety of mathematically distinct ways students reasoned about each open-ended response question; we continued analyzing additional responses, refining categories as we did so, until our categories were saturated. This process led to 6 categories of students' reasoning about Q1b, and 2 categories about Q1d (see Table 1 & 2). Items that did not fall into the categories described in the tables were labelled as "other" or marked if they were left blank. Student responses could be coded in multiple categories.

To gain insights into differences in students' conceptual understanding of span, particularly with regard to how they related span to other ideas, we examined students' responses to Q1b where they justify their choice on Q1a. As introduced above, conceptual knowledge is characterized in terms of relationships between ideas (Hiebert and Lefevre, 1986). The definition of span[1], in isolation, does not provide students with sufficient information to answer Q1a. To answer Q1a and provide a complete justification on Q1b, one must first have a way to reason about why the set of all linear combinations of the given pair of vectors would trace out **at least** a plane in three-space (e.g. linear independence of the two given vectors guarantees that nothing less than a plane is traced out). Then one must also have a way to think about why the set of all linear combinations of the given pair of vectors would trace out **not more than** a plane in three-space (e.g. you would need a third vector that didn't lie in the plane spanned by the first two vectors in order to span the entire three-dimensional space).

[1] Suppose $v_1, \dots, v_p$ are in $\mathbb{R}^n$. We define Span$\{v_1, \dots, v_p\}$ to be the set of all linear combinations of $v_1, \dots, v_p$. In other words, Span$\{v_1, \dots, v_p\}$ is the collection of all vectors that can be written in the form $c_1v_1 + \cdots + c_pv_p$ with $c_1, \dots, c_p$ scalars.

Code Name	*assigned when…*
Linear Independence	Student's response refers to whether or not the two given vectors are linearly independent. This includes responses that note things like: the vectors are not (scalar) multiples of each other, or something that gives that meaning (e.g. observation that the two vectors point in different directions).
Linear Combination	Student's response refers to a linear combination of the two vectors (in words, or by giving the formula $xv_1 + yv_2 = w$, or stating something like 'getting anywhere' – such as in a plane or 3-space)
Row Reduction	Student row reduces a matrix comprised of the given vectors (possibly augmented with a column of zeros).
Dimensionality	Student's response makes explicit reference to the number of vectors (2), entries (3), pivots (2), that the vectors are linear independent and exist/are/create a plane in $\mathbb{R}^3$; or claims that the two vectors are a basis AND uses these to form conclusion.
Vector as e.g. Point/Line/Plane	Student identifies each vector individually as corresponding to either a point, line, plane or 3-dimensional space.
Geometric or Graphical Representation	Response includes a drawing showing a geometric representation as a response or part of it.

Table 1: Codes for Q1b and their descriptions

Q1d (Span)	Augmented Matrix/Row Reduction	Student row reduces the matrix comprised of the given vectors and concludes the vector is/is not in the span if the result is consistent/inconsistent or there is / is not a solution.
	Linear Combination	Same description as in Q1b.

Table 2: Codes for Q1d and their descriptions

We developed a code that captured responses to Q1b that were "complete" in that they justified both why the span had to be **at least** and **at most** a plane. This was typically achieved by relating subsets of the following ideas to one another: linear independence, linear combinations, dimension, row reduction, or by coordinating with an appropriate geometric interpretation. For example, in Figure 2, Justification A is considered complete because it combined linear independence with dimension to conclude the span is a plane; Justification B used only linear independence ("not linear combinations of each other") to justify that the span of the two vectors is a plane.

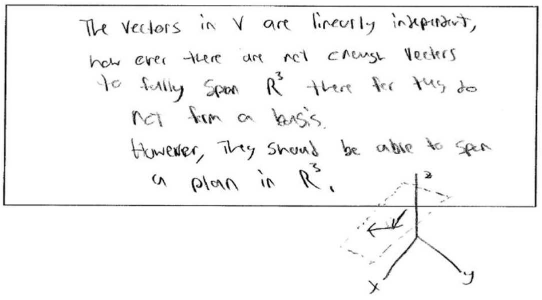
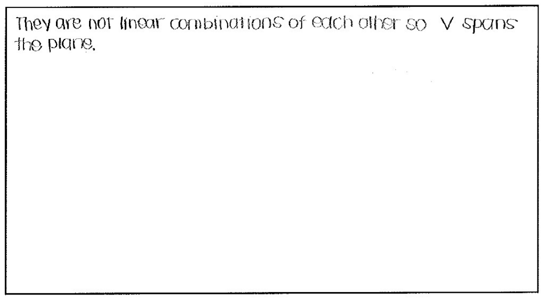
	
Justification A (Complete):	Justification B (Not Complete):

Figure 2: Assessment items related to span

Following our first round of coding, we also noticed that some responses were proof-like in nature in that they included deductive reasoning with logical connections between ideas. In order to capture this subtlety in relation to our codes, we conducted a second round of coding in which we identified when the mathematical idea corresponding to a particular code was employed in a mathematically *deductive* way. As we analyzed student responses, we noticed that some explanations were better structured than others as evidenced by both the leveraging of appropriate combinations of ideas, and by the presence of logical connections linking those ideas. We see these as important features of arguments that are logical and deductive in nature (e.g. similar to the way mathematical proofs are structured, (Rota, 1997; Johnson-Laird 1999; Ayalon and Even, 2007). To capture if students' reasoning is deductive in a systematic way, we looked at students' responses to identify if made use of logical deduction in their response. Responses that included terms like *since*, *because*, *therefore*, *this implies*, or *this leads to,* did receive the deductive reasoning code. For example, the response "since the two vectors are linear independent and they form a basis of dimension 2, they should be a plane,' was assigned a deductive reasoning code because this student used the term since, followed by two premises "linear independent and dimension," and then concluded deductively that the span of the set V is a plane.

FINDINGS

We first look at student reasoning about span based on response patterns on multiple-choice questions and our coding of their open-ended responses and interpret this through the lens of concept image and concept definition. We then examine students' open-ended responses in greater detail to consider their conceptual understanding and use of deductive reasoning.

To gain insight into differences in students' interpretations of the span of a given set of two linearly independent vectors in $\mathbb{R}^3$, we examine the choices selected by students from the two groups. Almost twice as many as IO students correctly picked a "Plane." Non-IO students picked other incorrect choices at a higher rate; in the case of choices "Two points", "A line", and "Two Planes" the differences were statistically significant at $p < 0.05$.

Choices	IO	% (IO)	Non-IO	% (Non-IO)	Significance (z-test)
i. A point	1	0.79	1	0.77	p=0.984
ii. Two points	0	00	5	3.9	**p=0.026**
ii. A line	4	3.2	12	9.3	**p=0.043**
iv. Two lines	6	4.8	8	6.2	p=0.617
v. A plane	94	74.6	53	41.1	**p<0.001**
vi. Two planes	5	4	17	13.2	**p=0.009**
ii. A 3-D space	12	9.5	14	10.9	p=0.726

Table 3: Popularity of choices of Q1a Picked by IO and Non-IO Students

When we qualitatively compared the reasoning of IO and Non-IO students, we noticed two main trends. First, IO students reasoned about span in terms of linear independence, dimensionality, or row reduction at significantly higher rates than Non-IO students. Second, significantly more Non-IO students, consistently with their selections in Q1a (Two Points and Two Planes), reasoned about the span of the set of two vectors by interpreting each vector individually as a geometric object as evidenced by more Non-IO student responses being assigned the Vector as Point/Line/Plane code.

Codes	IO students	% (IO)	Non-IO students	% (Non-IO)	Significance (z-test)
Linear independence	64	50.79%	30	23.26%	**p<0.00001**
Linear Combination	25	19.84%	22	17.05%	p=0.56868
Dimensionality	73	57.94%	44	34.11%	**p=0.00014**
Row Reduction	15	11.90%	0	0%	**p<0.00001**
Vector as Point/Line/Plane	18	14.29%	39	30.23%	**p=0.00222**
Geometric/Graphical	22	17.46%	20	15.50%	p=0.67448

Table 4: Codes for IO and Non-IO Students' Approaches to Q1b

When students were asked to identify whether or not given vectors lie in the span of a set of two vectors (Q1c), we noticed 2 trends. First, IO students correctly chose Q1c(iii); a scalar multiple of one of the vectors in the set, or Q1c(v); a linear combination of vectors in the set, at significantly higher rates than Non-IO students (see Table 6.) Second, Non-IO students incorrectly selected Q1c(iv); the vector $[1,0,0]$, and Q1c(vi); any vector in R^3, as being in the span of the given set of two vectors at significantly higher rates than IO students. Because the answer choices in this question offer insight into the ways in which a vector can be in the span of a given set of vectors, we interpret this to mean that IO students' concept image is better aligned with the concept definition of span (as compared with Non-IO students). By this we mean, IO students' concept image aligns better than Non-IO students with the concept definition. Note that the choice Q1c(iii) is similar to Q1c(v) in the sense that there should be a good understanding of the formal definition of span to see that the second scalar of the linear combination of the two vectors of V, should be 0 to get the choice Q1c(iii). Q1c(iii) and Q1c(v) represent two examples of vectors that belong to the span of the set of vectors V and they also relate to the formal definition of span by being written explicitly as linear combinations of vectors in the set. The selection of the vectors given in Q1c(iii) and Q1c(v) suggests that the students have an understanding of the formal definition of span and that made them recognize the elements that belong to span, which also suggest an alignment between the concept image and the concept definition. In other words, IO students showed a better sense of how to identify vectors in the span than Non-IO students.

Choices	IO	% (IO)	Non-IO	% (Non-IO)	Significance (z-test)
i. $[1,2,0]$	107	85%	110	85%	p=0.936
ii. $[1,2]$	19	15%	24	19%	p=0.453
iii. $[0,-2,-4]$	101	80%	78	60%	**p<0.001**
iv. $[1,0,0]$	13	10%	22	17%	p=0.119
v. $3.1[1,2,0]-\frac{4}{5}[0,1,2]$	90	71%	77	60%	**p=0.049**
vi. Any Vector in R^3	10	8%	23	18%	**p=0.019**

Table 5: Popularity of Choices of Q1c Picked by IO and Non-IO Students

Codes	IO	% (IO)	Non-IO	% (Non-IO)	Significance (z-test)
Linear Combination	99	79%	97	75%	p=0.522
Augmented Matrix (RR)	27	21%	10	8%	**p=0.002**
Other	5	4%	18	14%	**p=0.005**
Blank	5	4%	6	5%	p=0.787

Table 6: Codes for IO and Non-IO Students' Approaches to Q1d

We coded students' explanations of how they would check, in general, if some vector is in the span of a set of vectors (Q1d; see Table 6). Our findings suggest that IO and Non-IO students reasoned in terms of linear combinations at similar rates. However, the differences observed in Q1c provide evidence that students in the two groups have different interpretations of what is meant by linear combinations. IO students' selections suggest that their concept images of linear combinations tend to be more inclusive of scalar multiples and their sums and differences.

Complete justifications and use of logical reasoning

In addition to what we just presented about the reasoning of IO and Non IO students, we noted other differences between the two groups. We highlight two additional key distinctions when comparing the open-ended responses of IO students with those of Non-IO students. First, IO students exhibited more richly connected conceptual understandings of span. Second, we observed deductive reasoning at higher rates among responses of IO students.

Recall that we examined how students' ideas about span related to other ideas by examining the justifications (Q1b) for their choices on Q1a. According to our coding scheme, IO students provided "complete" justifications at significantly higher rates than Non-IO students (49.20% versus 18.60%, respectively). We interpret this to mean that IO students had more richly connected conceptual understandings of span as compared to Non-IO students.

As noticed above, when coding the data, we also noticed that IO students' responses seemed more proof-like when compared with the responses of Non-IO students; we used the deductive reasoning code to quantify this difference. Again, we noticed that 53.17% of IO students used deductive reasoning as compared to only 25.58% of Non-IO students in justifying their response to Q1a.

DISCUSSION

These results suggest that IO instructional approach and engagement in mathematical argumentation (in small group work and whole class discussion) could help explain why IO students gave better arguments. This is one possible explanation – basically that the IO learning environment is designed to give students more practice making mathematical arguments verbally in their problem-solving work and discussions and explanations during class time – so this could then be seen in improved written mathematical arguments on their assessment responses (Reinholz, 2015). Another possible explanation is that if they understood the ideas better, they would be better able to make arguments about them. A third possible explanation is the nature of this particular idea (span) is such that it cannot be understood in isolation but rather has to be coordinated with other understandings (especially linear independence and dimensionality). We suggest that a student who has these connections should have a good understanding of the formal definition of span, otherwise he/she would not have that rich connection as a result of a good conceptual understanding. This also is valid for students who used deductive reasoning in their response to the question about span. For more details on how the instructional design helps improve the relations between the concept image and concept definition, a better conceptual understanding and promotes better deduction reasoning of span see (Bouhjar et. all, 2020).

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Etude de l'enseignement du concept d'idéal dans les premières années post-secondaires : élaboration de modèles praxéologiques de référence

Julie Candy

HEP-VS, Suisse, Université de Montpellier, France, julie.candy@hepvs.ch

Cet article présente la construction et l'interprétation de modèles praxéologiques de référence pour l'enseignement du concept d'idéal dans les premières années post-secondaires (deuxième année de licence et classes préparatoires aux grandes écoles) en France, avant que ce concept ne soit enseigné de façon systématique en théorie des anneaux. La méthodologie est détaillée. Les modèles obtenus permettent une comparaison des choix opérés par les deux institutions et une première discussion de la mise en place d'une pensée structuraliste, dans l'optique de l'enseignement de l'algèbre abstraite en troisième année d'université.

Keywords: Teaching and learning of linear and abstract algebra, Teaching and learning of specific topics in university mathematics, Transition to and across university mathematics, Reference praxeological model, Structuralist praxeologies.

INTRODUCTION

Le concept d'idéal[1] est un concept d'algèbre abstraite enseigné à l'université. Généralement, il est introduit en France en deuxième année de Licence comme outil (Douady, 1992) dans le contexte de l'algèbre linéaire ou de l'arithmétique, puis étudié comme objet lors de la troisième année dans les cours d'algèbre abstraite. Les difficultés rencontrées par les étudiant-es lors de l'apprentissage de l'algèbre abstraite sont bien documentées dans la littérature (notamment Leron & Dubinsky, 1995). Ainsi, le concept d'idéal prend sa place à la transition de l'algèbre vers l'algèbre structuraliste (Hausberger, 2018, p. 82) dans laquelle le travail se fait au niveau des structures et non plus des éléments constituant ces structures. C'est donc une transition interne due à la nature épistémologique du savoir (Hausberger, 2018, p. 77). Étudier l'enseignement d'un tel concept permettra de documenter cette transition, mais aussi de mettre en lumière des éléments qui peuvent faciliter ou faire obstacle à la transition, pour l'étudiant-e, de l'algèbre vers l'algèbre structuraliste. Pour pouvoir faire une telle analyse, il convient déjà d'étudier quel est le rôle de ce concept au sein des différentes institutions que rencontrent les étudiant-es lors de leur cursus de Bachelor : comment vit le concept d'idéal ? Quelles sont ses fonctions ? Quels sont les liens entre ses différentes fonctions ? Quels éléments unificateurs sont mis en place par ces institutions pour préparer l'entrée dans la pensée structuraliste ?
Dans cet article, nous répondrons à ces questions dans le cas de la deuxième année des cursus post-secondaire en France. Il s'agit d'une première étude, la transition entre la

1 Un ensemble I d'un anneau commutatif $(A,+,\cdot)$ est un idéal si et seulement si $(I,+)$ est un sous-groupe de $(A,+)$ et $a\cdot x\in A$ pour tout $x\in I$ et $a\in A$. .

deuxième et la troisième année fera l'objet d'une communication ultérieure. Dans la première partie, nous décrirons le cadre théorique utilisé pour cette étude, ce qui permettra de raffiner nos questions de recherches. Puis, dans la deuxième partie, nous présenterons et discuterons la méthodologie de construction des modèles du savoir enseigné. Enfin, nous conclurons par l'analyse des modèles et les réponses apportées à nos questions de recherche.

CADRES THÉORIQUES, REVUE DE LITTÉRATURE ET QUESTIONS DE RECHERCHE

Comme nous l'avons mentionné, le concept d'idéal est un concept de l'algèbre abstraite enseigné dans plusieurs institutions différentes et à différents niveaux. Ainsi, l'étude de l'enseignement de ce concept nécessite de se placer dans un cadre théorique qui permette d'analyser la manière dont le concept est enseigné tout en tenant compte des caractéristiques institutionnelles. Ainsi notre choix s'est porté sur la théorie anthropologique du didactique (Chevallard, 1998). Nous compléterons notre étude grâce à des notions d'écologie des savoirs (Artaud, 1997) et la notion de praxéologie structuraliste (Hausberger, 2018).

Transposition didactique et praxéologies

Les savoirs tels qu'ils sont enseignés, même dans l'enseignement supérieur, ne sont pas exactement les savoirs savants tels qu'ils se développent dans la communauté scientifique. Ainsi Chevallard (1991) explique que le savoir savant est transformé en savoir à enseigner dans un processus appelé *transposition didactique externe*. Puis le savoir à enseigner subit lui-même un processus de *transposition interne* par lequel il est transformé en savoir enseigné. Dans cet article, nous présenterons l'analyse du produit de la transposition interne du concept d'idéal.

Le concept de praxéologie (ou d'organisation mathématique), qui est un concept central de notre étude, peut être défini comme suit (Chevallard, 1998) : l'activité mathématique peut se décrire en termes de quadruplets $(T, \tau, \theta, \Theta)$ où T est le *type de tâches*, τ est la *technique* qui permet de réaliser T , θ est la *technologie*, c'est-à-dire le discours qui décrit et justifie la technique, et enfin Θ désigne la *théorie*. Par exemple, un type de tâches T : « Montrer qu'un sous-ensemble d'un anneau est un idéal » peut être réalisé via la technique τ : « Montrer que cet ensemble vérifie les propriétés de la définition formelle » dont la technologie est θ : « Définition d'un idéal » et la théorie Θ sera résumée dans cet article par "théorie des anneaux".

Modèle praxéologique de référence

Afin de pouvoir analyser la place de ce concept dans les institutions en jeu, nous avons choisi de construire un *modèle praxéologique de référence*. Comme l'expliquent Chaachoua, Ferraton et Desmoulins (2017, p. 302) :

> L'identification de ces [organisations mathématiques] passe donc par la caractérisation des types de tâches institutionnels et peut être vue comme une « reconstruction » du chercheur. Notons que ce dernier, pour des raisons liées à sa problématique, peut bien entendu procéder à un autre découpage que celui de l'institution voire le compléter ; il construit alors un modèle praxéologique de référence (MPR) regroupant les praxéologies à enseigner, enseignées mais également enseignables. Le modèle rend ainsi possible l'analyse de ce qui a cours dans différentes instances d'un système d'enseignement.

Pour organiser ce découpage nous utiliserons le regroupement en organisations mathématiques proposé par Chevallard (1998) : les *organisations mathématiques ponctuelles* sont générées au sein de l'institution par un unique type de tâches T. Ces organisations mathématiques ponctuelles prennent place dans des *organisations mathématiques locales* résultant de l'intégration de diverses organisations mathématiques ponctuelles sous un discours technologique commun. Enfin, de manière analogue, les *organisations mathématiques régionales* résultent de l'intégration de diverses organisations mathématiques locales sous un discours théorique commun.

Ecologie des savoirs

L'écologie des savoirs rend le chercheur *attentif aux dépendances des objets qu'il étudie* (Artaud, 1997, p. 101) : elle amène à considérer les concepts mathématiques comme n'étant pas détachés de leur environnement mais bien faisant partie d'un *écosystème* sur lequel ils agissent et qui agit sur eux. Dans cet article nous étudions le concept d'idéal dans l'*écosystème didactique scolaire*, lieu d'enseignement du concept. Suivant Artaud (1997, p. 113), nous appellerons *habitats* les différents lieux de vie du concept et *niches* la fonction que le concept occupe au sein de chacun de ces habitats. Enfin, nous reprenons également le concept de *besoins trophiques* que nous utiliserons dans la suite de l'étude : en ce qui concerne les objets mathématiques, il s'agit des objets dont un objet mathématique donné a besoin pour vivre dans l'écosystème considéré.

Les praxeologies en algèbre abstraite

Pour modéliser l'entrée dans la pensée structuraliste, Hausberger (2018) introduit la notion de *praxéologie structuraliste* et de *niveau structuraliste* d'une praxéologie. Le point de départ est la reconnaissance du rôle fondamental joué par la *dialectique entre le particulier et le général*, que Hausberger appelle également *dialectique entre objets et structures*. Par exemple, plutôt que de démontrer que l'anneau des décimaux est principal, des apprenant-es de niveau avancé s'attacheront à démontrer le résultat plus général suivant : tout sous-anneau de $\mathbf{Q}$ est principal (Hausberger, 2018, p. 84). En d'autres termes, la méthode structuraliste vise à raisonner en termes de classes d'objets (les sous-anneaux), de propriété (la principalité) conservée ou non par des opérations sur les structures (ici, le passage à un sous-anneau), de façon à mettre en évidence les « ressorts » des preuves.

De plus, différentes généralisations d'un résultat sont possibles, avec différents niveaux de généralité. Lorsque le résultat engage des objets (i.e. il ne s'agit pas d'un résultat purement théorique), des preuves élémentaires sont souvent possibles, où les structures jouent essentiellement le rôle de vocabulaire. La praxéologie qui en résulte est dite de niveau structuraliste 1. Le niveau 2 est atteint lorsque sont utilisés des résultats généraux sur les structures, qui conduisent à des praxéologies dites structuralistes de niveau 2. Hausberger définit également un niveau 3, qui intervient rarement dans les premières années d'apprentissage, de sorte que nous ne le détaillerons pas ici. Donnons un exemple : le type de tâche "Montrer qu'un anneau donné est principal" peut être résolu "à la main" pour $\mathbf{Z}$ ou $\mathbf{K}[X]$ (niveau 1). Lorsque l'on utilise le théorème "Tout anneau euclidien est principal", il s'agit du niveau 2. La technologie de la praxéologie de niveau 1 peut également contenir des éléments du type "On procède comme pour $\mathbf{Z}$, la clef est l'existence d'une division euclidienne", en général apportés par l'enseignant. Ceci suggère une théorie *implicite*, celle des anneaux euclidiens, et constitue, en quelque sorte, un niveau intermédiaire dans l'entrée dans la pensée structuraliste.

Formulation des questions de recherche

Ce cadre théorique nous permet donc de formuler les questions de recherche suivantes relativement aux deux institutions à l'étude : quelles sont les organisations mathématiques dans lesquelles prend place le concept d'idéal ? Comment sont-elles structurées en organisations locales et régionales ? Quels liens met en évidence le point de vue de l'écologie des savoirs ? Que dire des niveaux structuralistes de ces praxéologies ? Les réponses apportées dans l'article à ces questions nous permettrons de conclure avec une analyse comparative des institutions intervenant dans l'étude.

MÉTHODOLOGIE

Description des institutions

Notre étude requiert l'identification des institutions en jeu. Les CPGE revêtent les caractéristiques d'une institution scolaire « classique » : il y a un programme officiel[2] et à la fin des deux ans de classe préparatoire, les étudiant-es passent, pour la majeure partie d'entre eux, des concours d'entrée aux écoles d'ingénieurs. Enfin, les étudiant-es de CPGE ont été admis-es à l'entrée sur dossier en fonction de leurs résultats scolaires et les plus faibles d'entre eux et elles ne sont pas autorisé-es à suivre la deuxième année « Mathématique-Physique ». Une dernière particularité est à prendre en compte dans cette institution : elle est séparée en deux classes, MP et MP*. Les étudiant-es qui ont obtenu les meilleurs résultats en première année intègrent la MP* qui a donc pour vocation de préparer aux concours des écoles d'excellence, comme par

2 https://prepas.org/index.php?module=Site&voir=document&id_document=397

exemple l'Ecole Normale Supérieure (ENS). Les programmes officiels étant les mêmes, nous avons choisi de ne pas séparer MP et MP* en deux institutions distinctes.

Parallèlement aux CPGE, se pose la question de l'Université. L'absence de programme officiel national, la rédaction des syllabus dans des commissions internes aux départements de mathématiques et la grande liberté laissée à l'enseignant nous amène à nous questionner sur la possibilité de considérer l'Université comme une institution générique. L'examen du syllabus de plusieurs universités françaises nous a permis de mettre en évidence une homogénéité et stabilité des programmes également relevée dans la littérature (Bosch, Hausberger, Hochmuth & Winsløw, 2019). Nous avons fait le choix de considérer la deuxième année de Licence (L2) comme une institution et, dans la poursuite de nos travaux, la troisième année (L3) comme une seconde institution car c'est en L3 que les étudiant-es se spécialisent en mathématiques. De plus, c'est également en L3 que des étudiant-es venant des CPGE intègrent l'Université. Ainsi, le choix d'étudier en parallèle le modèle praxéologique institutionnel de L2 et de CPGE et de les comparer prend sens puisque cette analyse permettra d'obtenir des résultats sur les transitions L2-CPGE/L3 et donc en particulier sur l'entrée dans la pensée structuraliste. Dans cet article, nous nous centrerons sur les résultats des analyses de CPGE et de L2.

Choix et constitution du corpus

L'étude de la transposition didactique du concept d'idéal en Licence implique de récolter des données sur la manière dont le concept est enseigné. Plutôt que d'aller récolter des données en classe, nous avons choisi une étude de corpus formé de polycopiés de cours et de feuilles de travaux dirigés (avec leurs corrigés) d'enseignants en algèbre. Ce choix est essentiellement un choix pragmatique permettant un recueil à plus grande échelle. De plus, la transmission du corpus peut sembler moins invasive qu'une présence dans la classe. Pour compléter ce corpus écrit, nous avons choisi de mener des interviews de ces enseignants dont les résultats ne seront pas utilisés dans cet article.

Pour l'étude en deuxième année du post-secondaire en France, notre corpus est constitué des documents de cours et interviews de deux professeurs de CPGE, l'un professeur de MP et l'autre professeur de MP*, et de deux professeurs de L2.

Les analyses praxéologiques et les modèles praxéologiques de référence

L'analyse praxéologique d'un savoir universitaire complexe comme celui d'idéal soulève plusieurs questions méthodologiques.

La première question concerne la méthodologie pour discriminer ce qui est une tâche de ce qui est un type de tâches au sein de l'institution. Pour ce faire, si le nombre d'occurrences des tâches analogues est supérieur à deux, nous avons considéré que l'on peut rattacher ces tâches à un type de tâche. Pour les tâches isolées dans les analyses praxéologiques, notre analyse épistémologique ainsi que la culture du sujet que nous avons développé à travers l'analyse de manuels classiques comme celui d'Escofier (Escofier, 2016 analysé dans Candy, 2020) nous permettent d'identifier la présence ou non d'un type de tâches sous-jacent au sein de l'institution. La présence de ces cas limites est à relier au statut du concept d'idéal en deuxième année du post-bac : le concept est un concept outil dont la dimension objet est peu travaillée, le nombre de tâches dans chaque corpus qui le mobilise est inférieur à 15.

Une fois que les types de tâches sont identifiés comme tels se pose une seconde question méthodologique qui concerne le degré de généralité ou la forme de la généralisation à adopter dans la formulation du type de tâches. Par exemple, l'on rencontre lors de l'analyse du corpus des tâches où l'on doit montrer que $\mathbf{K}[X]$ ou $\mathbf{Z}$ sont principaux. La question se pose alors de l'appartenance de ces tâches à un type de tâche T1 « montrer qu'un anneau est principal » ou T2 « montrer qu'un anneau euclidien donné est principal ». Si la technologie contient toujours (à chaque instanciation de la tâche) la division euclidienne (dont on n'attend pas, à ce niveau, de définition formelle unificatrice) alors on va opter pour T2. Ceci est lié également au niveau structuraliste de la praxéologie, que la description soignée des techniques, technologies et théories, en cohérence avec l'intitulé du type de tâche, va permettre d'identifier, voire de clarifier lorsqu'il s'agit d'un niveau intermédiaire (entre le niveau 1 et 2). Cette identification peut donc être discutée mais appartient à la construction d'un modèle praxéologique de référence du chercheur et donc doit aider à amener des réponses aux questions de recherche.

RÉSULTATS

Dans le cadre de cet article nous ne détaillerons pas, par manque de place, toutes les praxéologies du modèle praxéologique de référence.

Les modèles suivants présentent deux types d'organisations mathématiques régionales : des organisations mathématiques régionales *structuralistes* (représentées en vert sur les modèles) dans lesquelles sont développées les propriétés concernant les structures et des organisations mathématiques régionales *mixtes* (représentées en orange sur les modèles) dans lesquelles les propriétés des structures sont contextualisées à un domaine d'objet afin de produire de nouveaux résultats sur ce domaine. Par exemple, le concept d'idéal principal, vivant au sein de l'organisation mathématique régionale structuraliste *structures algébriques usuelles (anneaux)* est ensuite contextualisé dans le domaine d'objets $\boldsymbol{K}[X]$ pour nourrir les besoins trophiques de l'organisation mathématique locale *PGCD ou PPCM dans* $\boldsymbol{K}[X]$ en permettant la définition du concept de PGCD ou de PPCM de polynômes. La superposition d'une organisation mathématique mixte à une organisation mathématique structuraliste signifie que les propriétés du domaine d'objets sont déduites à partir des propriétés générales sur les structures. La circonscription de la notion de théorie dans le cadre de l'algèbre structuraliste est décrite plus en détail dans Candy (2020).

Les modèles praxéologiques de CPGE et de L2

Tout d'abord, nous pouvons commenter la structure globale des modèles ci-dessous. En L2, il vite une seule organisation mathématique ponctuelle *appartenance d'un ensemble à la classe d'objet idéal (via la définition)*. En dehors de cette organisation mathématique, le concept prend place au niveau de la théorie de deux organisations mathématiques locales distinctes : *polynôme minimal d'un endomorphisme*, qui s'inscrit dans la théorie mixte de *réduction des endomorphismes* et *PGCD et PPCM dans* $\boldsymbol{K}[X]$, qui s'inscrit dans la théorie mixte *anneau de polynôme sur un corps*. En CPGE vit en plus l'organisation mathématique locale *PGCD et PPCM dans* $\boldsymbol{Z}$. L'organisation mathématique locale *polynôme minimal d'un élément algébrique* ne vit qu'en MP* au sein de l'organisation mathématique régionale *A-algèbre*. On ne trouve pas mention des nombres algébriques dans le programme officiel de CPGE. Le fait que cette organisation mathématique locale y vive s'explique par la finalité de la formation : les étudiant-es doivent être préparé-es aux concours d'entrée aux grandes écoles. Or dans ces concours on trouve des sujets sur les nombres algébriques. Cette organisation mathématique locale est donc particulière car elle n'est pas une contrainte de l'institution CPGE elle-même ; elle existe sous l'influence d'autres institutions, en l'occurrence les grandes écoles, qui motivent ces concours.

Notons, dans le modèle de CPGE, la présence de l'organisation mathématique ponctuelle *Propriétés formelles des opérations sur les idéaux*. On y trouve des tâches comme « Montrer que $\sqrt{I \cdot J} = \sqrt{I \cap J}$ » ou «Montrer que $I \cdot J \subset I \cap J$». Cette organisation mathématique consiste en un travail sur des idéaux généraux. Elle se distingue car c'est une organisation mathématique ponctuelle qui n'appartient à aucune chaîne trophique dans l'institution en jeu, et vit donc détachée des autres. On peut expliquer cela par le fait que les raisons d'être de cette organisation mathématique prennent place, par exemple, en géométrie algébrique. Par contre, nous faisons l'hypothèse que sa présence est révélatrice d'un phénomène didactique : cette organisation mathématique est détachée de ses raisons d'être par l'institution pour entraîner les étudiant-es, sur des exemples classiques, à utiliser des définitions formelles, à mobiliser des techniques algébriques, logiques et ensemblistes.

En CPGE, on note la présence d'une organisation mathématique intitulée *principalité dans les anneaux euclidiens* qui n'existe pas en L2. En effet, on trouve dans les analyses praxéologiques des types de tâches dont la technologie repose sur des propriétés de principalité d'anneaux euclidiens tels que $\mathbf{Z}[i]$. Les éléments technologiques mobilisés en CPGE reposent sur la construction d'une division euclidienne dans ces anneaux (ce qui nécessite de raisonner par analogie avec le cas de $\mathbf{Z}$ pour envisager une extension dans un cas plus général). L'organisation mathématique régionale *anneaux euclidiens* est représentée en pointillés pour signifier que le professeur de MP* ne va pas jusqu'à l'introduction formelle de la théorie des anneaux euclidiens mais unifie les praxéologies développées dans *principalité des anneaux euclidiens* à l'aide d'un discours méta portant sur le rôle de la division euclidienne dans ces praxéologies.

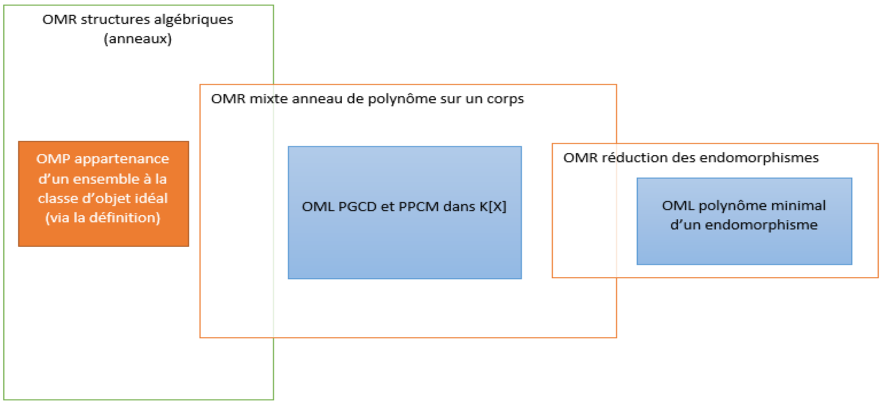

Figure 3 : modèle praxéologique de référence de l'institution L2

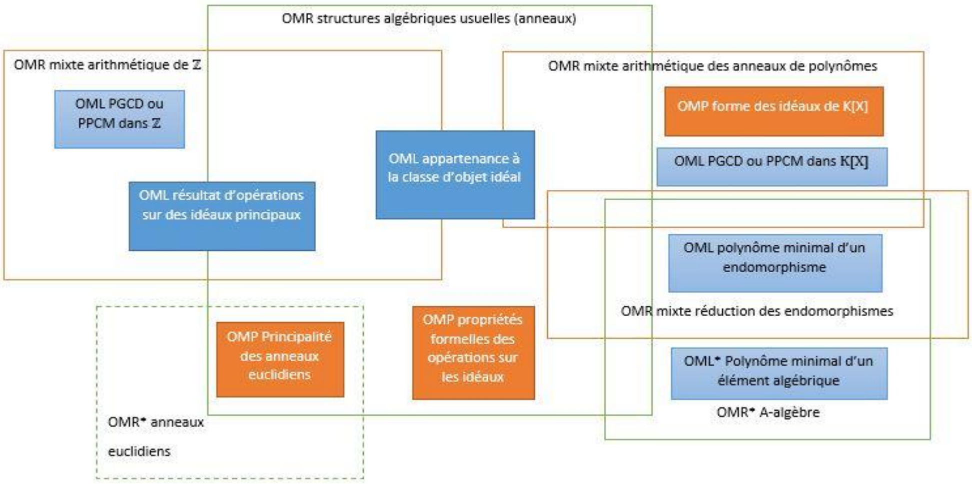

Figure 4 : modèle praxéologique de référence de l'institution CPGE.

On remarque enfin qu'en L2, le concept d'idéal est engagé majoritairement dans des organisations mathématiques très ciblées et réduites ; il existe un unique type de tâches qui fait vivre la définition du concept d'idéal : « montrer qu'un ensemble I d'un anneau A est un idéal ». On peut faire l'hypothèse (à confirmer lors des entretiens) que les enseignants-chercheurs de L2 ont considéré que la notion d'idéal vit, dans le cadre du programme de L2, en dehors de son écosystème naturel (la théorie abstraite des anneaux, dans laquelle la notion d'idéal est la "bonne" notion pour fabriquer des anneaux quotients). En d'autres termes, un parti-pris épistémologique et écologique les conduirait à limiter le travail possible autour de cette formalisation, notamment lorsque l'objectif final du module est centré sur la réduction des endomorphismes. In fine, ces choix différents de transposition didactique seraient également à relier à des contraintes institutionnelles différentes : la division du curriculum en CPGE est moins contraignante que celle en modules à l'Université. De plus, il faudrait comparer les temps d'enseignement alloués à ces concepts. Enfin, dans l'institution CPGE, un même enseignant prend en charge la totalité du programme de l'année (cours et travaux dirigés) ce qui permet un point de vue plus holistique et pourrait favoriser le développement d'organisations mathématiques plus étendues.

CONCLUSION

Malgré un nombre peu élevé de tâches qui concernent le concept d'idéal dans le corpus, nous constatons que le concept d'idéal est un outil d'introduction de nouvel objet au sein des habitats réduction des endomorphismes, nombres algébrique, arithmétique de ***Z*** et arithmétique de **K**[X] et c'est la raison d'être principale de sa présence au sein de ces institutions.

La présence de l'organisation mathématique ponctuelle isolée *propriétés formelles des opérations sur les idéaux* va pousser à introduire des objets (comme le radical par exemple), probablement pour des raisons d'entraînement de techniques algébriques, ensemblistes ou formelles, et qui ne pourront prendre du sens pour l'étudiant-e que bien plus tard dans sa formation (à la différence de l'enseignant-e qui a probablement choisi ces exemples pour leur portée mathématique, donc en un certain sens des raisons “esthétiques”).

Si l'on compare la L2 et la CPGE, on s'aperçoit qu'en L2 il n'y a qu'une organisation mathématique ponctuelle où le concept d'idéal est mobilisé dans le bloc de la praxis. De plus, le niveau structuraliste des types de tâches est différent entre les deux institutions. Par exemple, dans l'organisation mathématique locale *principalité dans les anneaux euclidiens* la description des techniques associées aux praxéologies de cette organisation mathématique permet de montrer que le travail se fait au niveau des objets sans utilisation de théorème plus généraux (par exemple la principalité des anneaux euclidiens). De ce fait, les praxéologies travaillées dans les deux institutions sont structuralistes de niveau 1. Cependant, en CPGE nous avons noté un travail sur différents anneaux euclidiens qui repose sur la technologie structuraliste implicite (souvent relevée dans les notes des professeurs) de principalité des anneaux euclidiens. Ainsi, en CPGE nous nous situons à un niveau intermédiaire entre le niveau 1 et le niveau 2 dans lequel le théorème structuraliste apparaît sous-jacent dans les remarques du professeur.

La comparaison du modèle praxéologique de référence de L3 aux MER de L2/CPGE nous permettra de documenter le phénomène de transition auxquels les étudiant-es sont soumis-es au passage à l'algèbre structuraliste : les organisations mathématiques locales présentes dans les modèles praxéologiques de L2 et de CPGE sont-elles encore présentes ? Y a-t-il une évolution de ces organisations mathématiques locales vers des organisations mathématiques locales structuralistes de niveau 2, c'est-à-dire que les technologies contiendront des théorèmes qui portent sur la structure ? Ou au contraire aura-t-on une construction de nouvelles organisations mathématiques locales structuralistes sans qu'elles soient reliées à celles construites dans les institutions L2 et CPGE ?

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BACKWARD REASONING AND EPISTEMIC ACTIONS IN STRATEGIC GAMES PROBLEMS

Marta Barbero[1,2] Inés M. Gómez-Chacón[2] and Ferdinando Arzarello[11]

Università di Torino (Italy), [2]Universidad Complutense de Madrid (Spain), igomezchacon@mat.ucm.es

This paper focuses on the epistemic and cognitive characterization of backward reasoning in strategy games resolution. It explores the use of AiC (Abstraction in Context) as a tool for the analysis of the epistemic actions involved in these processes. It is reported a first analysis developed by the research team in order to be used as protocol-guide in the analysis of a study carried out with PhD students in Mathematics Degree in a Spanish and an Italian University, who face problem solving games. The case study shows the process of discovery that a PhD student makes to formulate a general recursive formula. It is a key for understanding the interaction between the AiC model and the characteristics of backward reasoning. The analysis allows to combine the two models - backward reasoning and AiC - in a unified framework that allows to focus both short-term and long-term processes in students' activities.

Keywords: Teaching and learning of specific topics in university mathematics, Teaching and learning of logic, reasoning and proof, backward reasoning, epistemic actions, strategy games

INTRODUCTION

Backward reasoning has great potential in the study of mathematics since it can support students when engaged in tasks, where they are asked to pass from argumentations and inquiry to mathematical proofs. For deepening this issue we specifically developed some studies at the university level focussed on mathematical thinking, where learning the method of analysis is a critical issue (Antonini, 2011, Peckhaus, 2000).

In such studies, which analysed mathematics and engineering students involved in problem solving activities (Gómez-Chacón & Barbero, 2018 & 2019), it was noted that so-called regressive reasoning — as an emerging key process in the dialectics between inference processes — develops mainly in interrogative movements and is responsible for the generation of new ideas and elements in the solution process. This reasoning is used in its character of "ordering device": through it, the students manage to find elements necessary for the construction/definition of the objective. The backward reasoning, which is based on the return of reasoning to an informal context, helps to connect more intuitive aspects with the mathematical and computational context.

In standard mathematical problems, it is more difficult working backwards than forwards. So it is necessary to offer students a large class of problems to which the method of working backwards is appropriate, such as strategy games presented here. We also identified some factors in the cognitive and affect interplay, which would inevitably cause difficulties for students to construct and work backwards. These studies (Gómez-Chacón, 2017) showed how the epistemic emotions continually exert numerous so-called operator effects, both linear and nonlinear, on attentional activity and on the ability to perceive goal-path obstacles and to overcome them. Understanding is linked with the appraisal of their ability to influence (control dimension), with their ability to predict, and with mental flexibility (Gómez-Chacón, 2017; Gómez-Chacón & Barbero, 2019). The dectected taxonomy of obstacles suggests that the lecturer, as a mediator of knowledge, explicitly takes into account the nature of backward reasoning underlying the interplay between epistemological and cognitive models.

This paper focuses on the epistemic and cognitive characterization of backward reasoning in strategy games resolution. Strategy games allow for the natural development of backward reasoning. Players must make strategic choices to make their moves. These choices are triggered by typical implicit questions that players ask before making a new move: “What can I do in this situation? What is better to do?” To answer these typically strategic questions, they reflect both on the moves already made and on the possible moves to do and they activate the backward reasoning (Wickelgren, 1974).

We explore the use of AiC (Abstraction in Context) as a tool for the analysis of the epistemic actions involved in these processes of resolution. We try to understand how the process of abstraction evolves, analyzing the relationship established between the epistemic actions (categories) of the RBC model of Dreyfus and Kidron (2014), and it is based on the perspective of abstraction in AiC context (Dreyfus et al., 2001), as well as by the subcategories of analysis introduced in this investigation based on the specific characteristics of the regressive reasoning. Epistemic actions are understood as mental actions that develop during the abstraction process and explain the emergence of a new, more elaborate and complex construction. We report a first analysis developed by the research team in order to be used as protocol-guide in the analysis of a study carried out with 185 undergraduate students in Mathematics Degree. Further analysis can be found in Barbero, Gómez-Chacón & Arzarello (2020).

The structured as follows: first the theoretical frame underlying the analytical methodology of the study; second, the context of the study description and its particular goals; third, first results presentation, drawn from a case study micro-analysis, where the theoretical background is applied; finally, a discussion and some conclusions.

BACKWARD REASONING

In mathematics, progressive reasoning alone is not exhaustive to fulfil the tasks of solving problems. Great mathematicians like Pappus, Descartes, Leibniz, in their discussions about analysis and synthesis, emphasize this fact (Peckhaus, 2000). Backward reasoning is known by different denominations, each underlying some of its main features: regressive analysis, backward solution, method of analysis, etc. It is the practice that involves the making of a number of arguments from the bottom of the problem and proceeds through logical correspondences which allow to obtain something known or to be reached through other paths. This process includes different ways of proceeding in problem solving: Backward heuristics, Reductio ad Absurdum, Starting with the end of the problem, Assuming the problem solved (Beaney, 2018).

Pappus was the mathematician who has contributed substantially to the clarification and exemplification of the method. In the seventh book of his Collection he deals with the topic of Heuristics (methods to solve the problems). There he exemplifies the method of analysis as the method of synthesis, therefore making the development of this reasoning clearer. Pappus defines the method of analysis as follows: "In analysis, we start from what is required, we take it for granted; and we draw correspondence (ακολουθον) from it and correspondence from the correspondence, till we reach a point that we can use as a starting point in synthesis. That is to say, in analysis we assume what is sought as already found (what we have to prove as true)." (elaboration by Polya, 1965 and by Hintikka and Remes, 1974). Subsequently he points out: "This procedure we call analysis, or solution backward, or regressive reasoning." (Hintikka and Remes, 1974). And on the Method of Synthesis: "In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequents the former antecedents and linking them one with another, we in the end arrive at the construction of the thing sought. This procedure we call synthesis, or constructive solution, or progressive reasoning." (Hintikka and Remes, 1974)

The two processes are closely related and there is no analysis method without the synthesis one. Solving a problem is therefore a combination of the two procedures. Peckhaus (2000) studies this analysis-synthesis scheme and affirms that "The analytical [is] […] the procedure which starts with the formulation of the problem and ends with the determination of the conditions for its solution. The synthetical represents the way from the conditions to the actual solution of the problem. […] This branch of the scheme is deeply connected with the complementary [one]." Not only analysis can't exist without synthesis but also "synthesis can't be isolated and presupposes analysis."

The concept of Backward Reasoning involves characteristics that allow us to identify its development throughout the resolution of a task. Philosophers and mathematicians from the ancient Greeks, through the authors from the 17th and 18th centuries to the 20th one have studied its characteristics. The main features are the following:

- *Direction vs cause-effect*. In Pappus' definition, the backward direction of reasoning is highlighted. This entails going from the end of the problem to its beginning. By applying the method, the premises of a certain idea are sought. In the 17th and 18th centuries, authors such as Arnauld and Nicole interpreted the method as a search for cause-effect relationships between ideas. By these, the connection between the notions in background and the problem are identified. The knowledge of the development of the resolution of the task and the effects and causes of each notion involved in the process arise (Beaney, 2018; Peckhaus, 2002).

- *Decomposition*. According to Plato and Pappus, this kind of reasoning allows for the reduction of the problem to its simplest components. The properties that define the assignment and the relationships between the most complex and the simplest objects involved in it are identified by extracting and investigating the principles that are at the base of the task. Aristotles, for example, underlines the fact that "sometimes, to solve a geometrical problem, you can only analyse a figure", breaking it down into its basic components and understanding the different parts of it (Beaney, 2018).

- *Introduction of auxiliary elements*. Kant, Polya and Hintikka, focus their attention on a fundamental process part: the introduction of new elements (a known Geometry practice: the auxiliary constructions). In the progressive and deductive processes all the bases are given and from these, consequences are elaborated. Unlike the backward reasoning, new notions appear and develop throughout the resolution at specific moments, according to the solver needs (Beaney, 2018; Hintikka & Remes, 1974).

EPISTEMIC ACTIONS

The concept of epistemic action was introduced into cognitive sciences by Kirsh and Maglio (1994) to indicate those physical actions that facilitate cognition and allow problems to be solved more quickly. These actions help to acquire useful information for the resolution that are hidden or difficult to compute mentally, have the purpose to simplify the mental processes. In mathematics education the term was first used by Hershkowitz, Schwarz y Dreyfus, (2001), who derived it from Pontecorvo and Girardet (1993) in their research on abstraction. The mental processes that occur in the student when solving a problem are not directly observable but can be identified through the analysis of the students' verbalisation or their physical actions. Epistemic actions are those actions that allow to identify the mental progresses in which knowledge is used or built and to operationally describe the procedures. They develop within the argumentative processes and are the basis of the interpretative activities. The actions involve procedures of a high methodological and metacognitive level and include the explanation of those procedures used for the interpretation of particular events.

The research of the last twenty years has resulted in the development of the theory of Abstraction in Context (AiC) (Dreyfus and Kidron, 2014) which aims to provide a theoretical and methodological approach, at the micro level, on the processes of learning mathematical knowledge. From a theoretical point of view, AiC attempts to create a bridge between cognitive knowledge and theories of abstraction, constructivist theory and the theory of activity. From a methodological point of view, AiC is a tool that allows the analysis of thought processes. The central theoretical construct of AiC is a theoretical-methodological model, according to which the emergence of a new construct is described and analyzed by means of three observable epistemic actions: recognizing (R), building-with (B), and constructing (C).

- Recognizing. It consists in recognizing some previously learned knowledge as relevant for the resolution of the problem.
- Building-with. It consists of combining a set of knowledge with the aim of achieving a specific objective. Objectives can be: to implement a strategy, to meet a justification to a conjecture, to find the solution to the problem.

- Constructing. It consists in assembling and integrating the previous knowledge with the aim of producing a new construct.

RESEARCH AIM AND METHODOLOGY

Aim

The aim of this paper is to show the use of AiC combined with the characterization of backward reasoning as a tool for the analysis of the epistemic actions involved in discovery processes. Both epistemic and cognitive elements are highlighted to examine how university students develop backward reasoning.

Participants and instrument

Data were collected in 2018 from 185 Spanish and Italian mathematics students, aged between 19 and 30. The participants students are spread all over all the years of academic studies from the first year of Bachelor to the last year of PhD (Table 1). They have different mathematical notions with regard to solving problems, but they had not received any special training about backtracking heuristics. These data are summarised in Table 1.

Mathematics Bachelor Italy	Mathematics Bachelor Spain	Future High School teachers (Master-students)	Mathematics PhD	Total
99	50	28	8	185

Table 1. Participants

To study the epistemic and cognitive characterization of backward reasoning in strategy games resolution we choose the 3D Tic-Tac-Toe (Golomb and Hales, 2002). This is a finite 2 players game with perfect information. Generally, it is played with paper and pencil. The board of the k-dimensional Tic-Tac-Toe ($k>1$) is a k-dimensional cube of side n, i.e. a (n, k)-board. The two players choose to adopt "X" or "O" to indicate the position of their pawns on the board. The game version used for the this research project experiments consists of a (4,3)-board. The board was presented in its two-dimensional representation (Fig. 1). The objective is to place 4 marks in a row horizontally, diagonally or vertically while trying to block the opponent from doing so.

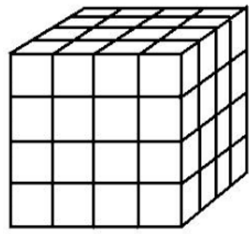
The given task (Fig. 1) consists in solving the game and finding a relationship between the number of winning lines and the board dimensions. Some mathematical notions acquired in university degree are necessary to solve it.

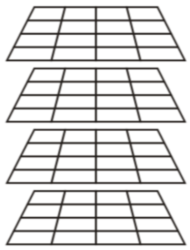
The methods for obtaining the data are direct observations during the working session, the recordings from the cameras, and the documents where students describe their approaches to the problem solution on protocols. The students worked in pairs or alone; we gave each pair of students paper and pencil and some “empty board”, using which they could elaborate a game strategy. Students were also asked to describe their approaches to solving the problem specifically describing: their thought processes in the resolution, the difficulties they encountered, and the strategies they would use in order to solve with paper and pencil. Students had two hours to do that.

3D Tic-Tac-Toe is the three dimensional version of the classic Three in a Skate game. The game board is a 4x4x4 cube.

The game is for two players. One player uses "crosses" and the other uses "zeros".

The objective is to place 4 marks in a row horizontally, diagonally or vertically while trying to block the opponent from doing so.





1. By helping yourself with the two-dimensional version of the game board, solve the game by developing your thinking process with a detailed solution protocol.

2. Mathematically express (formula, pattern, routine, ...) the relationships that can happen between the dimensions of the game board and the winning lines

Figure 1: Strategy game statement

A qualitative analysis was chosen to examine the resolution protocols of the students through the combination of the Backward Reasoning Epistemic Model and the AiC Model. We will illustrate it through a significant example in next section.

RESULTS: CASE STUDY

In this section we analyse a single student's resolution protocol of the 3D Tic-Tac-Toe. This allows us to get a deep understanding of the tendencies of the behaviour related to the sequences of actions during the discovery phase of resolution. The chosen student, whom we name A, is key informant of the PhD students group. A is an expert student, who solved the problem by investigating the mathematical relationships that are at the basis of this game using backward reasoning.

The student begins the game resolution by solving the 2D version of the game (3x3 board). First, he plays trying to remember the winning strategy, then he starts calculating mathematically the number of winning lines. Then he moves on to the 3D version of the game where he continues to reason about the number of winning lines until he obtains a general formula. Then it shows that the formula that he has found is valid for any cube of dimension (n, d) and finally he reasons again about the winning strategy, this time for the 3D case. The extract refers to the discovery process that the student makes to formulate the general recursive formula that allows to identify the number of winning lines knowing the size of the game board. Backward reasoning is predominant in this excerpt (Fig. 2).

12. I decide to move on to the 3D case. The previous strategy suggests me to count lines. I make a few drawings to test. There are 10 lines in each plan parallel to the axes and there are 12 planes parallel to the axes. I lack the "diagonal lines" as in the example. They seem more complicated.
13. I'm starting to do numerology: $10 = 4 * 2 + 2$ which is broken down as the number of pawns per dimension of the plane plus two diagonals. Will it be general?
14. I realize that $12 = 4 * 3$ that seems to follow the previous pattern. Hope. It looks like a nice combinatorial problem.
15. It reminds me of geometry calculations on finite fields. I think about shooting over there, but I realize that there are cyclic lines that come out on one side and appear on the opposite side. These movements are not allowed. I could rule them out but it seems too complicated. I abandon this strategy.
16. I think of a recursive pattern. I guess n pieces in d dimensions (the usual case is $(n, d) = (3,2)$ and this is $(n, d) = (4,3)$). Maybe the number of straight lines follows a pattern.
$$L(n, d) = cnt(n, d) * L(n, d - 1) + Diagonals$$
17. The constant must be the number of planes parallel to the axes. As in the previous case, these have to be nd, then I refine my formula to
$$L(n, d) = nd * L(n, d - 1) + Diagonals$$
18. Diagonals don't seem that simple. I start to play with the example of the cube and the plane. They seem to join opposite vertices of opposite faces. Will it be general?
19. I calculate that a hypercube has 2^d vertices, which gives me two faces with 2^{d-1} vertices. Thus, if my previous observation is correct, the formula is
$$L(n, d) = nd * L(n, d - 1) + 2^{d-1}$$

Fig. 2: Extract of student protocol

The student begins the resolution of the case in 3 dimensions thinking in analogy with the resolution of the case in 2 dimensions that he has previously carried out. The first objective is to count the winning lines on the board. To do so, he divides the game board into planes and counts the winning lines present on each plan. He then begins to think about the number of lines in each floor and breaks it down trying to identify the parts of the number with elements of the game (number of checkers for each winning line, size, number of diagonals). He then analyse and decompose each floor in the same way. At this point he introduces a recursive "auxiliary pattern" and conjectures the existence of a general recursive formula that relates the number of winning lines with the size of the table. He then analyses the formula and looks for a mathematical expression for each part of it. He then obtains the general recursive formula.

Analysing the extract, it is possible to identify different epistemic actions performed by the student. Using as definition of epistemic action: "that action in which knowledge is used or constructed". Each epistemic action can be characterized as an expression of the different characteristics of backward reasoning: in this extract we can see elements of decomposition (D) and insertion of auxiliary elements (E) and solution formulation (FS). In the same way, the same action can be classified according to the AiC model.

In the table below the second column identify the actions, the third identify the characteristics of backward reasoning and the last identify the AiC classification.

Protocols	Epistemic action	BR	AiC
12	Splitting the game board into planes Counting the winning lines in each plan Grouping winning Lines into a Scheme	D D E	B B R
13	Mathematically break down a number Identify each element of the decomposition	D E	C C
14	Mathematically break down a number	D	B
15	Analogy/ break motion		
16	Introduce a recursive pattern Conjecture: general recursive formula	E FS	R C
17	Break down the formula into its elements Analyse the constant element	D	C
18	Analyse the diagonal element	D	B
19	Representation of the diagonal in relation to the vertices of the hypercube Formulation of the general formula	D FS	C B

Table 2: Analysis of Epistemic action

Analysing the epistemic actions from the point of view of backward reasoning, one can observe how the student breaks down the problem and inserts auxiliary elements in an alternating way in order: first to conjecture the existence of a general formula and then to represent it mathematically. From the point of view of the analysis with AiC-model one can notice a certain regularity in the alternation of the AE (Table 2): Two sequences B-R-C-B-R-C characterize the formulation of the conjecture, while two sequences C-B-C-B characterize the formulation of the general formula. The actions that characterize the "decomposition" are actions that do not develop instantaneously in the resolution process but that suppose a longer time of realization. If you look at the introduction of auxiliary elements, these actions are instead instantaneous. Some actions, such as the introduction of a recursive pattern, can be a recognition of concepts belonging to the student's background, it happens after a structural analogy. During this analogy (line 15) the student remembers geometrical concepts that help him to identify patterns. In other actions, such as the identification of each element of the decomposition of the number 10 with an element of the game, the student creates a new construct from the processing of knowledge already encountered in the resolution.

CONCLUSION

In this investigation, we chose to use the AiC - model to understand how the process of abstraction develops in the construction of new mathematical knowledge using backward reasoning. The development of the different epistemic actions was analysed, with the help of the subcategories built in this research and the relationships they established among themselves. If we look at the whole process of the protocol (Fig. 3), we can see how the student passes through different contexts in order to achieve the general mathematical formulation. He begins working within the game context, then he moves to a mathematical context to interpret the example through this new lens, then he goes forward and explains the game in a more general mathematical context.

The transition between the three contexts happens with a complex back and forth process, where the different contexts are repeatedly activated, as illustrated in Fig. 3.

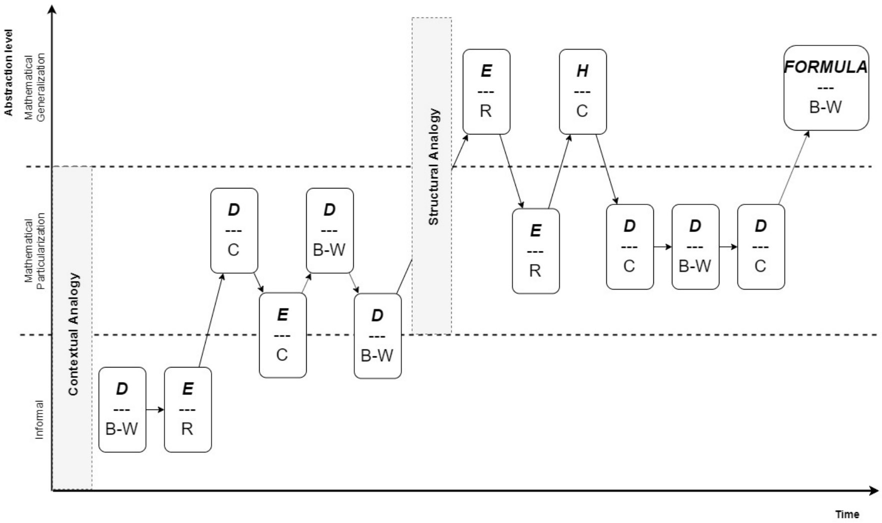


Fig. 3 Pattern in epistemic actions and context

Following the introduction of subcategories of analysis, built in a narrow link with the nature of backward reasoning it is possible to analyse, in detail, characteristics associated with the development of students' thinking processes. This helps in better understanding the connections between the different epistemic actions that can influence Building with and Construction.

We notice that the incorporation of Backward Reasoning-based categories allows to identify breaking elements and which they trigger the construction process of the formula. These actions occur and they cannot be determined only with specific actions defined according to the AiC-model. In this case it has been necessary to identify elements of the constructive epistemic action processes produced in the long term. The back and forth movement above is identified as a cognitive travel between the concrete and the abstract: in it the analogy processes — both contextual and structural analogy— have been crucial. In the process of conjecturing and justifying, complex chains of plausible reasoning are often elaborated, which may contain new nuances that enrich already known patterns. An exhaustive analysis of these processes requires an exploration not only of the punctual but in the long term.

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Introducing group theory with its *raison d'être* for students

Hiroaki Hamanaka[1], Koji Otaki[2] and Ryoto Hakamata[3]

[1]Hyogo University of Teacher Education, Japan, hammer@hyogo-u.ac.jp,
[2]Hokkaido University of Education, Japan, [3]Kochi University, Japan

This paper reports results of our sequence of didactic situations for teaching fundamental concepts in group theory—e.g., symmetric group, generator, subgroup, and coset decomposition. In the situations, students in a preservice teacher training course dealt with such concepts, together with card-puzzle problems of a type. And there, we aimed to accompany these concepts with their raisons d'être. Such raisons d'être are substantiated by the dialectic between tasks and techniques in the praxeological perspective of the anthropological theory of the didactic.

Keywords: 2. Teaching and learning of specific topics in university mathematics, 9. Teaching and learning of linear and abstract algebra, Group theory, Raison d'être, Praxeology.

INTRODUCTION AND MAIN THEORETICAL RESOURCES

Abstract algebra is one of the major areas of undergraduate and graduate mathematics. And it has been pointed out by several authors that many students have difficulty in transiting from elementary algebra to abstract algebra (e.g., Dubinsky et al., 1994; Hausberger, 2017, 2018; Bosch et al., 2018). In abstract algebra, we consider the algebraic structures, group, ring, field, etc, which inherit the property of the familiar calculations among numbers or equations as objects of elementary algebra. Among such algebraic structures, group is the simplest but difficult topic. The reason is that a group has only one operation from the first, and non-commutativity is primary in group theory: these properties cannot be observed in standard number systems. Then, we problematize that, in the teaching of abstract algebra, students usually do not experience inquiry where the notion of group with one operation can grow. In fact, Bosch et al. (2018) points out that, regarding the learning of group theory, the *raison d'être* of group theory is seldomly questioned—roughly speaking, *a raison d'être of knowledge* means a problematic situation the study of which naturally produces the knowledge as a significant tool. According to this problematization of the teaching of group theory, let us pose our research question as the follows: *what problems and situations could become raison d'être of group theory for students?* Larsen (2009) seems to be a representative previous work which studies this question. Larsen's approach aimed students' reinvention of the concept of group and isomorphism in geometric context. In this paper, we would like to propose another possible approach with above features. As well as Larsen's approach, we assume teacher's guide, and we expect students to develop the notion of group, subgroup, generators of a group, coset decomposition on their interest and with proactive motivation through this approach. Let us highlight that our approach does not aim the complete abstraction process of the concept of group. Rather, this approach may be the introductory program of the abstract group theory, which promotes students to notice the notion and to enhance the definition of group on their own consideration.

In the presentation and the analysis of our program, we make use of the *praxeology model* within the framework of the ATD, i.e., the *anthropological theory of the didactic* (cf. Chevallard, 2019). The praxeology in the ATD is a model for describing any bodies of knowledge, e.g., different mathematical domains, i.e. algebra, geometry, and so on. Such domains are based on their own theoretical foundations consisting of axioms, fundamental theorems, problematic questions, basic objects of study, etc. Each theoretical foundation is simply called a *theory* denoted by Θ in the ATD. In turn, a theory Θ describes and justifies many specific statements, particular objects of study, local problems, and so on. The system of such second-level theoretical entities is called the *technology* θ in the ATD. A theory Θ and a technology θ constitute a *logos* part of a given praxeology, i.e. $[\theta / \Theta]$. Let us emphasize here that any *logos* part originally comes from more concrete, specific, even *ad hoc* human actions which are called *praxis* parts of the given praxeology. The *praxis* part is reduced to two subparts of the type of tasks T and the technique τ, that is, $[T / \tau]$. A type of tasks T is any motivation of a given praxeology which is handled by some technique τ. The overall picture of a praxeology is denoted by $[T / \tau / \theta / \Theta]$. The order of emergence of each part of a praxeology depends on the program: a traditional group theory course may start from technology or theory. In our program, we propose to start the program from the extra-mathematical task of card puzzles explained below.

From perspective of the theory of praxeologies, we define the notion of *raison d'être* as any *system of interrelated tasks* (and each task of the system) satisfying the following two conditions: 1) it allows the inquirers to (re)produce a given whole praxeology (and praxeological elements); and 2) it is lively or familiar for the inquirers' viewpoint. Let us emphasize here that *raisons d'être* of praxeologies are *relative* and *changeable*. It depends on performers of the praxeologies like mathematicians and students.

THE CARD PUZZLE FOR TEACHING GROUP THEOTY

Task design

Let us recall the studying process of vector and vector space. Among the algebraic objects taught in secondary mathematics, vectors have exceptionally different algebraic structures from number systems. In secondary mathematics, vectors are introduced in a constructive way based on planer or spatial geometry, not in the axiomatic way, and the focus is on individual vectors and their calculations. It is on this basis that concepts of abstract vector spaces are studied later. Thus, the study on the vectors performs as the previous step towards abstract linear algebra. Likewise, we would like to propose giving opportunities first to experience constructive algebraic structures which have only one non-commutative operation. Studying planer vectors is a nice beginning towards linear algebra because the planer vector space is easy to grasp and possesses typical properties as a vector space. Then, what is the most typical group? We propose that the symmetric group S_n could be an appropriate candidate. In fact, it is well-known that any finite group is a subgroup of S_n for some large n. Moreover, S_n can be introduced in a constructive way, for example, S_4 can be introduced as permutating operations of 4 numbered cards.

The students in this experimentation belonged to a preservice teacher training course, therefore advanced group theory could not be included. We designed an introductory program dealing with symmetric groups, which would naturally introduce fundamental concepts in group theory without forcing their definitions from the first. This program was conducted in the first term in 2019 for three third-year undergraduate students as a seminar with the first author as a teacher. This seminar started in May and continues for two years; however, we focus on the first seven sessions. As the prerequisite knowledge, they had already studied fundamental set theory including concepts of map, injection, surjection, equivalence relation and quotient set in other courses.

The initial and central type of tasks $\boldsymbol{T}$ designed for the seminar is the following:

> We have n cards arranged in a row and each number from 1 to n is written on one of the cards. The objective is to rearrange them in the ascending order using particular available operations only. Less number of operations is preferred.

We call each task $t^{(i)}{}_{T}$ of $\boldsymbol{T}$, which consists of the number of cards and available operations, a *puzzle*. Based on $\boldsymbol{T}$, several puzzles are proposed to students and regarding each puzzle $t^{(i)}{}_{T}$, students struggled not only with analysing them, but also with related questions raised in the analysing processes. Such related tasks will form other derived task types $*\boldsymbol{T}$ and $**\boldsymbol{T}$, which shall be explained in later sections. The first example is Puzzle 1 ($t^{(1)}{}_{T}$) indicated in Fig 1. It deals with 3 cards and the available operations are A, switching first and second cards, and B, switching second and third cards. All puzzles proposed to students are indicated in Fig 2.

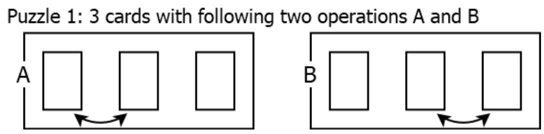

Fig 1: Puzzle 1

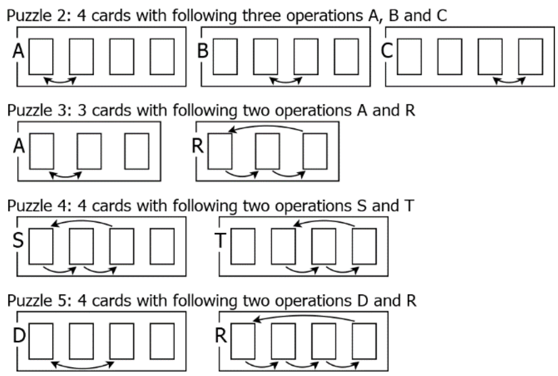

Fig 2: Puzzle 2, 3, 4, and 5

First session: From puzzle to non-numerical equational representation

In the first session, $\boldsymbol{T}$ was explained, and Puzzle 1 ($t^{(1)}{}_{T}$) was provided for students. Thus, this program started from an *extra-mathematical* context to analyse a puzzle, which was proposed without using any terms of group theory. As the solving method, it was supposed beforehand to draw the Cayley graph, which consists of vertices of all possible orders and edges each of which connects two orders possible to change one to the other by one available operation. Of course, students did not know Cayley graph, but we had expected them to spontaneously use such a graph representation. In fact, a student said:

Student A: Just 6 orders only are possible. Let's list them all.

Then they started to construct Cayley graph without knowing that term (Fig 3). Students produced successfully the method of drawing Cayley graph, which is a technical element $\boldsymbol{\tau}^{(1)}_{\boldsymbol{T}}$ in their praxeology (in this paper, we use "$\boldsymbol{\tau}$" for not only the whole technique but also its elements). This technique enables them to focus on and grasp the whole set of operations or orders. To emphasize that each operation corresponds to an order, the teacher stressed that 123 is the initial order.

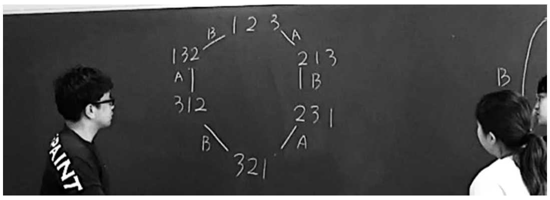

Fig 3: Cayley graph drawn by students

The following are students' remarks regarding this graph representation.

Student A: Operation B once and A twice are needed (to change 321 into 123).

Student B: Operations A and B should be done alternatively. Doing A twice is useless.

At this moment, they did not realize the binary operation among puzzle operations and their representations were based on the natural language. However, being asked how to change 321 into 123, they answered "BAB or ABA"; they used the composition operation unconsciously. It was after that time that the definition and the notation of the composition operation was confirmed: for two operations X and Y, we denoted the composition of them, doing X and then Y, by XY. One student said "it's like the multiplication", and we discussed that $AB \neq BA$. Here, it should be remarked that these discussions were not rigorous because it was not defined what these operations mean mathematically. Strictly speaking, operations should be formulated as maps from $\{1,2,3\}$ into itself, however such theorisation was postponed until some later sessions, that is, the teacher did not intervene in the students' spontaneous praxeology to avoid developing praxeologies without any *raison d'être*.

Also, in this session, based on the above Student B's remark, we tried to express his remark as equations. This is a new derived task $\boldsymbol{t}^{(1)}_{*\boldsymbol{T}}$. Specifically, being asked what happens when the operation A is repeated, they considered how to express $2n+1$ times repetition of operation A:

Student A: May be $(2n+1)A$ … oh, it's no good.

Asked by the teacher, they discussed that doing twice and thrice of A are AA and AAA respectively and it was pointed out by students that they look like powers of A. Thus, we negotiated that we note n times repetition of A by A^n and obtained the equation $A^{2n+1} = A$. Then we discussed how to proceed the task $\boldsymbol{t}^{(1)}_{*\boldsymbol{T}}$, that is, representing their findings that $2n$ times repetition of A is the same as doing nothing:

Student B: Doing nothing is 0 times repetition and it may be written as 0?

Teacher: If it's written as 0, A and doing nothing equals A, then $A0 = A$, right?

Students: Oh, it's 1. 1 is better!

In this way, they faced the new type of tasks $*\boldsymbol{T}$, to express relations between operations and to find new relations between them, and with need in coping with these tasks $\boldsymbol{t}^{(1)}{}_{*T}$, they developed a technical part $\boldsymbol{\tau}^{(1)}{}_{*T}$ of representing the relations between puzzle operations in algebraic way, even though they are not algebraic objects for them at this point. This enables them hereafter to use elementary algebraic representations and techniques to express and deal with properties among puzzle operations paying attention to the non-commutativity.

Second and third session: inverse element and order of elements

At the end of the first session, we agreed what puzzle to analyse next, i.e., Puzzle 2 ($\boldsymbol{t}^{(2)}{}_{T}$). Thus, at the beginning of the second session, students presented their result (Fig 4). Compared to puzzle 1, the structure of Puzzle 2 is rather complex, however, they engaged rather lively in elaborating the Cayley diagram, because it enables them to see the whole perspective of the permutations in $\boldsymbol{t}^{(2)}{}_{T}$: $\boldsymbol{\tau}^{(1)}{}_{T}$ functions as a tool to carry out $\boldsymbol{t}^{(2)}{}_{T}$. In the process, they naturally began to consider 1234 as the initial order to operate, and to regard operations as permutations as well as in the case of puzzle 1.

After they elaborated this graph, they found many relations like $(AC)^2 = 1$, $(AB)^3 = 1$ and so on. Referring these equations, it was natural to ask the following question: for any operation X, does a natural number n exist, such that $X^n = 1$? This is a new task $\boldsymbol{t}^{(2)}{}_{*T}$ of $*\boldsymbol{T}$. After some empirical attempts, they could understand that, if we assume that an operation X had no natural number n such that $X^n = 1$, there would be different natural numbers i and j such that $X^i = X^j$. Thus, we focused on the question whether the statement "$YX = ZX$ implies $Y = Z$" is true or not; if it is true, we can have $X^{j-i} or\ X^{i-j} = 1$. In this context, we defined the inverse of an operation, and confirmed all operations have its inverse. Here, we can point out that the inverse element was introduced not for the axiom of group, but for the solution to the emerged question.

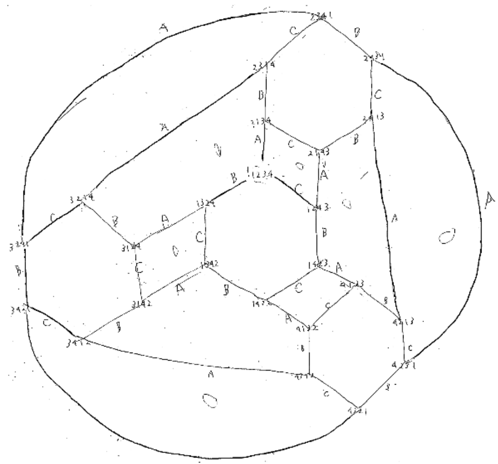

Fig 4: Cayley graph of puzzle 2 drawn by students.

In the third session, we dealt Puzzle 3 ($\boldsymbol{t}^{(3)}{}_{T}$), which includes operation R whose inverse R^{-1} is different from the original. Thus, the notion of inverse became more clarified. Puzzle 3 was selected by the teacher for this purpose.

Through carrying out $\boldsymbol{t}^{(2)}{}_{*T}$ and $\boldsymbol{t}^{(3)}{}_{T}$ in these sessions, the students elaborated the concept of inverse elements ($\boldsymbol{\tau}^{(2)}{}_{*T}$). In more detail, the task $\boldsymbol{t}^{(2)}{}_{*T}$ required the technique $\boldsymbol{\tau}^{(2)}{}_{*T}$ and this technique enabled them to carry out $\boldsymbol{t}^{(3)}{}_{T}$. Besides this, the students became to use algebraic expressions frequently and naturally in their discourse. It seemed that permutations had become algebraic elements, which can be dealt in algebraic ways through the composition operation.

Fourth session: Symmetric group and its generators

Until the third session, every analysed puzzle involved all permutations, however Puzzle 4 ($\boldsymbol{t^{(4)}_{T}}$) had been assigned at the end of the third session and they had found that is not always true. Puzzle 4 is also selected by the teacher to highlight this phenomenon. In the fourth session they drew the Cayley graph of Puzzle 4 from the initial order 1234 as usual and found it involves only 12 permutations. Moreover, students, as it was expected, added an optional graph which involves permutations that do not appear in the original graph (Fig 5).

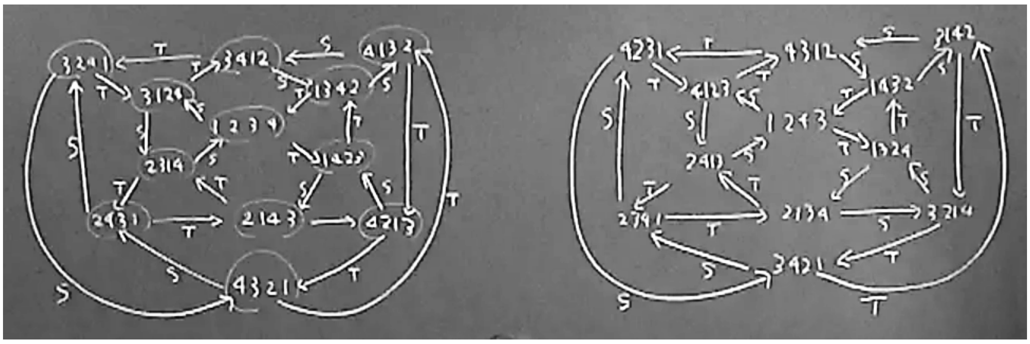

Fig 5: Cayley graph of puzzle 4

Then, they faced the question: why this diagram split into two parts, what is the difference from the previous puzzles? This is a new task $\boldsymbol{t^{(1)}_{**T}}$ belongs to the new type of tasks $**\boldsymbol{T}$ involving the structure of the group. To discuss them, it was necessary to formulate what operations are mathematically and to make the discussion more rigorous. On this account, we negotiated the notation representing a bijection f from $\{1,2,3,4\}$ to itself: f is indicated by the result (written in framed numbers) of the corresponding permutations of 1,2,3,4. (This is different from the standard notation of permutation, however, their recognition is based on the permutation of 4 cards, this notation was rather acceptable for them.) For example, $\boxed{1234}$ means the identity permutation and $\boxed{1243}$ means the transposition of 3 and 4. And here the term "symmetric group of degree *n*" and the notation S_n was introduced by the teacher. Also, it was confirmed that the vertices of the graphs we have elaborated correspond to the elements of S_3 or S_4. Thus these formulation technique $\boldsymbol{\tau^{(1)}_{**T}}$ was carried out for the need to proceed the consideration on $\boldsymbol{t^{(1)}_{**T}}$.

Next, students were asked to explain the difference between the situation of Puzzle 4 and those of previous ones, for example Puzzle 2. They tried to express it:

Student B: In the case of puzzle 2, all elements of S_4 can be *made* by A, B, C, however it isn't in this case.

Teacher: What do you mean by "can be *made*"? Please explain more precisely.

Finally, they could not express it clearly, however, they entirely agreed on the description given by the teacher: "in the case of puzzle 2, all elements of S_4 can be expressed as a finite composition of A, B, C, and their inverses in some order". It was this moment that the definition of generators is provided. Here we remark that, though the definitive expression was finally given by the teacher, their cognitive process was rather different from the usual studying process which starts from providing definitions. In fact, they *could* consider how to express it, since they could distinguish the case of being generated from otherwise. These notions of generators and property of being generated should be the new technique to cope with the task $\boldsymbol{t}^{(1)}{}_{**\boldsymbol{T}}$ and at the same time they could be a technological element $\theta^{(1)}{}_{\boldsymbol{T}}$ within group theory, relative to the type of tasks $\boldsymbol{T}$ (we also use "θ" for the technological elements like in the case of τ).

Fifth and sixth session: Coset decomposition and subgroup, at the same time.

In the previous session, we realized the difference between Puzzle 2 and 4, however, it remained mysterious that in the case of Puzzle 4, whole elements of S_4 split into two *congruent* graphs, that is, $\boldsymbol{t}^{(1)}{}_{**\boldsymbol{T}}$ was not solved completely. To consider its reason, they were suggested by the teacher to understand this splitting phenomenon as a quotient set by an equivalence relation: what equivalence relation is behind this splitting? This is the new emerged task $\boldsymbol{t}^{(2)}{}_{**\boldsymbol{T}}$, based on the application of some notions within set theory ($\theta^{(2)}{}_{\boldsymbol{T}}$). Since it was difficult to consider from one example only, they were encouraged to analyse one more case, Puzzle 5 (Fig 6).

Considering these cases, students noticed that, in the case of Puzzle 4, two permutations P and P' are in the same connected graph, if and only if "P can *become* P' using S and T " (this is student's exact description). Again, the student's description was not accurate. Then, with teacher's help, they managed to elaborate the description: "it is if and only if P'can be expressed by the composition of P and a finite composition of S, T, S^{-1}, and T^{-1}." Here, we denoted this binary relation as $P \sim P'$. This is the basic concept of coset decomposition included in $\theta^{(1)}{}_{\boldsymbol{T}}$.

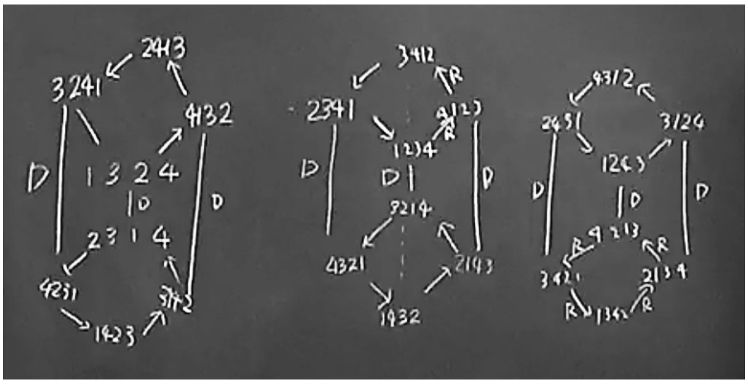

Fig 6: Cayley graph of puzzle 5 (the edges connecting the top and bottom are omitted)

Then, the above notion led them to the task $\boldsymbol{t}^{(3)}{}_{**\boldsymbol{T}}$ of considering whether this binary relation is an equivalence relation or not and its reason. We checked the three properties: reflexivity, symmetry, and transitivity. The following is the result of our discussion.

– Reflexivity holds, since 1 is one of finite compositions of S, T, S^{-1}, and T^{-1}.

– Symmetry holds, since the inverse of a finite composition of S, T, S^{-1}, and T^{-1} is also a finite composition of the same elements.

– Transitivity holds, since the composition of two finite compositions of S, T, S^{-1}, and T^{-1} is also a finite composition of S, T, S^{-1}, and T^{-1}.

Through this consideration, we negotiated to denote the set of finite compositions of S, T, S^{-1}, and T^{-1} by $\langle S, T\rangle$, and realized that $\langle S, T\rangle$ has nice properties: it is these properties that make the corresponding binary relation $\sim$ to be an equivalence relation. Also, we reflected that what happens in cases of $\langle D, R\rangle$ in S_4 (Puzzle 5) and $\langle A, B, C\rangle$ in S_4 (Puzzle 2), and found that the corresponding relation of $\langle D, R\rangle$ classifies S_4 into 3 classes and that of $\langle A, B, C\rangle$ classifies S_4 into just one class.

Then, it was this moment that the definition of subgroup of S_n was provided: a subset K of S_n is a subgroup if and only if K includes 1 and is closed under the composition operation and the inversion. This definition was not forced from above, but rather naturally raised from the discussion to understand the splitting of Cayley graphs, and would be acceptable for students. These notions of subgroup, coset decomposition, are certainly elements of $\theta^{(1)}{}_{T}$ in their praxeology. These were endowed with their *raisons d'être* from the first: the properties in the definition of subgroup were raised to support the concept of coset decomposition which was observed through the type of tasks $**T$.

Seventh session: Coset and Lagrange theorem

It was natural for students to focus on the question "Is there any subgroup whose corresponding coset decomposition consists of 4 classes or 5 classes?" In this manner we investigated the structure of a coset and found that each coset has the same number of elements (Lagrange theorem in symmetric groups).

DISCUSSION

The following is the list of praxeological elements at stake in the whole program.

T: To find properties regarding puzzle operations	
	$t^{(1)}{}_{T}$, $t^{(2)}{}_{T}$, $t^{(3)}{}_{T}$,…: Puzzle 1, Puzzle 2, Puzzle 3, …
***T**: To express relations between operations and to find new relations between them	
	$t^{(1)}{}_{*T}$: To express the relation found in the first session
	$t^{(2)}{}_{*T}$: To consider whether every operation has a finite order
****T**: To consider and (or) explain the principle regarding the structure of the group	
	$t^{(1)}{}_{**T}$: To consider the principle of the splitting of Cayley graphs
	$t^{(2)}{}_{**T}$: To consider what equivalence relation is behind the splitting
	$t^{(3)}{}_{**T}$: To consider if the found binary relation is equivalence relation or not, why and what is behind it
$\tau^{(1)}{}_{T}$: To draw Cayley graphs	
$\tau^{(1)}{}_{*T}$: To represent the relations between puzzle operations in algebraic way	
$\tau^{(2)}{}_{*T}$: To deal with the inverse operations	
$\tau^{(1)}{}_{**T}$: To formulate symmetric groups using set theoretical terms	
$\theta^{(1)}{}_{T}$: Symmetric group theory—especially, notions of S_3, S_4, generator, subgroup, coset	
$\theta^{(2)}{}_{T}$: Elementary set theory —maps, equivalence relation, and quotient set	
(Θ_{T} : General group theory)	

Table 1: Praxeological elements

Up to this point, we have described the study process in the program, where different technical and technological entities in group theory emerged from a sequence of lively tasks. The bundle of these tasks constructs a *raison d'être* of group theory for the students. Let us clarify here the relationship between such tasks and technical-and-technological elements, as the first property of our experiment (Fig. 7). On the one hand, each task brings about a derived technical-and-technological element. On the other hand, such an element produces a certain new task and is applied to some tasks. In Fig 7, the arrows indicate these deriving, producing, and applying relations. This *dialectical interplay between tasks and techniques* is the first property of our implementation. Note that our analysis relies on a basic assumption of the ATD, that is, the *postulate of the relativity of praxeological entities*. $\theta^{(1)}{}_{\boldsymbol{T}}$ and $\theta^{(2)}{}_{\boldsymbol{T}}$ are *technological for* $\boldsymbol{T}$—which is the motivation of this praxeology—, but *technical for* $^{**}\boldsymbol{T}$. Type of tasks, techniques, technologies, and theories are not the natures of praxeological elements but their functions.

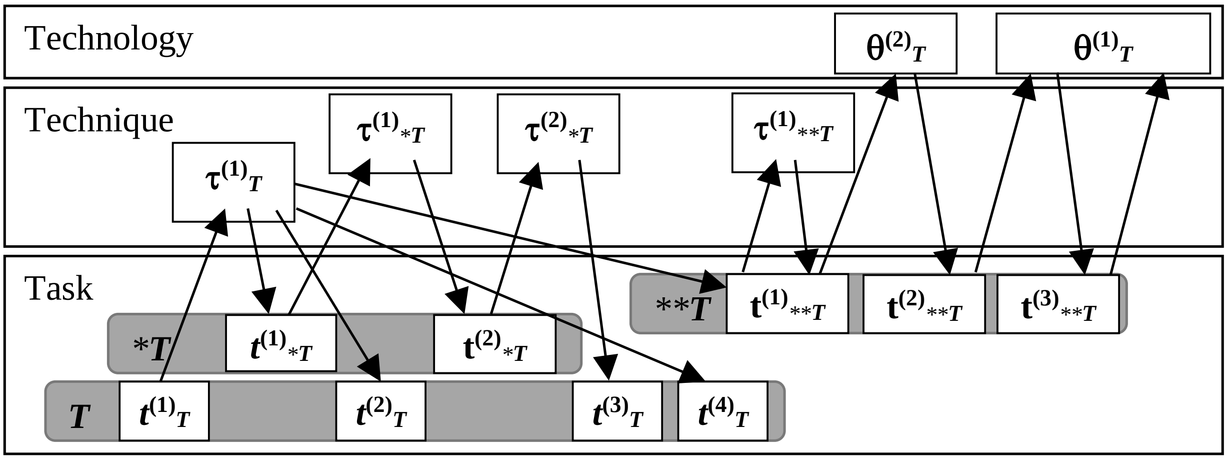

Fig 7: The dialectic between tasks and techniques

The second property is related to the *incompleteness of theoretical elements* in the program. The praxeology has the algebraic nature but excessively focuses on symmetric groups, that is, not get access to the general group theory. This is a main reason why we call this seminar an *introductory* program of abstract algebra. However, the praxeology at stake could reach to standard structural results in symmetric group such as coset decomposition and Lagrange theorem, which can be easily extended to general groups. This potentially extendable structural results might be a gate for proceeding to general group theory with *raison d'être.*

The third property related to the second property is the *implicitness of associativity*. The definition of group involves three conditions: associativity, existence of an identity element, and existence of inverse elements. In our program, the identity element was required to make equational representation of found relations ($\tau^{(1)}{}_{*\boldsymbol{T}}$), and inverse elements also emerged to accomplish the task $\boldsymbol{t}^{(2)}{}_{*\boldsymbol{T}}$. However, in this program all objects in consideration were maps and associativity was always satisfied from the first. Students *knew* that associativity holds but never focused on it. It was used with no special consciousness. Associativity seems often trivial and tends to be transparent for students. Thus, the teacher had no chance to highlight associativity regarding group theory with *raison d'être*.

FINAL REMARKS

Winsløw et al. (2014) points out the two types of institutional transitions in university mathematical praxeologies. The first type is the transition where a *praxis* part of a praxeology without the *logos* part gets its *logos* part. On the other hand, in the second type, a *logos* part of a mathematical praxeology changes to a *praxis* part of another advanced praxeology. Our program should fit as the first type transition. Taking above three properties of our program into consideration, we might be able to proceed to a further development of didactic situation for teaching group theory as follows: one is a situation where their understanding of potentially extendable technology elements derives the general group theory. Such situation would require group-like objects in entirely other context, to which the existing technology elements can be applied. Also, to focus on associativity group-like objects, in which associativity is not trivially satisfied, would be needed.

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Designing a Geometry Capstone Course for Student Teachers: Bridging the gap between academic mathematics and school mathematics in the case of congruence

Max Hoffmann[1] and Rolf Biehler[1]

[1]Paderborn University, Institute of Mathematics, Germany,
max.hoffmann@math.upb.de

At Paderborn University, a new 6th semester geometry-course for upper secondary student teachers has been designed and taught by the first author of this paper. To show links between academic mathematics and school mathematics we established so-called interface weeks. These are weeks during a course in which lecture, exercises and homework focus on topics that are related to the normal canon of content but specially chosen for their relevance in school contexts. In this article, we want to present our design for an interface week on the topic of congruence. In order to do so, we first illustrate how so-called interface aspects are used to systematize the mathematical background of the topic, thus giving future mathematics teachers the chance to act professionally. We then show examples of learning activities and first results of the accompanying research.

Keywords: Transition to and across university mathematics, Teaching and learning of specific topics in university mathematics, Geometry, Student Teacher, Capstone

INTRODUCTION

In his well-known quote, Klein (1908, p. 1) describes two discontinuities, which must be clearly distinguished. The *first discontinuity* is a perceived disconnectedness between school mathematics and the academic mathematics that students encounter when they enter their university studies. Klein focusses exclusively on the aspect of mathematical knowledge. In current transition research, also differences at the level of teaching/learning methods or social challenges inherent in the transition are seen as responsible factors for the difficulties which students experience when they start their studies. These difficulties can even cause some students to completely drop the study of mathematics. Interventions that specifically address the transition problems at the aspect of mathematical knowledge explicitly build on existing knowledge and previous mathematical experience from school when designing teaching/learning processes in university mathematics. The focus of these efforts is the acquisition of competences in university mathematics. Although was formulated by Klein in the context of teacher education, the first discontinuity is also relevant for students who aim to major in mathematics. The *second discontinuity* assumes that students have acquired knowledge in university mathematics. However, students often perceive this mathematical knowledge as not very relevant for their future professional work. Solutions therefore aim, to identify the contents of academic mathematics that can be connected (in the sense of supporting professional teaching) to school mathematics and in a second step to develop learning opportunities that support students in discovering connections. The aim is to better enable students to use their background in university mathematics as a basis for professional acting as a teacher. In most (German) universities, such linkage is provided in additional courses on the didactics of mathematics. Our new approach is to enhance the mathematics course itself so that the specific course on the didactics of geometry can focus on pedagogical content knowledge. The mathematics course is to be enriched by learning opportunities that help students to take a mathematical perspective on a profession-oriented situation (e.g. reacting to a student's contribution or analysing a textbook page) and to act with the necessary professional knowledge.

THEORETICAL BACKGROUND

Talking about a discontinuity between school mathematics and academic mathematics requires analysing differences between the two. Dreher, Lindmeier, Heinze and Niemand (2018) summarize how these differences have already been described by Klein and, more recently, by Wu (2011) and other authors:

> Mathematics as the scientific discipline taught at university has an axiomatic-deductive structure and focuses on the rigorous establishment of theory in terms of definitions, theorems, and proofs. It usually deals with objects that are not bound to reality […]. [M]athematical objects [as taught at schools] are often introduced in an empirical manner and bound to a certain context. Concept formation […] is […] often done in an inductive way […]. Mostly intuitive and context-related reasoning is more in the focus than rigorous proofs. (Dreher et al., 2018, p. 323)

In order to link these two types of mathematics, we follow the idea of *mathematical background theories* (e.g. Vollrath, 1979). Topics of school mathematics are characterized by several locally ordered domains, which are mostly unconnected to each other. They are often built up from an empirical phenomenon and are at the end networks of terms and concepts that are logical in themselves. (Freudenthal, 1973). Background theories phrased in the language of academic mathematics can now contribute to the foundation of these locally ordered domains in two ways: On the one hand, they can help to connect the domains with each other and thus clarify conceptual relationships and bring statements from the different areas into logical connections. On the other hand, such background theories provide the basis for the local ordering and selection of content foci within the individual domains. Based on extensive research, Ball and Bass (2002, p. 11) give a list of typical mathematical job tasks that a mathematics teacher has to master in everyday teaching. This list has been extended by Prediger (2013). Four of these job tasks that are important for our project are the following:

A teacher must …

- be able to master requirements set for students by her- or himself at different levels
- analyse and evaluate approaches (used in e.g. textbooks)
- analyse and rate student contributions and react to them in a way which is conducive to learning

- analyse mistakes of students and react in a way which is conducive to learning

(Prediger, 2013, p. 156)

The identification of the corresponding mathematical background theory is a prerequisite for professional teaching, since it enables a teacher to correctly analyse a given situation. With sufficient background knowledge, a mathematical perspective on a typical professional problem can be taken and a solution can be worked out in the context of this perspective. In the last step, this solution then must be didactically transferred back into school mathematics and adapted to the mathematical horizon of the respective school students. At this point, we would like to emphasize that this is just one of several perspectives that can be taken on typical professional situations, but it is the one in which the mathematical background plays the most important role, which makes it relevant for the design of lectures in mathematics. In terms of teachers' professional knowledge, our aim is to establish links between school mathematics and academic mathematics in the sense of school-related content knowledge (SRCK) (Dreher et al., 2018). This construct supplements the known facets of content knowledge and pedagogical content knowledge with a profession-specific component. The latter consists of three facets, namely: Knowledge about the school curriculum and its structure as well as the understanding of its mathematical legitimation, secondly the knowledge of the interrelations between school mathematics and academic mathematics both top-down, and thirdly bottom-up. We call learning opportunities that evoke the conscious passing through the described three-step process (Take the mathematical perspective. - Solve your problem. – Didactically transfer your solution) *interface learning opportunities (to bridge the second discontinuity)* and thus generalize the term *interface task* as used for example by Bauer (2013). Our research interest lies in the development and evaluation of interface learning opportunities with the aim of identifying generalizable principles for success and failure and formulating general design principles for interface activities.

ABOUT THE GEOMETRY COURSE

The course in which our project takes place is located in the 6th semester of the degree programme for future upper secondary math teachers. Since it was newly introduced as part of a change in the study regulations, we had many design options for implementing the requirements of the module manual, which are: an axiomatic system for Euclidean geometry should be treated according to the module description of the course and, the role of the parallel postulate should be discussed using a model of hyperbolic geometry. For this purpose we use an axiom system developed by Iversen (1992). His system is equivalent to the known axioms of Hilbert, but it needs fewer axioms, which are of course more charged with content (as for example, the axioms for $\mathbb{R}$ are already included.), but also more intuitively understandable. In addition, we also consider the Euclidean plane by means of analytical geometry using Euclidean motions and thus include the geometry as it is usually treated in upper secondary school. In this way we can take different views on typical terms of school geometry (straight lines, orthogonality, reflections, etc.). The course took place for the first time in the summer semester 2019 with about 25 active participants. The presence part consisted of a weekly two-hour lecture and two two-hour tutorial groups of about 13 students each. As usual in mathematics lectures in Germany, weekly homework assignments were set for the students to work on. Within this context, students also get tasks for linking school and university mathematics, which are part of a semester-accompanying so-called *interface ePortfolio*. We follow Bruder, Scholz and Menhard (2012) with this concept of an accompanying e-portfolio. It must be emphasised that the course differs significantly from the usual mathematics courses in university teacher education, which are usually the standard bachelor-of-science courses and thus do not specifically address the specific needs of student teachers. Most of the existing projects in which special courses for student teachers are developed are placed at the start of university education**.** The topic of elementary geometry is well suited for an exclusive teacher education lecture, because although it is an important topic for student teachers, it usually does not play a role in the subject studies.

RESEARCH DESIGN

We develop and study our interface activities within the framework of a design research approach following the methodology of Prediger et al. (2012). In our project we go through the following cycle three: (Step 1) *specifying and structuring the interface topic*, (Step 2) *(re)designing interface learning activities*, (Step 3) *use and research interface activities* and (Step 4) *developing and refining (local) theories*. The initial run just took place in summer semester 2019: In order to *specify and structure the interface topic (step 1)*, we must link a school mathematics topic with corresponding academic mathematics. The challenge now is to systematize the background theory so that it can be used as a basis for professional teaching. It is utopian to assume that teachers will later think, “Oh yes, lemma 4.2 can help me here.” Our approach is to work out so-called *interface aspects* that channel the work with the mathematical background in a typical professional situation. In the next section this will be explained by an example. We decided to focus our project on the interface topics *symmetry* and *congruence*. For *design* and later *redesign (step 2)*, we had to develop three types of interface learning activities: The lecture itself, tasks for the weekly tutorial groups and tasks for the weekly homework. In the first run, we were primarily dependent on experience from other mathematics lectures for student teachers, other projects on interface tasks and our conceptual considerations, which are described above. In the next two runs, we will also be able to build on our research results according to the principles of design research. For the *research on our interface learning activities (step 3)*, we have collected a lot of data both at the level of the cohort as a whole and in case studies. Details about the data collection will be presented in the next section. Based on the research results from step 3, we can then *develop and refine local teaching-learning theories (step 4)* that relate to the developed activities belonging to the selected interface topic. In this article we would like to illustrate the implementation of the different steps of the cycle using various examples from the *interface week "congruence"*.

DATA COLLECTION

The students' solutions of the interface tasks were scanned and will be evaluated with methods of qualitative content analysis. A small sub-group from each tutorial group was videographed in an extra room while working on the tasks. By evaluating the students' discussions, we hope to gain insights into the conception of interface tasks as well as into how students use the newly acquired background theory in their communication with peers. Some students were interviewed about their experiences during their work on the homework and their view on the tasks. This enables us to gain insights into the subjective perception of the learning processes during the interface week. In the following week, we used an acceptance questionnaire in the whole group with mainly closed items, which gives us a general picture of the perceived difficulty, comprehensibility, motivation, etc. of the interface tasks.

DESIGN OF THE INTERFACE WEEK ON "CONGRUENCE"

For one of the interface weeks, we chose congruence, which is an important topic of geometry teaching in lower secondary school. Among other things, the topic forms an important background for constructions with compasses and rulers and represents an essential method of geometric argumentation in school mathematics.

Mathematical Background

The first part of the lecture on the *interface week on "congruence"* deals with the clarification of the underlying academic mathematics.

We call a figure (that means subsets of $\mathbb{R}^2$) F congruent to G, if there is an (bijective) isometry $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $\varphi(F) = G$. We prove that congruence is an equivalence relation. That leads us to the question of what information is needed to unambiguously determine the equivalence class of a plane figure. Prominent tools for this are the congruence theorems for triangles, which are also proved in the lecture.

Congruence in school mathematics

Weigand et al. (2018, p. 202) describe *congruence* as an important basic concept for topics of lower secondary geometry. This includes constructions with compasses and rulers, justifications and proofs as well as the determination of lengths and area contents via congruent subfigures. Congruence can be introduced as a basic concept explained on the enactive level (in the sense of fitting when laying one figure on top of the other) or based on the theory of Euclidean isometries (Weigand et al., 2018, p. 203).

Interface Aspects

In accordance with step 1 of our research design, we worked out the following four *interface aspects* for *congruence* as a kind of meta-knowledge for constituting a systematized view of the mathematical background. The notion of "interface aspect" is an important theoretical concept of our approach.

1. *Aspect of quantities with identical size:* Because there is always an isometry (by definition) between a figure and a congruent figure, it is guaranteed that congruent figures will match in several geometric quantities. This applies especially not only to the border of the figure and the distance between corner points, but also to the dimensions of other objects that can be constructed from the figure (diagonals, intersections, incircles, …) and their equivalent objects in the congruent figure.
2. *Aspect of relation:* Congruence is an equivalence relation on the power set of $\mathbb{R}$. The statement delivers characteristics that are connected intuitively with the concept of congruence: Each figure is congruent to itself (reflexivity). If A is congruent to B, then B is also congruent to A (symmetry). It is only in this way that the phrase "figures are congruent *to each other*" is meaningful. If two figures are congruent to a third one, they are also congruent to each other (transitivity).
3. *Aspect of classification:* The aspect of relation provides a disjunctive classification of all subsets of $\mathbb{R}^2$ into congruence classes. The classification aspect now emphasizes the typical question of identifying and describing particularly relevant congruence classes, as well as working out common properties of all figures within these classes. The latter is a specification in the sense of the aspect of quantities with identical size. The following question about the smallest possible amount of information for the unambiguous assignment of a figure to its congruence class leads to the classical congruence theorems.
4. *Aspect of mapping:* While the aspect of quantities with identical size statically compares the measurable properties of congruent figures, the question as to how one figure can be “transformed” into the other is part of the mapping aspect: (1) For every two congruent figures there exists by definition a mapping (bijective isometry) which transfers the figure into the other. (2) We can always express these mappings by the composition of a maximum of three straight line reflections (three-reflections theorem). (3) This mapping is always a glide reflection, a rotation or a translation. All this is especially valuable if you have proven the congruence of two figures over a congruence theorem. Automatically we already know about the existence of such a mapping.

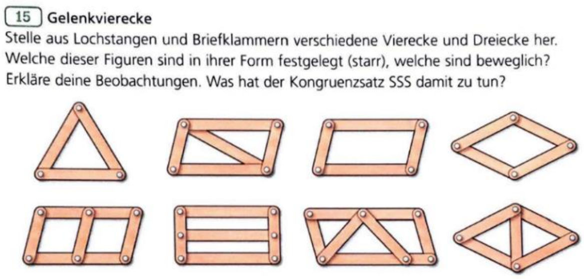
We can now use the *interface aspects* to illustrate links between school mathematics and academic mathematics. Many activities in dealing with congruence in school use congruence in the sense of the *aspect of classification*. This includes in particular constructions: Unambiguous constructability of triangles or other figures means that all figures that can be constructed from a given set of sizes are in the same congruence class. Also, the solving of plane triangles and the associated question of a minimum number of determining characteristics for a congruence class of triangles fall under this aspect. The determination of lengths or area contents using (sub)figures is based on the view of the concept of congruence described in the *aspect of quantities with identical size*. Proofs can use congruence in the sense of a mixture of both aspects: Congruence theorems are used to identify congruent figures with the aim of connecting different quantities with identical sizes. The *aspect of relation* is contained in the usual wording in textbooks: "congruent to *each other*". This is well-defined, only because congruence is a symmetrical relation. And finally, when the enactive action of laying congruent figures on each other, through reflections etc., is formalized, the *aspect of mapping* always comes into play.

The interface aspects take up all areas of the mathematical background of the topic “congruence” and systematize them on a level independent of the degree of abstraction. Our hope is that the aspects are suitable to support the taking of a meaningful mathematical perspective in a typical professional situation. We have shown that for each aspect there are areas of school mathematics in which it can be addressed.

Examples for interface tasks on the topic of congruence

The following task was used in the exercise group on congruence.

Consider the following textbook task *(Neue Wege 7, NRW (2014), p. 195)*:



a) For each of the figures shown, consider whether their shape can be changed or not. Explain your observations and discuss which role the theorem SSS plays.

b) Discuss, the role of the interface-aspects of *congruence* in the given task.

c) Based on your considerations in a), consider at least one congruence theorem for squares in your group and justify its validity. (our translation).

The professional relevance of the task lies in the domains (in the sense of Prediger, 2013) "be able to master requirements set for students by her- or himself at different levels " and "analyse and evaluate approaches (used in e.g. textbooks)". The content-related focus of the task lies in the *aspect of classification*, since it is a question of a sufficient number of given quantities for the unambiguous description of congruence classes (of quadrangles). The *aspect of quantities with identical size* also plays a role, as the modifiable figures are characterised by the fact that there are lengths (e.g. of diagonals) which are not clearly defined by the given lengths.

Quotes from students working on the task

As already described, we have video recordings from the tutorial meetings at our disposal, which we evaluate using methods of qualitative content analysis (Kuckartz, 2018). This analysis is still work-in-progress. One of our foci is students’ use of the interface aspects that the students have learned in the previous lecture. We now want to illustrate three of our categories using examples from the discussion of a group of three students on the above task (Table 1).

Category name	Category description	Example
proper and expected use of an interface aspect	The students use an interface aspect in an appropriate manner, in accordance with our preliminary considerations	(referring to the aspect of quantities with identical size) “[…] If we look at the [undrawn] diagonals, the lengths change if they (the figures) can move.”
proper but unexpected use of an interface aspect	The students adequately address an interface aspect in a way not anticipated by us.	Student uses the aspect of mapping in the context of the moving figures as follows: In the case of a moving figure, the different possibilities do not go into each other through reflection, rotation or translation and are therefore not congruent.
Incorrect use of an interface aspect	The students use an interface aspect in the wrong way.	(referring to the aspect of classification) *“[...] Then we have in principle two classes, one where something can move and one where nothing can move. [...]”*

Table 1: Categories of content analysis of video recordings of tutorial groups.

Especially the second two categories (proper but unexpected use and incorrect use) are very valuable for redesigning the learning activities according to our Design Research cycle. Passages which fall in the second category expand our knowledge about anticipated learning processes and thus indirectly also about possible problems. The example of the third category described in the table provides a reason to reconsider the formulation of the *aspect of classification*. The aim of the redesign is to make it clear to the students during the course that the classes are congruence classes and not sets defined by other features such as "All figures that are created by deformation of a moving figure". The first category shows us where the aspects seem to function exactly in the intended sense.

FURTHER SELECTED RESEARCH RESULTS

As described above, we also studied the interface week with a questionnaire with closed items. At this point, we like to present first results of two scales that deal with the self-assessment of students after the interface week (Table 2). Figure 1 shows a histogram of both scales. It becomes clear that the maj-ority of the students in both scales assess themselves rather positively and - after the interface week - are confident

that they can work with the background theory in the field of congruence and act professionally on this basis. The interesting question is what role the interface aspects play here. The item 'The interface aspects to congruence have helped me to better structure the mathematical background of the topic congruence.' was answered by all 21 surveyed students on a Likert scale from 1 to 5 with 4 (rather true, 14 students) or 5 (completely true, 7 students).

Scale	Description	Example Item	scale consistency
Background theory self-perception (5 items)	Self-perception of the ability to act in the background theory of congruence.	*I think that after the interface week, I can precisely define school mathematical terms from the area of congruence.*	$\alpha = .85$
Professional acting self-efficacy expectation (9 items)	Expectation to self-efficacy to act professionally in the area of congruence as a teacher (Item formulations are based on the job-tasks described by Prediger (2013)	*I think that after the interface week, I can analyse and evaluate a textbook excerpt on the topic of congruence.*	$\alpha = .79$

Table 2: Two scales of the acceptance questionnaire to the interface week congruence.

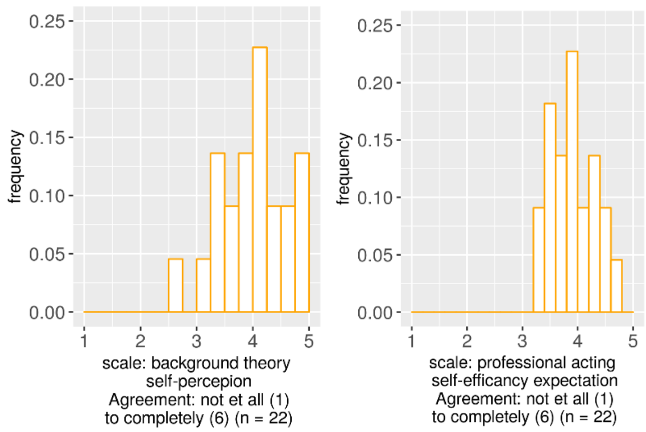


Figure 1: Histograms of the two scales from Table 2.

This is first and foremost a positive assessment, but it also shows that there is still room for improvement. Here, the results of the video evaluation described above, and the analysis of homework provide a good basis for a reformulation of the aspects in order to increase their accessibility of. The fact that the aspects are perceived as helpful is also supported by first results of our interview study, as the following original quote of one of the interviewed students shows:

> [...] The aspects already were very useful. So especially the whole thing of getting a structure like that in there. That you say: I have the aspects and can then relate them to them [to situations] and could think about which aspect I could use where or emphasize where.

FURTHER PERSPECTIVES

In the next step, we will continue the descriptive evaluation of the collected data and connect them with each other in order to be able to make more profound statements about learning processes in the interface weeks. However, it is already clear that the idea of interface aspects is positively received by the students, but it is still unclear, how such aspects can be schematically set up for other mathematical concepts.

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Relation between understandings of linear algebra concepts in the embodied world and in the symbolic world

Mitsuru Kawazoe

Osaka Prefecture University, Faculty of Liberal Arts and Sciences, Japan, kawazoe@las.osakafu-u.ac.jp

For the use of embodied notions in teaching linear algebra, some studies indicate that it is helpful, but another study indicates that it is sometimes problematic. Hence more study is needed. In this study, linear (in)dependence and basis were focused on, and the relation between understandings of them in the embodied world and in the symbolic world was investigated. The effectiveness of an instruction emphasizing geometric images of them was also investigated. The main results of the study were the following: conceptual understanding of linear dependence of four spatial vectors such that any three of them do not lie on the same plane was positively associated with understanding of basis in the symbolic world; however, understanding of linear dependence of such vectors had not been improved by a geometrical instruction.

Keywords: linear algebra, teachers' practices, linear independence, basis.

BACKGROUND AND THE PURPOSE OF THE STUDY

It is widely recognized that linear algebra is a difficult subject to learn due to its abstract and formal nature. Dorier and Sierpinska (2001) stated that "linear algebra remains a cognitively and conceptually difficult subject." It has been a big challenge to overcome the difficulty in teaching linear algebra. Some researchers pointed out that the use of embodied notions, namely the use of visual images, helps students to understand concepts in linear algebra (cf. Stewart & Thomas, 2007; Hannah et al., 2014; Donevska-Todorova, 2018, p. 268). However, there is another study indicating that using visual images is sometimes problematic in teaching linear algebra (Sierpinska, 2000, p. 244). These studies indicate that the use of visual images in teaching linear algebra and its effectiveness should be more investigated. That is a motivation of our research to investigate students' conceptions of linear algebra concepts in the context of geometric vectors.

In our previous studies, we observed the following: (1) there are many students who fail to determine linear dependence of four spatial vectors such that any three of them do not lie on the same plane (Kawazoe et al., 2014); (2) some of those students take a longer time to image that three spatial vectors not lying on the same plane span the whole space (Kawazoe & Okamoto, 2016; Kawazoe, 2018). However, we have not investigated how these observations are related to understanding of concepts and procedures in linear algebra.

In this study, we focused on concepts of linear (in)dependence and basis, and studied the following research questions: (1) Is geometrical understanding of linear (in)dependence in the embodied world related to understanding of linear (in)dependence and basis in the symbolic world?; (2) Can geometrical understanding of linear (in)dependence in the embodied world including the case of four vectors be improved by an instruction emphasizing a geometric image of linear (in)dependence?

THEORETICAL FRAMEWORK

We use Tall's model of three worlds (Tall, 2013) combined with APOS theory (Arnon et al., 2014) to distinguish students' understanding for linear algebra concepts, following Stewart and Thomas (2007). Tall (2013) described the development of mathematical thinking in terms of three worlds: *embodied* world, *symbolic* world, and *formal* world. Tall stated that "the combination of embodied and symbolic mathematics can be seen as a preliminary stage to the axiomatic formal presentation of mathematics." In linear algebra, the embodied world is a world of geometric vectors (arrows), the symbolic world is a world of numerical vectors, matrices, polynomials, and operations using symbols. APOS theory enables us to distinguish students' conceptions into four levels: *Action*, *Process*, *Object,* and *Schema*. Then, students' conceptions in linear algebra can be described in each of three worlds (cf. Stewart & Thomas, 2007). As for linear (in)dependence, Action-Process-Object conceptions in the embodied world are described as follows. Students having *Action* conception draw a linear combination explicitly in a discussion of linear (in)dependence. Students having *Process* conception can use a set of linear combinations but cannot use a spanned space correctly. Students having *Object* conception can completely understand that any two non-parallel geometric vectors are linear independent and they span a plane, any three geometric vectors not lying on the same plane are linear independent and they span the whole space, and any four geometric vectors are always linearly dependent.

We view some linear algebra concepts from the viewpoint of Lakoff and Núñez (2000). For an example, we regard a role of basis of a vector space as the 'discretization' of a space, following the explanation given by Lakoff and Núñez (ibid., p. 260-261). To give a basis for a vector space is equivalent to give a coordinate for the space. In the embodied world, it means to represent every point in a plane or a space as a pair or a triple of numbers. Moreover, we apply the 'Basic Metaphor of Infinity' (ibid., p. 158) to students' image of spanned space, according to an observation of our previous study (Kawaoze & Okamoto, 2016) that many students image a space spanned by linearly independent three spatial vectors as a 'gradually expanding three-dimensional object' which finally fills the whole space. We used these viewpoints in designing linear algebra lessons in this study.

CONTEXT: THE COURSE, STUDENTS, DESIGN OF LESSONS AND TASKS

The study was conducted in a linear algebra course aiming at engineering students at our university, but in a special class for students who failed to pass it when they were in the first-year. The course consists of a spring semester class and a fall semester class. The former is a 2-credit class, meeting for 90 minutes each week for 15 weeks. The latter is a 4-credit class, meeting for 180 minutes each week for 15 weeks. Each of them is followed by an examination period. The course covers usual linear algebra topics: matrix, gaussian elimination, system of linear equations, and determinant, etc. in the spring semester; formal vector space, spanned space, linear (in)dependence, basis, dimension, linear map, inner product, orthogonal basis, eigenvalue, eigenvector, and diagonalization, etc. in the fall semester. This study was conducted during the first five weeks in the fall semester class. In these weeks, students learned formal vector space, spanned space, linear (in)dependence, basis, and dimension.

Design of lessons

Each lesson consisted of a lecture part and an exercise part. Lectures were given in the first half, and exercises were given in the second half. The lecture part was designed as to emphasize geometric images of linear algebra concepts especially by using the image of a spanned space in the embodied world. In the lecture part, the teacher introduced linear algebra concepts in the following way.

First, the notions of linear combination and spanned space were introduced. A space spanned by three linearly independent spatial vectors was shown to students by using teacher's fingers, and it was emphasized that linear combinations with negative coefficients were contained in the spanned space. The teacher stressed the importance of imaging a part of the space consisting of linear combinations with some (or all) coefficients being negative in order to grasp the correct image of the spanned space.

The notions of linear independence and dependence were introduced by using usual formal definitions, but the meaning of linear independence and dependence of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ in a vector space were explained in terms of spanned space as follows:

> Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly dependent if and only if one of the n vectors can be represented by a linear combination of the other n-1 vectors, that is, one of the n vectors is contained in the space spanned by the other n-1 vectors.
>
> Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly independent if and only if none of the n vectors can be represented by a linear combination of the other n-1 vectors, that is, none of the n vectors is contained in the space spanned by the other n-1 vectors.

It was also explained that linearly independent vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ give an ascending sequence of vector spaces $V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_n$ where V_k (k=1, 2, …, n) is the space spanned by $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$.

Then, the notion of basis was introduced by using a usual formal definition:

> Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ in a vector space V is a basis of V if and only if they are linearly independent and any vector in V can be represented as a linear combination of them.

It was explained that the second condition is equivalent to that V is spanned by $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$. In the introduction of basis, the role of basis was explained as to give a coordinate system, and a basis was explained as a set of 'axes.' It was explained that the second condition means that it contains a sufficient number of axes to represent the whole space and that the first condition means that there is no extra axis in the set.

In the exercise part, students worked on paper-based exercises given by the teacher. Exercises mainly consisted of questions in the symbolic world and some of them can be viewed as questions in the embodied world: determining linear (in)dependence of vectors in $\mathbb{R}^n$ (n=2,3,4) or in polynomial spaces, determining whether a given set of vectors in $\mathbb{R}^n$ (n=2,3,4) or in polynomial spaces is a basis or not, finding a basis and the dimension of given subspaces in $\mathbb{R}^n$ (n=2,3,4) or in polynomial spaces, etc. Many of the questions were computational ones. Some of them were related to the geometric instruction given in the lecture part and they can be answered with geometrical reasoning.

Design of tasks

The following four tasks, which were translated from Japanese, were designed in order to investigate students' understanding of dimension, linear (in)dependence and basis.

Task 1: Answer the following questions. If you do not know (or if you have not learned), write your answer as "I don't know."

(1) Describe your image of an example of one dimension, two dimension, and three dimension, respectively, using figures and words freely.

(2) For vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n, \boldsymbol{v}_{n+1}$, assume that vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ span an n-dimensional space V, and that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n, \boldsymbol{v}_{n+1}$ span an (n+1)-dimensional space W. When you draw a picture showing this situation, what kind of picture do you draw? Draw a picture of your image.

Task 2: Determine whether spatial vectors given in each picture are linearly independent or not. Note that each vector lies on a line or a plane shown in the picture. (If there are multiple planes, each vector lies on one of them.)

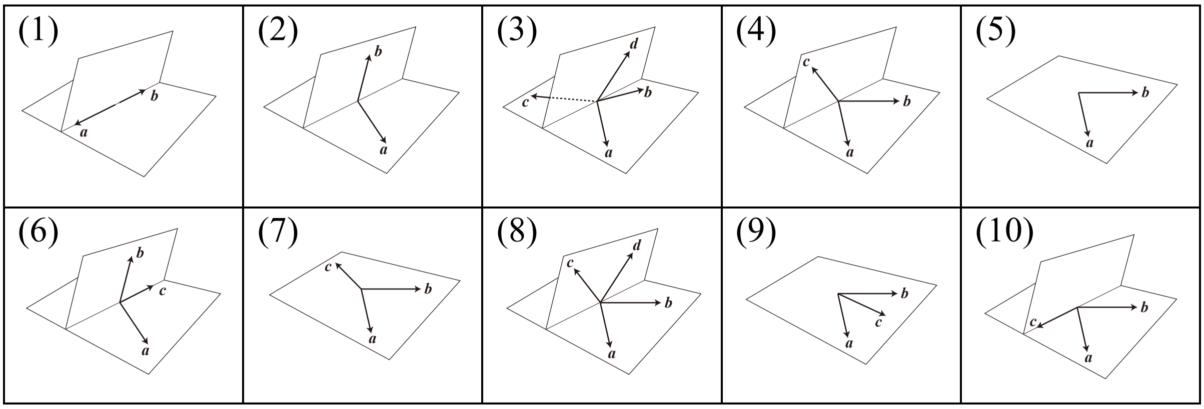


Figure 1: Test items in Task 2

Task 3: (Q1) For vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ in a vector space V over K, describe two conditions (in the definition of basis) for $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ to be a basis of V. Write your answer in the answer columns (A) and (B). (Q2) Determine whether the following set of vectors is a basis or not. If it is not a basis, answer which condition that you described in Q1 is not satisfied. In the latter case, write your answer by using the symbol A or B, and write "A, B" in both cases. (Vector spaces are as follows: (1) $\mathbb{R}^4$, (2) $\mathbb{R}^3$, (3) $\mathbb{R}^2$, (4) $\mathbb{R}^3$, (5) the space of polynomials $f(x)$ with degree less than 3 whose coefficients are in $\mathbb{R}$, (6) the space of polynomials $f(x)$ with degree less than 2 whose coefficients are in $\mathbb{R}$.)

(1) $\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ (2) $\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\-1\end{pmatrix}$ (3) $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\-1\end{pmatrix}$ (4) $\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\2\end{pmatrix},\begin{pmatrix}-1\\-1\\-5\end{pmatrix}$

(5) $x+1, x^2$ (6) $x-1, x+1$

Task 4: (Q1) Determine whether spatial vectors given in each picture are linearly independent or not, and describe the reason. (Q2) Determine whether the given vectors in $\mathbb{R}^3$ are linearly independent or not, and describe the reason.

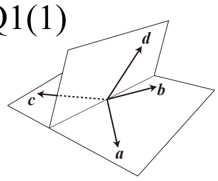
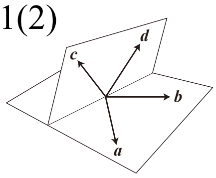
Q1(1)	Q1(2)	Q2(1)	Q2(2)
		$\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}-2\\-3\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}$

Figure 2: Test items in Task 4

A priori analysis of tasks

Task 1 and Task 2 are pre-tests conducted at the beginning of the semester. Task 1(1) can be answered as ‘line’, ‘plane’, and ‘space’. Task 1(2) is a non-routine task to examine whether students have an image that *V* is contained in *W*, or *W* extends outside of *V* as a space. Task 2 includes all of the important cases of less than or equal to four spatial vectors regarding linear (in)dependence. Task 2 is the same one that we used in our previous study (Kawazoe & Okamoto, 2016). According to our previous result (ibid.), Task 2 (8) was expected to be difficult for the participants. Task 2 (8) contains four vectors and any three of them do not lie on the same plane, hence it cannot be reduced to the case of less than or equal to three vectors. Task 2 (3) also contains four vectors, but it can be reduced to the case of three vectors because the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ lie on the same plane. The terms ‘dimension’, ‘span’, and ‘linearly independent’ were used in the texts in these tasks. Since the participants were in the second-year or higher, they had already learned them when they were in the first-year.

The aim of Task 3 is to investigate students’ understanding of the definition of basis. For any set of vectors listed in (1)-(6), one can determine their linear (in)dependence without computation. Only (2) and (6) are basis, and the others are not.

In Task 4, Q1 is a task in the embodied world, and Q2 is a task in the symbolic world. The two pictures in Q1 was taken from Task 2. According to the result of our previous study (ibid.), determining linear (in)dependence of four spatial vectors is problematic. Q1(1) and Q2(2) present essentially the same situation, and Q1(2) and Q2(1) present essentially the same situation. Q1(1) and Q1(2) can be answered by drawing vectors representing linear combinations, or by using the fact on vector subspaces spanned by two or three vectors. Q2(1) and Q2(2) can be answered by using numerical computation (with or without the use of the Gaussian elimination), but they also can be answered with geometrical reasoning.

METHODOLOGY AND DATA COLLECTION

We implemented four-weeks lessons whose design is described in the above. Task 1 and 2 were conducted at the beginning of the first lesson. Task 3 was conducted at the third week, and Task 4 was conducted at the beginning of the fifth week lesson. Participants' answers for Task 1 were analyzed whether they have an image of dimension less than or equal to three and whether they have an image of increment of dimension. Participants' reasoning for Task 4 Q1 were analyzed with APOS theory. Participants' reasoning for Task 4 Q2 were classified into two types: algebraical reasoning, and geometrical reasoning. For other tasks, participants' answers were evaluated depending on their correctness. Then, the relations between the results of these tasks were investigated.

The study was conducted in the fall semester in the academic year 2018. All data were collected during the first five weeks in the linear algebra class for engineering students who had failed in the previous year or before. The number of students in the class were 58. Among the 58 students, 38 of them worked out all the tasks from Task 1 to Task 4. In this study, the data of the 38 participants was statistically analyzed.

RESULTS

The result of each task and setting of groups

Task 1. For Task 1 (1), almost all participants described their images for dimension 1, 2, 3, as 'line', 'plane', 'space', respectively. For Task 1 (2), only 11 (28.9%) of them could draw their image of increment of dimension as extending outside the space. We set two groups according to the result of Task 1 (2): *GI+* is the group of 11 participants having a geometric image of increment of dimension, *GI-* is the group of the others.

Task 2. The percentages of correct answers for Task 2 were as follows: (1) 97.4%, (2) 94.7%, (3) 65.8%, (4) 97.4%, (5) 89.5%, (6) 94.7%, (7) 86.8%, (8) 52.6%, (9) 89.5%, (10) 86.8%. The percentages of correctness for (3) and (8) were much lower, compared with the others. The pictures of (3) and (8) contain four vectors. The number of vectors in the others is less than four. The result of Task 2 was almost the same as the one in our previous study (Kawazoe & Okamoto, 2016), except for the result of (3). In the previous study, the percentage of correct answers for (3) was 84.5%. The median of the number of correct answers per participant was 9. We set two groups according to the result of Task 2: *GV+* is the group of participants who answered correctly to more than 8 questions, and *GV-* is the group of the others.

Task 3. For Q1, the number of participants who could describe two conditions in the definition of basis correctly was 23 (60.5%). While 34 (89.5%) of the participants could describe linear independence of the vectors correctly as one of the conditions, 24 (63.2%) of them could described correctly that the vectors span V or that any vector in

V can be represented as a linear combination of the vectors. 8 (21.1%) of the them described 'dim $V=n$' as one of the conditions, which is a wrong answer because 'dim V' is defined after the definition of basis is introduced.

For Q2, while the percentages of correct answers for (2), (3), (4) were high, those of (1), (5), (6) were relatively low: (1) 78.9%, (2) 97.4%, (3) 94.7%, (4) 94.7%, (5) 78.9%, (6) 65.8%. As for reasoning in (1), (3), (4), and (5), we evaluated whether a participant could answer correctly based on the necessary and sufficient conditions to be a basis. Hence, for a participant who described 'dim $V=n$' in Q1, we evaluated his/her answer for Q2 whether it was logically correct based on his/her answer in Q1. The percentages of correct answers for reasoning were as follows: (1) 65.8%, (3) 63.2%, (4) 36.8%, (5) 65.8%. The median of the number of errors in Q2 (including errors in reasoning in the case of non-basis) per participant was 2. We set two groups according to the number of incorrect answers for Task 3 Q2: *B*+ is the group of participants whose incorrect answers were less than or equal to 2, and *B*- is the group of the others.

Task 4. The percentages of correct answers for Task 4 were as follows: Q1(1) 89.5%, Q1(2) 55.3%, Q2(1) 86.8%, Q2(2) 89.5%. The pictures in Q1(1) and Q1(2) are same as in Task 2 (3) and Task 2 (8), respectively. While the percentage of correct answers for Q1(2) remained still low, the one for Q1(1) was much improved from the result of Task 2 (3). Though Q1(2) is essentially same as Q2(1) from a geometrical viewpoint, the results of them were different. According to the reasoning in Q1, we set the following groups: For j=1, 2, O_j+ is the group of participants showing *Object* conceptions in the reasoning for Q1(j), O_j- is the group of participants showing *Action/Process* conceptions or giving no reason in the reasoning for Q1(j). According to the reasoning in Q2(j), we set the following groups: For j=1, 2, GR_j+ is the group of participants using geometrical reasoning for Q2(j), GR_j- is the group of the others.

The relations between the results of each task

In the following analysis, we used Fisher's exact test instead of the Chi-square test because there were small numbers in cross-tabulation.

Relation between understanding in the embodied world and understanding of basis: Fisher's exact test indicated that having a geometric image of increment of dimension (Task 1 (2)) and the result of Task 3 Q2 were positively associated (p<0.05, Table 1). Fisher's exact test also indicated that showing Object conceptions in reasoning for Task 4 Q1(2) and the result of Task 3 Q2 were positively associated (p<0.05, Table 2). On the other hand, we could not find any significant relation between O_1+/- and B+/-.

	B+	*B*-
GI+	9	2
GI-	11	16

Table 1: Relation between the results of Task 1(2) and Task 3 Q2

	B+	*B*-
O_2+	8	1
O_2-	12	17

Table 2: Relation between having Object conception and the result of Task 3 Q2

Relation between understandings in the embodied world and in the symbolic world: Fisher's exact test indicated that showing Object conception in reasoning for Task 4 Q1(2) and the number of correct answers in determining linear (in)dependence in Task 4 were positively associated ($p<0.01$, Table 3), where NC means the number of correct answers in determining linear (in)dependence in Task 4. On the other hand, we could not find any significant relation between O_1+/- and the result of Task 4. Fisher's exact test also indicated that the use of geometrical reasoning for Task 4 Q2 and the number of correct answers in determining linear (in)dependence in Task 4 were positively associated ($p<0.05$, Table 4), where $GR+= GR_1+\cup GR_2+$, $GR-= GR_1-\cap GR_2-$, and NC is the same as in Table 3. Fisher's exact test also indicated significant correlations for GR_1+/- ($p<0.05$) and for GR_2+/- ($p<0.05$).

	NC=4	NC<4
O_2+	8	1
O_2-	9	20

Table 3: Relation between having Object conception and the result of Task 4

	NC=4	NC<4
GR+	10	5
GR-	7	16

Table 4: Relation between the use of geometrical reasoning and the result of Task 4

Difference of understanding of linear (in)dependence between before and after of four-weeks lessons: The picture in Task 4 Q1(1) and Q1(2) are same as the one in Task 2 (3) and (8), respectively. McNemar's test indicated that there was a significant difference between the results of Task 2 (3) and Task 4 Q1(1) ($p<0.05$, Table 5), where the participants were divided into two groups depending on whether their answers for Task 2(3) were correct ($T_{2(3)}$+) or not ($T_{2(3)}$-), and they were divided into two groups depending on whether their answers for Task 4 Q1(1) were correct ($T_{4Q1(1)}$+) or not ($T_{4Q1(1)}$-). On the other hand, Fisher's exact test indicated that the result of Task 2 and the number of correct answers in determining linear (in)dependence in Task 4 Q1 were positively associated ($p<0.01$, Table 6), where NC_{Q1} means the number of correct answers in determining linear (in)dependence in Task 4 Q1.

	$T_{4Q1(1)}$+	$T_{4Q1(1)}$-
$T_{2(3)}$+	23	2
$T_{2(3)}$-	11	2

Table 5: Relation between the results of Task 2 (3) and Task 4 Q1(1)

	NC_{Q1} =2	NC_{Q1} <2
GV+	16	7
GV-	3	12

Table 6: Relation between the result of Task 2 and the result of Task 4 Q1

DISCUSSIONS

As for the first research question, we observed some relations between understanding in the embodied world and understanding in the symbolic world. The analysis for Table

1 indicated that having a geometric image of increment of dimension and understanding of basis in the symbolic world were positively associated. The analysis for Table 4 indicated that the use of geometrical reasoning in the symbolic world and understanding of linear (in)dependence in both embodied and symbolic world were positively associated. The analysis for Table 2 and 3 indicated that having Object conception for linear (in)dependence in the embodied world, especially for the case of four spatial vectors such that any three of them do not lie on the same plane (as in the picture of Task 2(8) and Task 4 Q1(2)), was positively associated with understanding of basis in the symbolic world (Table 2), and also positively associated with understanding of linear independence in both embodied and symbolic world (Table 3).

As for the second research question, we observed that the effectiveness of the implemented instruction emphasizing geometric images was limited. The analysis for Table 5 indicated that understanding of linear dependence of four spatial vectors in the picture of Task 2 (3) had been improved during the four-weeks lessons. On the other hand, the result of Task 4 and the analysis for Table 6 indicated that understanding of linear dependence of four spatial vectors in the picture of Task 2 (8) had not been improved. Improving students' understanding of Task 2 (8) was more important because conceptual understanding of linear dependence in the case of Task 2 (8) was related to understanding of basis and linear independence in the symbolic world. How should we consider this result? There may be the following two possibilities: one is that the geometrical instruction implemented in this study was insufficient and it can be more improved; the other is that there is a limitation of students' perception even in the embodied world and it is cognitively hard to overcome such limitation. In the latter case, we should take into account of such limitation in teaching linear algebra, and it may lead us to reconsider how to design a linear algebra course under the framework with Tall's model of three worlds, especially to reconsider the balance and integration between geometric and algebraic presentation. However, the two possibilities need to be carefully examined in the future study.

Finally, we should mention the limitations of the study. First, the sample size was small. Second, the participants were not ordinary because they were students who had failed to pass the subject in the earlier years. Hence, further studies with a larger number of first-year students are needed.

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Investigating high school graduates' personal meaning of the notion of "mathematical proof"

Leander Kempen[1], Sandra Krämer[1], and Rolf Biehler[1]

[1]University of Paderborn, Germany, kempen@khdm.de

In this paper, we report on the results of a pilot study to investigate high-school graduates' personal meaning of mathematical proof. By using proof tasks and a following interview phase with meta-cognitive questions, we will describe students' personal meaning of the notion of mathematical proof and show that some students hold different meanings of the word "proof" simultaneously.

Keywords: reasoning, proof, generic proof, personal meaning, example.

INTRODUCTION

Mathematical proof can be considered being a major hurdle for mathematics freshmen (Selden 2012, p. 293). However, when trying to teach mathematical proof and proving to first-year students, their previous knowledge on the topic has to be taken into account (ibid., p. 414). Besides learners' competencies concerning proof construction, reading, and evaluation, the personal meaning they assign to the notion of mathematical proof seems to play a crucial role in their mathematical behaviour (Harel & Sowder 1989). While different studies have investigated university students' proof competencies, we focus on students' knowledge concerning mathematical proof after graduating from high school. Accordingly, we aim at investigating students' personal meaning of proof as a part of their existing knowledge of mathematical proof when entering university. For this purpose, we rely on the study of Recio and Godino (2001). These authors elaborated on different personal meanings of mathematical proof. We expand their investigation to today's high-school graduates and deepen their approach making use of qualitative methods. When clarifying high-school graduates' personal meaning of mathematical proof, the socialisation process concerning proof in school mathematics can be elaborated. Moreover, the consensus on the meaning of mathematical proof has to be considered as an inevitable requirement for teaching mathematical proof at university. In this sense, it might become possible to link university studies to previous experiences from school mathematics and the enculturation process to higher mathematics can be undertaken more consciously. Finally, it might get possible to conceptualize or to expose popular misconceptions concerning mathematical proof. In this paper, we report on the design and the results of a pilot study, where four high-school graduates were asked to prove two mathematical claims and to explain and to validate their performances afterwards.

THEORETICAL BACKGROUND

Empirical findings from the literature

Kempen and Biehler (2019) evaluated first-year pre-service teachers' proof validation. In their study, 29.7% of the 37 first-year students rated a purely empirical verification as "correct proof" when starting their university studies. Selden (2012, p. 398 ff.) summarizes several problems of first-year students with mathematical proof and highlights i. a. a nonstandard view of proof (e.g., what constitutes a proof and how the proof process is interpreted). Following Kempen and Biehler (2019, p. 246 ff.), first-year students mainly link the concept of proof with some prototypes of proof, like the proof of Thales' theorem or the proof of the binomial formulas. Also, beginning students do not have much experience with proof construction. While proving at university is associated with the use of definitions, the application of theorems about abstract concepts and deductive reasoning (Selden & Selden 2007), the named examples from school mathematics display another 'concept' of proof: In school geometry, proofs make use of a figure to perform reasoning. In elementary arithmetic, many proofs utilize simple calculations using variables (e.g. in the proof of the binomial formulas). Besides, following the TIMS-Study in 1998, German high-school students showed only low abilities concerning the construction and evaluation of mathematical proof (compare Reid & Knipping, 2010, p. 68).

Categorization of students' proof productions

Recio and Godino (2001) investigated the proof competencies of first-year university students in Spain. In their study, i. a. 429 students entering university were asked to work on two elementary proving tasks. The authors conclude that the percentage of students giving a substantially correct mathematical proof to each problem is less than 50%. Only 32.9% of the students gave correct answers to both proving tasks. Interestingly, about half of the students (53.8%) formulated a purely empirically based answer to at least one of the given tasks. The authors classified student's proof attempts using the following set of categories: (1) "The answer is very deficient (confused, incoherent)", (2) "The student checks the proposition with examples, without serious mistakes", (3) "The student checks the proposition with examples, and asserts its general validity", (4) "The student justifies the validity of the proposition, by using other well-known theorems or propositions, by means of partially correct procedures", and (5) "The student gives a substantially correct proof, which includes an appropriate symbolization". Finally, the authors tried to link the named categories with personal proof schemes in reference to Harel and Sowder (1998). Answers of type (2), the mere empirical confirmation of a proposition, are connected with the "explanatory argumentative scheme", because "There is neither a true intention to validate the proposition, nor an intention to affirm the validity of the proposition for all possible cases." (ibid.). Answers of type (3) are considered to be in line with the "empirical-inductive proof scheme"; these answers are based on verifying the proposition by using particular examples, without the intention of justifying the general validity. In contrast to the former categories, answers of type (4) and (5) include the intention of verifying the general validity of the proposition by using deductive reasoning. Accordingly, answers of type (4) are connected with the so-called “informal deductive proof scheme”. Finally, the answers of type (5) follow a more formal approach, making use of a symbolic and algebraic language. These answers are assigned to the “formal deductive proof scheme”. When students’ performances in the two tasks seemed to correspond to each other, the authors interpreted the proposed categories as personal schemes of mathematical proof to describe students’ personal meaning.

Reid and Knipping (2010, p. 130 ff.) distinguish four kinds of proof or argument according to the representations involved. (1) “Empirical”: Those arguments in which specific examples are used but do not represent a general case, (2) “Generic”: Those arguments in which specific examples are used to highlight a general idea, (3) “Symbolic”: Those arguments in which words and symbols are used as representations, and (4) “Formal”: Those arguments in which symbols are used without semantic reference. In this paper, we make use of this categorization of proofs to categorize students’ proof productions. Since we are dealing with high-school students’ proof attempts, the fourth kind of proof will not appear in the analysis. We split the third type of proof to distinguish an increased used of words (“narrative proof”) and an increased application of symbols and variables (“symbolic proof”). An example of each type of argument is given below.

Research Questions

Based on the theoretical considerations above, we formulate the following two research questions: (1) How do upper secondary school students prove claims from elementary arithmetic and geometry? (2) Which personal meaning of the notion of mathematical proof can be assigned to the students?

METHODOLOGY

Interview design

In our study, four students from upper secondary school were supposed to work on two proving tasks, one from elementary arithmetic and one from elementary geometry (see below). Afterwards, the students were asked to explain their solutions and to answer metacognitive questions. We used a combination of task-based interviews according to Goldin (2000) and the Precursor-Action-Result-Interpretation (PARI) method (Hall et al., 1995) modified by Kortemeyer and Biehler (2017) for the use within mathematics education research. Both methods intend two main stages: the solving of mathematical problems and a following interview concerning the participants’ approach, their reasons for choosing it and the interpretation of their results. Mainly, the modified PARI methodology was added to develop and organise the interview questions in three phases (working individually on the task, recapitulating one’s solution with the interviewer, reflecting on one’s strategies and decisions).

In accord with Goldin (2000, p. 522 f.), the study was split into the following stages: (i) posing the questions with sufficient time for working on the tasks, (ii) minimal heuristic suggestions and assistance, if the participants display serious problems (e.g., "Can you tell me what the claim is about?"), (iii) questions concerning students' approaches (e.g., "What did you do?"), and (iv) metacognitive questions. In this last part, the students were questioned i. a. about their satisfaction with their solutions and the reasons for choosing the respective approach. We also asked the participants to evaluate their solutions in order to see if they consider them as correct proofs.

Task analysis and expected solution

The proving tasks used in the study should be accessible to all students and allow for different approaches. We followed the idea of Recio and Godino (2001) to use one claim from elementary arithmetic and one from geometry. This way, we also wanted to investigate if students' proving approaches and personal meaning on the notion of mathematical proof differ with respect to the subject. We replaced the first claim from Recio and Godino (2001) with a proving task from Biehler and Kempen (2013), because this task seemed to be more suitable for us. The second claim was taken from the original study. The named tasks were slightly modified for the use within this study. Finally, the participants were supposed to solve the following tasks:

Task 1: *Prove that the sum of an odd natural number and its double is always odd.*
Task 2: *We consider two adjacent angles α and β. Prove that the bisectors of α and β always form a right angle.*

In Table 1, a non-exhaustive set of expected solutions with regard to the categorisation of arguments according to Reid and Knipping (see above) is presented.

	Task 1	**Task 2**
empirical argument	$1 + 2 \cdot 1 = 3$ $3 + 2 \cdot 3 = 9$	$120° + 60° = 180°$ $60° + 30° = 90°$
generic proof	$1 + 2 \cdot 1 = 3 \cdot 1 = 3; 3 + 2 \cdot 3 = 3 \cdot 3 = 9$ Comparing the equations, one can recognise that the result must always be three times the initial number. Since three times an odd number is always odd, the result is an odd number (see Biehler and Kempen 2013, p. 89).	$130° + 50° = 180°; \frac{130°}{2} + \frac{50°}{2} = \frac{1}{2} \cdot (130° + 50°) = 90°$ The sum of the adjacent angles α and β is 180°. The bisectors split α and β into two equal angles. Accordingly, the sum of the half-angle of α and the half-angle of β is always 90°.
narrative proof	The double of an odd number is always even. Since the sum of an odd and an even number is always odd, the statement is proven.	The sum of the adjacent angles α and β is 180°. The bisectors split α and β into two equal angles. Accordingly, their sum equals half

		of 180°. Therefore, the sum of the half-angle of α and the half-angle of β is always 90°.
symbolic proof	Let a be an odd number. Then $a + 2a = 3a$. Since three times an odd number is always odd, the statement is proven (see Biehler and Kempen 2013, p. 90).	Let α and β be adjacent angles. Then $\alpha + \beta = 180°$. Accordingly, we have: $\frac{\alpha}{2} + \frac{\beta}{2} = \frac{\alpha+\beta}{2} = \frac{180°}{2} = 90°$ (see Recio and Godino 2001, pp. 85).

Table 1: Possible solutions of the proving tasks in accordance with the categorisation of Reid and Knipping (2010).

The two tasks comprise mathematical content from middle school and are meant to be easy to understand. A diagram was added to the second task, where the angles have been drawn and named for clarifying the given claim. Both tasks allow for different approaches, which might give a hint to students' personal meaning of proof.

Data collection and data analysis

This pilot study was conducted with four students in their last year in a high school (two females, average age was 20; two students attending an advanced course in mathematics and two students attending a basic course). Students' proof construction and the following interview phase were filmed in order to be able to base the analysis of the proving process not only on the participants' description of their approach but also on observations of the filmed process.

The analysis of the concrete proofs created by the participants focusses on the type of proof corresponding to the participants' approaches (see Table 1) and on the mathematical correctness. We consider a proof being correct when the arguments given are mathematically correct and link the given data with the formulated claim in a deductive manner. For describing participants' personal meaning of the notion of mathematical proof, we made use of the categories proposed by Recio and Godino (2001) (see above). However, we did not want to assign the personal meanings of the notion of proof only based on students' proof productions. We also carried out a qualitative analysis of students' responses from the interview to increase the validity of our research.

For answering the first research question, we will categorize students' proof productions and rate their correctness. (We consider a proof being correct, if the claim is logically derived from the premises. We call a proof incorrect, if at least one of the arguments used is not true (in general) or if part of the whole chain of argument is missing.) We will combine the results of students' proof productions, their proving process and their answers from the interview to describe their personal meaning on the notion of mathematical proof (research question 2).

RESULTS

In this section, we will describe and discuss the results of each participant separately to work out a uniform description for each participant. Due to the size of the paper, we had to look for a selection from students' responses to analyse their performances.

Students' proof productions and personal meanings

Participant 1 creates a generic proof for the first task and a narrative proof for the second (see Figure 1). However, the argument explicated in the generic proof is incomplete, because the fact that one addend is odd does not explain the parity of the result. In the second task, the student's use of the word parallel is not correct. Moreover, she seems to give some kind of intuitive argument, why the angle stays the same.

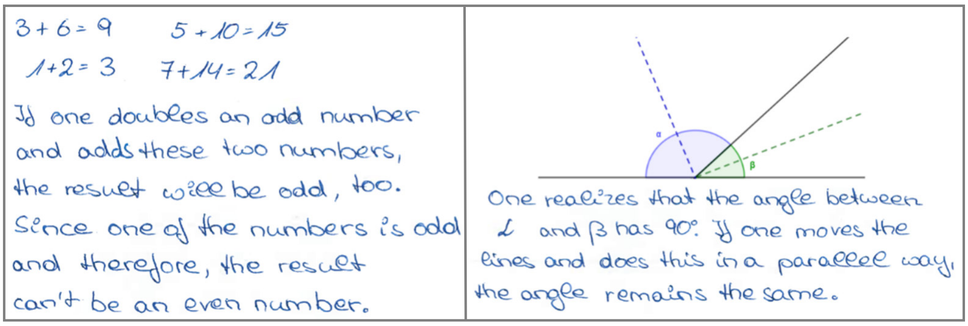

Fig. 1: Participant 1's proof productions (left: task 1; right: task 2; our translation)

When working on the first task, Participant 1 checks several examples and tries to find and to assemble arguments. She recalls her proving process as follows:

> Participant 1: So, I started with 3, added 6, because 6 is the double, and then you have 9. And then I was thinking, what will happen with other numbers, and there will always be an odd sum because when you have an odd number, when you take its double, which is an even number, and when you add an odd and an even number, you obtain an odd number. (Transcript 1; our translation)

This student uses concrete examples to find a pattern that might constitute a generic argument. Accordingly, she constructs a generic proof (see Figure 1, left). Unlike her explanation given in the interview, the explication of the argument in the written proof is incomplete. Concerning task 2, the student uses her set square to measure the angle in the graphic on the exercise sheet:

> Participant 1: I took the set square to check if the angle is 90°. […] Then I was thinking about, how one might prove this, because, one sees that the angle has 90°. I did not come up with a calculation or an equation […]. And then I was thinking: When you move this [the angle bisectors of α and β], they stay parallel; you always move them parallel; accordingly, this angle stays the same. (Transcript 1, our translation)

This approach of measuring the angle seems to be interesting, because the fact about the right angle has been mentioned explicitly in the task. Afterwards, as she recalls, she was looking for some kind of calculation. Not coming up with an equation, she tries to describe some kind of "dynamic" argument about moving the bisectors. Her 'narrative proof' is based on a visual impression and not convincing.

Participant 1's meaning of proof is beyond purely empirical evidence. She is trying to find an argument to verify the given claims in general. However, she expressed her satisfaction with her solution, because she "proved the claim given in task one". To sum up, participant 1's personal meaning of proof seems to be in line with the informal deductive-proof scheme. While looking for deductive reasoning in order to verify the given claim in general, she makes use of rather informal arguments.

Participant 2 has serious problems to understand the first proving task. Accordingly, we will not discuss his solution for task one here. In the second task, the student is finally making use of algebraic variables and equations to verify the given claim in general and he constructs a symbolic proof.

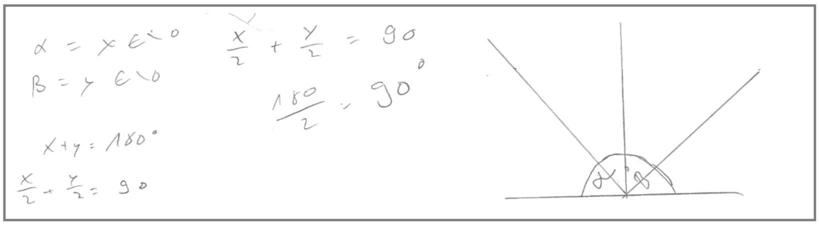

Fig. 2: Participant 2's solution to task 2

Like Participant 1, also this student uses his set square to measure the angle in the graphic on the exercise sheet. Afterwards, he creates a special case for the given claim. He explains his use of this example (the special case shown in Figure 2) as follows:

> Participant 2: OK, that's true. But why? Stop, I'll take another example. […] When I do it like this; I take 90° and both halves have 45°, and that's again 90°. A Coincidence? I didn't know. Good to know! That's really true.

Afterwards, he transmits the idea of the bisectors α and β to variables x and y:

> Participant 2: OK, α has a certain angle, called x. And for β we take y. I would say: x divided by two plus y divided by two equals 90. […] I know why it's true. Since half of 180 is 90, true? Yes, here we have 180° and half of it is 90°

Participant 2 is engaged to give a general argument and an explanation of why the claim holds in every case. He uses a special case to find a general argument. Since he is particularly making use of algebraic variables, we assign his personal meaning of proof with the formal-deductive proof scheme.

Participant 3's solution for the first task can be considered as a sketch for a generic proof; his answer to the second question seems to be a description of the given facts, not containing any argument (see Figure 3.)

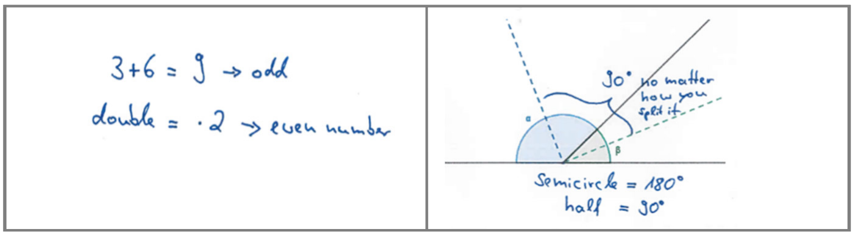

Fig. 3: Participant 3’s proof productions (left: task 1; right: task 2; our translation)

In the interview, Participant 3 explains her use of the concrete example in the first task:

Participant 3: You have … to prove something … you need examples. You have to prove it and that’s the evidence.[1] […] Yes, I succeeded, the presentation … to prove it. But to prove it logically, that was harder for me.

Interviewer: OK. Accordingly, how to prove it logically?

Participant 3: Yes […]. Not just based on one example. But to say that it is always like this. I guess, there’ll be a logical explanation, but I just don’t know it.

Here, the example is used to show that the statement is true in this special case. Interestingly, the student calls this a proof. But she also displays a second notion of proof when talking about „to prove it logically“. In the second case, a proof is „not based on one example“ but concerned with generality.

In the second task, this student determines the 90° angle (see Figure 3). However, she expresses dissatisfaction with her solution and her preference for mathematical symbols in the context of proving:

Interviewer: What did you not do well?

Participant 3: Maybe, I did not explain the connection, so... proving. […] I guess, there will be something like a rule or something like this. […]

Interviewer: You just mentioned that a general rule was important for you. Of what importance is the use of mathematical symbols and variables for proving?

Participant 3: It is important. It is easier to understand, the proof. So, I think, it is always important.

Participant 3 seems to be aware of the general character of mathematical proofs. However, there seems to exist two different kinds of notion of mathematical proof for her. One kind of “proof” is about empirical evidence about the truth of a statement in in some concrete cases. This view is in line with the meaning called “explanatory argumentative scheme” (see above). On the other hand, there is the “logical proof”, as she calls it. This kind of proof is concerned with generality. To perform the corresponding kind of proving, the student is trying to use valid arguments and rules or formulas. In this case, the meaning of proof seems to be in line with a deductive proof scheme.

[1] In original language (german): „Man muss das ja belegen, das ist sozusagen der Beleg.“

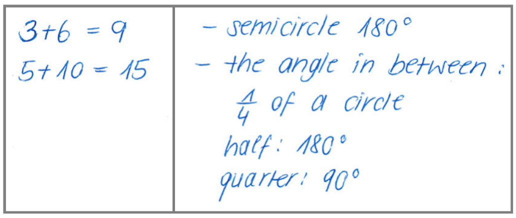

Fig. 4: Participant 4's proof productions (left: task 1; right: task 2; our translation)

Participant 4 checks two concrete examples when working on the first task. Like Participant 3, his solution of the second task is a description of the given facts without any argument (see Figure 4). He comments his approach in the first task as follows:

Participant 4: Here [in the claim], it says "always". That's what bothers me with my solution because I do not cover this, because I have only a limited number of examples. I have two, but there are infinite. In this manner, I cannot prove [it for] all. I would need a paper, where I could write down all of them. […]

Interviewer: To sum up: Would you consider your solution being a proof?

Participant 4: Yes, because I showed it with two examples. That it is…yes… as long as there is no counterexample, then I would say: "It is like this".

Concerning the task 2, the student mentions some dissatisfaction with his solution:

Participant 4: I think, it is weaker than the other […], because here again, it is written "always". And I have shown that the angle in the diagram has 90°. And when you …, yes, I did not do so well, that the angle is always about 90° […]

Participant 4 is aware of the limitation of his approach for task 1. Like Participant 3, this student seems to distinguish two different meanings of "proof". The first is concerned with illustrating the truth of a given claim with some concrete examples. The latter is in line the generality of mathematical statements. In this case, the testing of a finite number of examples cannot form a mathematical proof.

To sum up, all participants seemed to be aware of some kind of generality when dealing with the given mathematical claims. In this sense, a deductive proof scheme could be assigned to all students, when mentioning that the mere use of concrete examples without further argument cannot prove a given claim in general. However, two students also held a second view on proof simultaneously, being somehow in line with the explanatory argumentative scheme or the empirical-inductive proof scheme.

FINAL REMARKS

As one output from this pilot study, we want to highlight that the combination of proof productions and the following interview phase to reflect on one's solutions seems to be valuable to investigate students' personal meaning of mathematical proof. While some students only tested some concrete examples when working on the task, they clearly mentioned the limitation of their approach in the interview. Interestingly, two participants in this study displayed several different personal meanings of the notion of mathematical proof. Therefore, a one-to-one assignment of student and personal meaning does not seem to be possible nor desirable, because it would lead to an oversimplification of the issue. This result might give a hint, why some students call an empirical argument a “proof”. These students might use the word “proof” in a special way, being nevertheless aware of the limitation of an empirical-inductive approach. This first result has to be checked in the upcoming main study.

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Récurrence et récursivité : analyses de preuves de chercheurs dans une perspective didactique à l'interface mathématiques-informatique[1]

Nicolás León[1], Simon Modeste[1], et Viviane Durand-Guerrier[1]

[1]IMAG, Université de Montpellier, CNRS, Montpellier, France,
nicolas.leon@umontpellier.fr

Nous présentons l'analyse de preuves de chercheurs sur l'équivalence de deux définitions du concept d'arbre en théorie des graphes, l'une des deux définitions étant de type récursif et l'autre non. L'analyse vise à mettre en lumière la relation entre les notions de récurrence et de récursivité, telle qu'elle est perçue par les experts, afin d'éclairer les questions didactiques que soulève leur apprentissage.

Keywords : Epistemological studies of mathematical topics, Teaching and learning of logic, reasoning and proof, Teaching and learning of number theory and discrete mathematics.

INTRODUCTION

Contexte et problématique

Cette communication s'inscrit dans le cadre du projet ANR DEMaIn, qui explore les interactions entre mathématiques et informatique d'un point de vue épistémologique à visée didactique, et s'intéresse aux relations entre le raisonnement par récurrence et la récursivité dans la perspective d'une ingénierie didactique (Artigue, 2014). Les travaux existants montrent que la récurrence et la récursivité s'avèrent problématiques, de nombreux étudiants ayant du mal à comprendre les preuves par récurrence et les structures et algorithmes récursifs (Michaelson, 2008 ; Rinderknecht, 2014). Certains travaux semblent indiquer que la relation entre ces deux notions pourrait être l'une des clés pour aborder les difficultés qu'elles suscitent (voir par exemple Leron et Zazkis, 1986 ; Polycarpou, 2006).

Nous présentons dans ce qui suit une étude exploratoire de cette relation en analysant des preuves produites à notre demande par des experts ayant recours à la récurrence et à la récursivité dans leur travail. À travers cette analyse de preuves, nous cherchons à identifier des invariants opératoires pouvant attester des conceptions (au sens de Vergnaud, 2009) relativement à ces notions. Nous avons retenu pour cette étude la preuve de l'équivalence de deux définitions en théorie des graphes, l'une donnée sous forme classique par une propriété et l'autre sous forme récursive. Après une explicitation de notre point de vue sur l'analyse de preuves d'experts, nous présentons et analysons l'énoncé que nous avons soumis aux chercheurs et présentons ensuite les résultats des analyses de preuves menées.

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Point de vue sur les preuves d'experts dans une perspective didactique

L'analyse de preuves mathématiques fait partie de l'activité du mathématicien ou de l'informaticien, mais également des professeurs, des élèves et des étudiants. Suivant Durand-Guerrier & Arsac (2009) nous considérons que l'analyse logique de preuves remplit trois fonctions principales : contrôler la validité ; comprendre la stratégie de l'auteur ; comprendre et s'approprier les preuves d'une part en tant que constitutif de l'étude des contenus en jeu et d'autre part comme moyen de comprendre ce que sont les preuves mathématiques et leurs spécificités éventuelles dans un domaine donné. En outre, l'analyse logique de preuve est un moyen de repérer les implicites inhérents à la rédaction de preuves dont nous faisons l'hypothèse qu'ils peuvent éclairer les pratiques expertes. En accord avec Weber & Alcock (2004) et Wilkerson-Werde & Vilensky (2011) nous faisons l'hypothèse que l'étude du travail d'experts avec les preuves est de nature à éclairer nos questions didactiques.

Pour conduire les analyses de preuves, nous prenons comme logique de référence le calcul des prédicats du premier ordre avec un point de vue sémantique (Durand-Guerrier, Meyer & Modeste, 2019). Par ailleurs, Durand-Guerrier & Arsac (2009) identifient un certain nombre de questions à se poser lors de l'analyse de preuves. Nous listons ici celles que nous avons retenues pour nos analyses : Quels objets sont introduits tout au long de la preuve et quel est leur rôle ? À quelles évidences est-il fait appel et comment être sûr de pouvoir contrôler la validité de la preuve ?

Qu'entendons-nous par récurrence et récursivité ?

Dans cet article, le mot « récurrence » renvoie au sens habituel du terme, c'est-à-dire au type de raisonnement inductif portant sur les entiers naturels. Nous nous intéressons également au raisonnement dit « par induction structurelle », qui généralise en quelque sorte le raisonnement par récurrence.

La « récursivité » s'applique dans de nombreux contextes et avec des significations qui peuvent différer légèrement (par exemple, on associe la récursivité à des définitions, des algorithmes, des types de données, des suites, etc.). Pour penser aux articulations entre récurrence et récursivité (León, 2019), nous nous intéressons au cadre de la construction récursive (ou inductive) de structures, consistant à prendre un certain ensemble d'éléments de base et à en construire un autre par l'application réitérée de certaines fonctions, appelées « constructeurs » – c'est le cas, par exemple, des structures des listes et des arbres, en informatique, ou de celle des formules bien formées, en logique des propositions. Les raisonnements par induction structurelle permettent de prouver les propriétés des éléments de ces structures récursives, en montrant que ces propriétés tiennent pour les éléments de base, et que les fonctions constructeurs les préservent.

ÉNONCE SELECTIONNE ET ANALYSE MATHEMATIQUE *A PRIORI*

L'énoncé choisi est le suivant :

> Démontrez l'équivalence des deux définitions ci-dessous :
>
> Définition 1 : « Un graphe *G* est un arbre *si et seulement si* entre deux sommets quelconques de *G*, il existe un unique chemin ».
>
> Définition 2 : « Un graphe *G* est un arbre *si et seulement si G* est soit un sommet isolé, soit un arbre auquel on ajoute un sommet pendant* ».
>
> * Un sommet pendant d'un graphe est un sommet qui n'est adjacent qu'à un seul autre sommet.

Il s'agit de démontrer l'équivalence de deux définitions, l'une récursive (définition 2) l'autre par une propriété (définition 1). Nous faisons l'hypothèse que la rédaction de la preuve conduit à expliciter des connexions entre récurrence et récursivité.

Notre choix s'est porté sur la théorie des graphes car c'est un champ à l'interface des mathématiques et de l'informatique, qui peut être enseigné au niveau universitaire, et dans lequel la recherche est encore très active. Dans ce domaine, les différentes définitions ou caractérisations des objets sont souvent nécessaires pour pouvoir ensuite choisir le point de vue le plus adéquat, en fonction des besoins (preuves, définition, manipulation, traitement informatique, etc.). L'arbre est un objet récursif par excellence, suffisamment élémentaire pour produire des preuves accessibles, mais dont la construction récursive est suffisamment complexe pour être représentative d'une grande partie des situations récursives. Il existe de nombreuses définitions équivalentes de l'objet arbre, Ouvrier-Buffet (2015, p. 344) en identifie au moins 10, dont les définitions 1 et 2 ci-dessus, que nous avons retenues parce que la preuve de cette équivalence n'est pas classique dans l'approche courante sur la caractérisation des arbres. De plus, le fait que la récursivité ne soit apparente que dans l'une des deux définitions laisse ouverte la possibilité de voir apparaître des raisonnements par récurrence ou par induction structurelle.

Ouvrier-Buffet liste également la définition « *G* est un arbre *si et seulement si G* est un graphe connexe sans cycle », qui apparaît dans les démonstrations des chercheurs que nous avons analysées (voir la stratégie S2 ci-dessous).

Preuve de l'équivalence des définitions

Nous proposons d'abord une preuve complète de l'énoncé sélectionné.

Posons A_1 l'ensemble des graphes vérifiant la définition 1 et A_2 l'ensemble des graphes vérifiant la définition 2. On veut montrer $A_1 = A_2$.

Montrons $A_2 \subset A_1$:

Montrons cela par induction structurelle, c'est-à-dire, montrons que les éléments de base de A_2 sont dans A_1 et que l'opération de construction de A_2 préserve le fait de vérifier la définition 1.

- Soit $a \in A_2$, un sommet isolé. Alors $a \in A_2$.

- Montrons que l'ajout d'un sommet pendant à un graphe de A_1 le maintient dans A_1.

  Soit $a' \in A_1$ et a obtenu par l'ajout d'un sommet pendant à a'.

  Dans a', pour tous sommets s et s', il existe un unique chemin de s à s'.

  Soient s_1 et s_2 deux sommets de a, montrons l'existence d'un unique chemin entre eux dans a.

  Existence :

  - Si s_1 et s_2 sont des sommets de a' : c'est garanti par $a' \in A_1$.
  - Si s_1 est le sommet pendant ajouté à a', notons s' son voisin dans a.
    Comme s' et s_2 sont dans A_1, il existe un chemin $(s',\ldots,s_2)$ dans a'.
    On construit alors un chemin $(s_1,s',\ldots,s_2)$ dans a entre s_1 et s_2.
  - Si s_2 est le sommet pendant ajouté à a' : idem.

  Unicité : Soit $C=(s_1,\ldots,s_2)$ chemin de s_1 vers s_2 dans a. Montrons son unicité.

  Soit s' le sommet pendant ajouté et s son voisin.

  Comme $\deg_a(s') = 1$, si $s' \in C$, alors $s' = s_1$ (ou de façon similaire $s' = s_2$).

  - Si $s' = s_1$, alors $C = (s_1,s,\ldots, s_2)$ et le chemin entre s et s_2 est unique car on n'a pas ajouté d'autres arêtes dans a' que (s_1,s). Et alors, le chemin C est unique dans a.
  - Sinon, il existe un unique chemin entre s_1 et s_2 dans a', car on n'a pas ajouté d'autres arêtes dans a' que (s_1,s).

On en déduit que $A_2 \subset A_1$.

Montrons $A_1 \subset A_2$:

Soit $a \in A_1$, montrons que l'on peut construire a à partir d'un sommet isolé en ajoutant des sommets pendants.

- Si a a un seul sommet : alors $a \in A_2$.
- Sinon, il existe s un sommet pendant (donc de degré 1). (Sinon, tous les sommets seraient de degré supérieur ou égal à 2 et il existerait un cycle dans a, donc il n'y aurait pas unicité des chemins entres deux sommets).

  On retire le sommet s à a (et l'arête adjacente) pour obtenir un graphe a'.

  a' est un élément de A_1, puisque tout chemin entre deux sommets de a (différents de s) est préservé et reste unique.

  En réitérant le processus, et puisque le nombre de sommets diminue strictement, on obtient une déconstruction de a, jusqu'à un sommet isolé, par élimination de sommets pendants. En inversant cet algorithme, on peut construire a à partir d'un sommet isolé par l'ajout de sommets pendants.

Donc $a \in A_2$. $\square$

Variante de $A_1 \subset A_2$ par « élément minimal ».

Supposons qu'il existe des éléments de A_1 qui ne soient pas dans A_2, et soit a un plus petit élément de A_1 (au sens du nombre de sommets) qui n'est pas dans A_2.

a a au moins 2 sommets. De plus, dans a il existe s un sommet pendant (donc de degré 1) : sinon, tous les sommets seraient de degré supérieur à 1 et il existerait un cycle dans a, donc il n'y aurait pas unicité des chemins entres deux sommets.

On retire le sommet *s* à *a* (et son arête adjacente) pour obtenir un graphe *a'*.
a' est un élément de A_1, puisque tout chemin entre deux sommets de *a* (différents de *s*) est préservé et reste unique.
Et $a' \in A_2$ par minimalité de *a*. Et donc, par construction, $a \in A_2$. Contradiction.
Donc $A_1 \setminus A_2 = \emptyset$, autrement dit $A_1 \subset A_2$.

Cette preuve nous servira de référence dans l'analyse des preuves de chercheurs, en particulier pour identifier des implicites.

Preuve générique de l'équivalence de deux définitions

Comme d'autres preuves sont envisageables, et afin de permettre une analyse plus complète, nous présentons une étude de la preuve générique de l'équivalence entre une définition par propriété et une définition récursive. De façon générale, on a deux définitions l'une sous forme d'une propriété, l'autre récursive :

Soit E un ensemble.
Def 1 : Un $x \in E$ est de type 1 si et seulement si $P(x)$.
Def 2 : Un $x \in E$ est de type 2 si et seulement si $x \in \{x_0,\ldots,x_k\}$ ou x est obtenu par application d'un opérateur $f_1,\ldots, f_n$ à un élément de type 2.

L'analyse mathématique et logique *a priori* des preuves possibles de l'équivalence de deux telles définitions nous permettra d'identifier certains types de stratégies et de les repérer dans les preuves des chercheurs.

Pour prouver que Def 1 équivaut à Def 2, il faut montrer que *x* est de type 1 si et seulement si *x* est de type 2.

type 2 ⇒ type 1 :

Il faut prouver que les x_i sont de type 1 (i.e. vérifient P) et que les opérateurs f_j préservent le fait d'être de type 1 (si *a* est de type 1 alors $f_j(a)$ est de type 1).
C'est une induction structurelle, que l'on pourrait aussi transformer en récurrence classique sur $\mathbb{N}$.

type 1 ⇒ type 2 :

Il y a plusieurs alternatives pour parvenir à cette preuve. Par exemple :

1. On peut montrer que si un élément *a* satisfait $P(x)$, alors
   i. Soit il est égal à un x_i, et donc il est de type 2.
   ii. Soit il existe un constructeur f_j tel que $a = f_j(b)$, avec et $b \in \{x_0,\ldots,x_k\}$ ou $P(b)$. Par un argument de finitude de cette « déconstruction » on s'assure qu'il existe un x_i à partir duquel on peut construire *a* par application réitérée d'opérateurs de type f_j. Ceci implique que *a* est de type 2.
2. On peut raisonner par l'absurde et par élément minimal : on suppose qu'il existe un élément *a* de type 1 qui n'est pas de type 2. Sans perte de généralité, on suppose qu'il est minimal. Ensuite, on montre qu'on arrive à le construire à partir d'un élément « plus petit » *b* par application de l'un des constructeurs f_j.

> Par l'hypothèse de minimalité de a, b est de type 2. On en déduit que a est aussi de type 2, ce qui est contradictoire avec l'hypothèse de départ.

Quelle que soit la méthode choisie, il faut noter l'importance d'avoir un ordre bien fondé, plus ou moins explicite, assurant que la « déconstruction » se « termine ».

Sur la base de ces analyses, nous avons étudié les preuves des chercheurs. Nous rendons compte d'une partie de nos analyses dans la section ci-dessous.

ANALYSE DES DEMONSTRATIONS DES CHERCHEURS

Méthodologie et données recueillies

Dans le cadre plus général de notre étude, nous avons réalisé des entretiens avec des chercheurs en mathématiques et en informatique. Les chercheurs ont été invités, à la fin de l'entretien, à démontrer l'énoncé que nous avons présenté et à nous envoyer leurs preuves plus tard. Nous leur avons indiqué seulement qu'elles seraient analysées pour mieux cerner les pratiques des experts : en particulier, nous n'avons pas spécifié qui pourrait être le lecteur hypothétique de la preuve. Les chercheurs n'ont eu aucune contrainte de temps pour fournir leur réponse. Parmi les 10 interviewés, nous avons reçu des preuves de trois chercheurs (C1, C2, C3). En particulier, C1 nous a envoyé non seulement sa démonstration finale, mais aussi trois brouillons que nous avons également étudiés. C1 est un informaticien ayant fait des recherches en algorithmique et complexité, C2 est un mathématicien spécialisé en mathématiques discrètes et C3 est un mathématicien qui développe des recherches en informatique théorique.

L'analyse détaillée des preuves des experts, qui ne sera pas présentée ici, a été menée à la lumière de l'analyse mathématique *a priori*. Cela nous a permis d'identifier et d'expliciter leurs stratégies de preuve. Nous voyons ces stratégies comme de potentiels invariants opératoires, pouvant rendre compte des conceptions des chercheurs autour de la récurrence et de la récursivité. En outre, nous avons étudié les implicites dans les preuves conformément à notre positionnement didactique.

Stratégies des chercheurs

Pour la suite, nous introduisons les notations suivantes :

- $UC(G)$:= entre deux sommets quelconques de G, il existe un unique chemin.
- $R(G)$:= soit G est un sommet isolé, soit G contient un sommet pendant x tel que $R(G \setminus \{x\})$.
- $AC(G)$:= G est acyclique et connexe.

S1 : *Chercher une injection vers* $\mathbb{N}$ *ou vers* $\mathbb{N}^m$.

Il s'agit de la recherche d'une injection de la structure de départ (ici, l'ensemble des graphes non orientés) vers $\mathbb{N}$, ou plus généralement vers $\mathbb{N}^m$, permettant d'effectuer un raisonnement par récurrence. Pour cela on peut considérer la fonction qui associe à un graphe G son ordre $|G|$, comme le font C1 et C2. Notre analyse mathématique *a* *priori* montre l'utilité de cette injection pour prouver que $UC(G)$ implique $R(G)$ dans la variante « par élément minimal ».

Nous avons observé un autre choix dans le premier brouillon de C1, où il cherche une injection vers $\mathbb{N}^2$, associant à un graphe G le couple (v,l) constitué du nombre de chemins de taille maximale dans G et de la longueur de chacun de ces chemins. L'injection peut également permettre d'effectuer un travail par disjonction de cas, comme le fait C2 quand il considère d'abord les graphes d'ordre 1 et ensuite ceux d'ordre supérieur. C1 utilise lui aussi une injection vers $\mathbb{N}$, mais qui envoie chaque sommet vers son degré, pour réaliser une disjonction de cas.

S2 : *Passage vers une définition alternative équivalente*

Les chercheurs se servent de définitions équivalentes du même objet pour prouver ses propriétés plus aisément. Dans notre cas, le couple de propriétés « acyclique et connexe » joue ce rôle de médiateur qui facilite le passage d'une définition à l'autre. C1 rend explicite l'équivalence $UC(G) \Leftrightarrow AC(G)$, dont il se sert à plusieurs reprises, dans un sens et dans l'autre. Pour C3 il est suffisant d'établir que $UC(G) \vee R(G)$ implique que G est connexe, et puis de faire appel implicitement à l'implication $UC(G) \Rightarrow AC(G)$ et explicitement à l'implication $R(G) \Rightarrow AC(G)$.

S3 : *Déconstruire/reconstruire le cas générique*

La déconstruction du graphe générique G peut se faire à plusieurs niveaux, selon le but recherché. Puisque la définition récursive fait référence à la présence d'un sommet pendant, il semble naturel de déconstruire le graphe en enlevant ce sommet pendant, pour explorer les relations entre le sous-graphe ainsi obtenu et le graphe de départ. Bien entendu, dans le cas où l'on présuppose $UC(G)$, l'existence d'un sommet pendant doit être prouvée au préalable. Autant C1 que C2 recourent à cette déconstruction. Nous trouvons une approche différente chez C3, qui fournit un algorithme de construction d'un graphe G tel que $UC(G)$, à partir d'un sommet quelconque lui appartenant s_0. Cet algorithme ajoute les sommets de G par « couches » selon leur distance par rapport à s_0 et est conçu pour montrer qu'effectivement G peut être obtenu à partir du sommet s_0 par des ajouts successifs de sommets pendants. Cette façon de procéder est proche de celle que nous avons conçue lors de l'analyse mathématique *a priori* ($A_1 \subset A_2$) ; la seule différence étant que dans notre algorithme les sommets sont ajoutés un par un.

S4 : *Décrire un algorithme construisant un objet pour prouver son existence*

Pour prouver que si G est tel que $|G| \geq 2$ et $UC(G)$, alors G contient au moins deux sommets pendants distincts, C2 élabore un algorithme permettant de trouver un chemin de taille maximale à partir d'un chemin quelconque dans G. Les deux sommets pendants correspondent alors aux extrémités du chemin maximal. Le contenu d'une itération de l'algorithme est présenté par l'ajout d'un sommet au chemin de départ, sous l'hypothèse que ce chemin n'est pas déjà de taille maximale. Le choix de montrer cet algorithme de construction nous semble intéressant. Il aurait été envisageable d'argumenter qu'un chemin maximal doit exister, les sommets du graphe étant finis, sans avoir à détailler comment on l'obtient – d'ailleurs, C2 fait appel à la finitude du nombre de sommets pour justifier que son algorithme s'arrête.

S5 : *Dérécursiver une définition récursive*

C3 nous met en garde contre les « dangers » des définitions récursives : « Les définitions récursives sont toujours un peu dangereuses dans le sens qu'elles font parfois penser aux fausses preuves qui supposent le résultat pour le démontrer. ». Pour s'en débarrasser, il propose une définition non-récursive qui est censée être équivalente à la définition 2 : « Un graphe G est un arbre *si et seulement si* G peut être obtenu à partir d'un sommet isolé en insérant successivement ses autres sommets comme des sommets pendants. »

Cette nouvelle définition « algorithmique » pose cependant deux questions : celle de valider son équivalence à la définition 2, et celle de décider si la nouvelle définition n'est pas elle-même récursive de façon détournée, cachée par l'expression « successivement ». Quoi qu'il en soit, dès que l'équivalence entre une définition récursive et une définition non récursive a été établie, on peut ensuite faire appel au point de vue le plus adéquat en fonction des besoins.

S6 : *Décrire/évaluer des cas simples d'une construction inductive ou de l'hérédité d'une récurrence*

C3 étudie des cas particuliers de l'hérédité d'une propriété, par exemple $\mathrm{P}(1) \Rightarrow \mathrm{P}(2)$, avant d'aller vers la preuve de $\mathrm{P}(n) \Rightarrow \mathrm{P}(n+1)$. On peut interpréter que cette redondance a des fins purement explicatives, ou supposer que c'est par l'exploration de ces cas élémentaires que C3 parvient à rédiger ensuite la preuve du cas général, voire s'assurer que les cas élémentaires y correspondent bien. Par ailleurs, il applique une procédure analogue pour définir une suite finie de sommets $(s_i)_{i=0,\ldots,|G|}$ et une relation de filiation qu'il utilise pour prouver que $\forall G\ R(G) \Rightarrow UC(G)$.

Identification d'implicites

Le chercheur peut choisir de sauter une étape de la preuve pour plusieurs raisons. Par exemple, il peut considérer que cette partie a peu de valeur explicative ou trouver qu'elle est fastidieuse pour le lecteur, ou pour lui-même. Mais on peut supposer que, dans certaines situations, le chercheur peut sauter une étape d'une manière moins volontaire, parce qu'un certain geste répétitif lui est devenu si familier qu'il n'a plus besoin de franchir les pas intermédiaires pour arriver à sa conclusion. Même s'il est difficile d'identifier les causes des ellipses rencontrées dans les démonstrations, nous pensons qu'elles nous renseignent sur les possibles raccourcis dans le raisonnement des chercheurs. Parmi les implicites, nous pointons ceux qui sont en lien direct avec les raisonnements de type inductif :

Faire l'ellipse du (des) cas de base

On trouve cette sorte d'ellipse dans le travail de C1, avec des omissions plus ou moins drastiques : au premier brouillon, il qualifie le cas de base de « trivial », mais il consacre quand même une ligne à l'expliquer ; au deuxième brouillon, il continue à qualifier le cas de base de « trivial », mais ne donne plus aucune explication ; et la récurrence de la version finale ne mentionne même pas le cas de base. Nous voyons donc, que même chez un seul chercheur la pratique consistant à omettre la preuve du cas de base n'est pas systématique et pourrait dépendre de la difficulté relative qu'il y attribue, ou de sa familiarité avec ce cas, parmi d'autres facteurs.

Faire l'ellipse du raisonnement inductif en entier

Nous voulons dire par là que le chercheur affirme sans démonstration un fait qui *pourrait* être démontré par un raisonnement inductif, et que ce raisonnement est soit mentionné mais non développé, soit complètement éclipsé. Nous trouvons un exemple du premier cas dans la preuve de C1 où ce qui serait formellement énoncé comme un raisonnement par induction structurelle est résumé par la phrase « $R(G) \Rightarrow AC(G)$ trivial, car le procédé de construction maintient invariant les propriétés de connexité et d'acyclicité ». Nous soulignons ici que ce raisonnement inductif, associé à la preuve de $UC(G) \Leftrightarrow AC(G)$, que C1 qualifie également de triviale, constitue l'une des deux implications ($R(G) \Rightarrow UC(G)$) de l'énoncé à prouver. Nous pensons que C1 est prêt à résumer cette procédure inductive aussi brièvement car il est très habitué à ce qu'il suffise de trouver un invariant du (des) constructeur(s) pour en déduire l'implication universellement quantifiée associée.

Nous trouvons aussi des ellipses complètes, où un fait mathématique pouvant être prouvé par récurrence est affirmé, sans aucune allusion à la preuve. Par exemple, dans un brouillon, C1 affirme sans démonstration qu'il ne peut y avoir de cycle dans les arbres selon la définition 2. On peut prouver ce fait par induction structurelle, en montrant que le graphe à un seul sommet est acyclique, et que si l'on ajoute un sommet pendant à un graphe acyclique, il reste acyclique. L'acyclicité des arbres est une propriété classique présente dans plusieurs définitions et il peut être difficile de travailler sur les arbres sans utiliser ces propriétés bien connues.

Nous voyons un autre exemple dans l'argument que C3 donne pour affirmer la connexité de chaque graphe vérifiant sa version de la définition 2 : « si un graphe s'obtient à partir d'un sommet isolé en y insérant des sommets pendants, c'est que le sommet isolé initial est relié à tous les autres sommets pendants insérés par la suite ». Ici, l'existence d'un chemin allant du dernier sommet ajouté vers le sommet initial pourrait faire l'objet d'une preuve par récurrence.

CONCLUSIONS ET PERSPECTIVES

Nous avons analysé des pratiques d'experts en lien avec la récurrence et la récursivité, afin d'envisager leurs conceptions sur ces notions. Notre analyse mathématique *a priori* s'est révélée utile pour montrer le rôle que peuvent jouer les notions ciblées dans l'activité de preuve en étude, ainsi que pour anticiper certaines stratégies que les chercheurs ont adoptées par la suite. Nous faisons l'hypothèse que les stratégies ayant émergé lors des analyses sont des invariants opératoires. De plus, nous avons relevé certains implicites qui pourraient également contribuer à éclairer la conceptualisation. Nous nous appuierons sur els éléments dégagés dans cette étude lors de la conception de séquences didactiques visant à faire travailler la récurrence et la récursivité et leurs interactions.

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Utilisation de l'articulation entre les points de vue syntaxique et sémantique dans l'analyse d'un cours sur le raisonnement

Virginie Deloustal-Jorrand[1], Michèle Gandit[2], Zoé Mesnil[3], Mickael Da Ronch[4]

[1]Université de Lyon, Université Claude Bernard Lyon 1, EA4148 S2HEP; [2]Maths à modeler, Université Grenoble Alpes; [3]LDAR, Université de Paris, zoe.mesnil@univ-paris-diderot.fr, [4]Institut Fourier, Université Grenoble Alpes.

This paper highlights the relevance of the articulation between syntax and semantics in proof and proving activities. In the first part, we define what we call syntax and semantics for a proof. In the second part, we present a logical and didactical analysis of a university course entitled "Mathematical Reasoning". This analysis relies on three types of data: interviews with teachers, worksheets and an assessment test.

Keywords: Teaching and learning of logic, reasoning and proof ; Teaching and learning of specific topics in university mathematics ; syntax ; semantics.

INTRODUCTION

À leur arrivée à l'université, les étudiants sont confrontés à la nécessité d'étudier et d'élaborer par eux-mêmes des raisonnements et des preuves de plus en plus complexes, ce qu'ils ont peu eu l'occasion de faire dans leurs études secondaires, y compris dans les sections scientifiques. De nombreuses recherches, tant au niveau national qu'international, mettent en évidence qu'ils ne peuvent pas s'appuyer sur une bonne maîtrise des connaissances et des compétences logiques nécessaires pour affronter la formalisation et la complexification de la structure logique des énoncés mathématiques. Dans certains cursus de l'enseignement supérieur des dispositifs de formation au raisonnement et à la preuve sont mis en place. Cependant, nous prenons pour hypothèse qu'il n'y a pas de savoir de référence pour de tels enseignements qui ferait consensus dans la communauté, le rôle de la logique mathématique dans l'apprentissage du raisonnement et de la preuve étant notamment l'un des éléments sur lequel les positions divergent (Durand-Guerrier, Boero, Douek, Epp et Tanguay, 2012). C'est pourquoi, au sein du groupe de travail « Logique, langage, raisonnement, preuves » du Groupe de Recherche DEMIPS (Didactique et Épistémologie des Mathématiques, interactions Informatique Physique, dans le Supérieur), nous cherchons à étudier les différentes épistémologies relatives à la preuve des professeurs proposant ces cours à l'université. Celles-ci peuvent transparaître de manière plus ou moins explicite dans leurs choix didactiques ou dans leur discours sur ces cours. Nous cherchons aussi à mettre en lien ces choix avec les difficultés et apprentissages effectifs des étudiants.

Pour caractériser ces différentes épistémologies de la preuve, et sans aller pour l'instant jusqu'à l'élaboration complète d'une telle modélisation, nous souhaitons construire une grille d'analyse autour de différents aspects de la preuve, en prenant comme première entrée, présentée ici, l'articulation entre les aspects syntaxique et sémantique. Nous faisons l'hypothèse que chacune de ces dimensions sera plus ou moins mise en avant dans un cours sur la preuve, et nous nous servons alors de cette grille pour analyser des documents de cours, des entretiens avec des enseignants, des productions d'élèves relatifs à un cours donné. Nous présentons d'abord plus précisément ce que nous entendons par aspects sémantiques et syntaxiques puis nous présentons quelques résultats de cette première étude de cas.

ASPECTS SYNTAXIQUE ET SÉMANTIQUE DE LA PREUVE

Le point de vue de la logique

L'objet de la logique est l'étude de la validité des raisonnements, c'est-à-dire qu'elle vise à fournir des outils pour établir quels sont les raisonnements corrects et pour débusquer ceux qui ne le sont pas. Pour cela, elle liste des *schémas de raisonnement*, ou des *règles de déduction* et modélise alors les raisonnements comme étant des mises en relation entre des propositions conformes à ces schémas ou à ces règles. La validité du raisonnement peut donc être attestée par sa structure et celle des propositions qui y interviennent, c'est-à-dire par des arguments d'ordre syntaxique.

Mais bien sûr, il est attendu de ces schémas ou de ces règles valides qu'elles préservent la vérité (on ne reviendra pas ici sur la distinction vérité/validité, voir Durand-Guerrier, 2008) : une proposition obtenue comme conclusion d'un raisonnement valide dont les prémisses sont vraies *doit nécessairement* être vraie. Ce critère permet de justifier quels sont les raisonnements valides, et d'établir la non-validité d'un raisonnement par des arguments d'ordre sémantique.

Cette articulation entre syntaxe et sémantique est au cœur de la démarche des logiciens, depuis les travaux « fondateurs » d'Aristote dans l'Antiquité Grecque jusqu'à la récente naissance de la logique mathématique contemporaine (Blanché, 1970). Dans la théorie du syllogisme d'Aristote (voir par exemple Aristote, 2007) le point de vue sémantique est bien présent (par exemple dans la justification des syllogismes de la première figure), mais finalement, les syllogismes sont décrits d'un point de vue syntaxique, même si la variété des formulations utilisées par Aristote montre qu'il « ne pousse pas le souci formel jusqu'au formalisme » (Blanché, 1970, p.48). En revanche, bien plus tard, Frege a ce souci du formalisme en construisant son *Idéographie*, qui est un système de signes permettant d'exprimer les déductions dans un langage entièrement formalisé, seule façon de garantir leur validité (Frege, 1999).

Nous retiendrons, pour notre propos, la caractérisation de la syntaxe et de la sémantique de Duparc (2015) :

> La syntaxe est le monde des symboles, de ces « coquilles vides » que manipulent les ordinateurs. C'est le lieu des opérations grammaticales indemnes de tout contenu, dépourvues de sens.
>
> La sémantique c'est au contraire le lieu des interprétations et des réalisations, des modèles ou des mondes possibles là où toute la syntaxe « prend corps ». C'est le lieu de la signification. (p. 12)

Duparc précise que « cette dichotomie ne renvoie pas à deux visions de la logique », mais plutôt à « deux faces d'une même feuille de papier » (p.13). Dans les travaux évoqués d'Aristote et de Frege, nous pouvons observer un phénomène de sémantisation de la syntaxe, et la relation syntaxe/sémantique, au départ intuitive dans les travaux des logiciens, est finalement formellement décrite depuis la définition sémantique de la vérité (au sens de la satisfaisabilité dans un modèle) de Tarski en 1933. Il est ainsi possible de vérifier pour un système syntaxique de déduction qu'il préserve la vérité, cela revient à montrer que si une proposition est prouvable (au sens syntaxique) dans une théorie, alors cette proposition est vraie dans tout modèle de cette théorie. Et un pas décisif a été fait par Gödel en 1930 qui, avec le théorème de complétude de la logique du premier ordre, montre la réciproque, à savoir que si une proposition F est vraie dans tout modèle d'une théorie T, alors F est prouvable à partir de T.

Articulation syntaxe/sémantique dans les recherches sur la preuve

Pour insister sur le fait que, dans la métaphore de Duparc, il s'agit d'une même feuille de papier, plutôt que d'insister sur le fait qu'il y a deux faces, nous préférons utiliser le terme *articulation.* En effet, pour qui s'intéresse à l'enseignement et l'apprentissage, l'important n'est pas tant une différence formelle que l'on peut faire entre syntaxe et sémantique, mais bien de voir comment ces deux aspects, leur articulation et éventuellement leur distinction, contribuent à la compréhension. On retrouve d'ailleurs ce point de vue dans de nombreuses recherches sur la preuve dans l'enseignement supérieur. Par exemple, dans une des premières recherches sur ce thème, Moore (1994) distingue trois usages des définitions et des théorèmes pertinents pour produire des preuves : nous situons le premier « produire et utiliser des exemples » du côté sémantique, alors que nous situons le troisième « utiliser les définitions pour structurer la preuve » du côté syntaxique. Et plus récemment, dans une liste de difficultés des étudiants identifiées par la recherche établie par Selden (2012), nous retrouvons également des difficultés en lien avec l'aspect syntaxique « *the proper use of logic, the neccessity to employ formal definitions* », alors que d'autres sont en lien avec l'aspect sémantique « *the need for a repertoire of examples, counterexamples, and nonexamples, the need for strategic knowledge of important theorems* ».

Nous reprenons la caractérisation de Weber et Alcock (2004) qui distinguent une approche syntaxique et une approche sémantique dans la production des preuves :

> We define a *syntactic proof production* as one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. […] We define a *semantic proof production* to be a proof of a statement in which the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws. (p. 210)

La séparation entre ces deux aspects n'est cependant pas toujours si claire : dans la phase de recherche d'une preuve (*proving*), certains moments peuvent relever d'un traitement uniquement syntaxique, par exemple lors de manipulations d'énoncés, et d'autres d'un traitement sémantique (par exemple, considérer l'instanciation d'un théorème universel par des éléments singuliers, et essayer de comprendre ce qui se passe pour ces éléments-là). De même, en ce qui concerne la preuve dans son aspect produit (*proof*), les textes de preuve ne sont pas des produits formels élaborés uniquement selon un traitement syntaxique mais montrent au contraire des choix de l'auteur sur les arguments et leur formulation (Hache & Mesnil, 2015). Gandit (2008) a d'ailleurs montré les effets néfastes pour l'apprentissage d'une approche se focalisant sur une mise en forme codifiée. En outre, Deloustal-Jorrand (2004) a montré, que la connaissance des tables de vérité de l'implication ne suffisait pas à la bonne appréhension et utilisation de celle-ci dans un raisonnement. Cependant, les liens entre structure logique d'un énoncé et structure logique de sa preuve peuvent servir de guide, et Selden et Selden (1995) ont montré que les étudiants étaient en difficulté pour exhiber la structure formelle d'un énoncé, et conséquemment, pour produire ou contrôler la structure d'une preuve de cet énoncé.

ÉTUDE DE CAS : LE COURS « *RAISONNEMENT MATHÉMATIQUE* »

À la suite de ces recherches, nous faisons donc l'hypothèse que chacune de ces dimensions syntaxique ou sémantique sera plus ou moins mise en avant dans un cours sur la preuve, selon l'épistémologie relative à la preuve propre à l'enseignant. Nous présentons ici une première étude de cas menée sur un cours intitulé « Raisonnement Mathématique », proposé en première année de licence de mathématiques à l'Université Paris Diderot en 2017. Nous avons donc cherché à repérer ces deux entrées syntaxique et sémantique sur différentes données issues de ce cours. En effet, nous avons, d'une part, conduit et analysé des entretiens avec les deux responsables du cours et, d'autre part, nous avons analysé le polycopié de référence du cours, les feuilles d'exercices, et le sujet d'examen. Pour voir si les choix des enseignants pouvaient être mis en lien avec les difficultés des étudiants, nous avons complété ces analyses du cours par des analyses des réponses des étudiants à l'examen.

Étude de la mise en œuvre du cours : polycopiés et entretiens de professeurs

Lors des entretiens, à la question « selon toi est-ce que c'est important qu'il y ait un enseignement spécifique sur le raisonnement mathématique ? », l'une des professeurs évoque spontanément des éléments de logique mathématique (quantificateurs, connecteurs, variables...) et donne à voir le choix d'une entrée syntaxique dans son enseignement : il s'agit de « décortiquer des énoncés », d'apprendre « comment on s'exprime en mathématiques, comment lire un énoncé, le comprendre », même si elle mentionne la nécessaire articulation avec la sémantique. D'autre part, elle choisit de faire travailler des démonstrations pour lesquelles « il n'y a pas besoin d'avoir d'idées ». On retrouve aussi chez l'autre professeur la volonté de donner aux étudiants des automatismes de rédaction de preuves selon la structure de l'énoncé : « on a tel type d'énoncé, donc la forme A implique B par exemple, et donc pour montrer ça qu'est-ce qu'on fait en général, on suppose A on montre B ».

L’analyse du polycopié donne également à voir cette entrée syntaxique dans l’enseignement de la preuve. Par exemple, dans une des sections intitulée « Stratégies de preuve en fonction de la forme de l’énoncé à prouver », où « forme » renvoie bien à l’aspect syntaxique, l'auteur s’efforce de fournir aux étudiants une méthode permettant de générer une preuve et cela quelle que soit la forme des énoncés considérés. Cette méthode vise à analyser, puis à modifier les énoncés en fonction de leur structure, suivant des règles issues des tables de vérité.

Regardons enfin l’analyse des deux premiers exercices de la feuille d’exercices n°1 :

Exercice 1. Précisez les variables libres et les variables liées qui interviennent dans les noms suivants:

$(a)\ 3q+r$	$(b)\ f(x_1+x_2+x_3)$	$(c)\ \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto ax \end{cases}$
$(d)\ \sum_{i=1}^{N}(i+j)^2$	$(e)\ \sum_{i=1}^{N}\sum_{j=i}^{M}(i+j)^2$	$(f)\ \int_0^x (3t+1)^2\,dt$
$(g)\ \{x \in \mathbb{Z} : x \text{ est un nombre pair}\}$	$(h)\ \#\{i \in \mathbb{Z} : a \leq 2i < b\}$	

Exercice 2. Décrire avec des symboles mathématiques les objets suivants, et précisez les variables libres:

1. L’ensemble des entiers relatifs multiples de m.
2. L’ensemble des solutions réelles de l’équation $ax^2+bx+c=0$.
3. L’ensemble des solutions complexes de l’équation $ax^2+bx+c=0$.

Figure 1 : Exercices de la première feuille

Dans l'exercice 1, il est question de *noms*, ce qui renvoie à la distinction noms/énoncés parmi les expressions mathématiques. Le point de vue est syntaxique : on s'intéresse à l'expression, non pas à ce qu'elle désigne. Au contraire, dans l'exercice 2 il est question d'*objets*, le point de vue est ici sémantique. Par ailleurs, dans l'exercice 1, il est demandé de préciser les variables libres et les variables liées. Nous faisons l’hypothèse que la technique attendue est purement syntaxique : par exemple, pour les items (d) et (e), les variables qui sont sous le symbole $\sum$ sont muettes, les autres sont libres, aucune reformulation de ces noms n'est demandée (ces sommes sont difficiles à interpréter pour des étudiants de première année). Au contraire dans l'exercice 2, il est d'abord question de reformulation, ce qui nécessite de comprendre les sens des mots utilisés dans la désignation de l'objet.

L’exercice 1 donne pourtant la possibilité d'articuler les dimensions syntaxiques et sémantiques : bien que le critère d'identification des quantificateurs soit le plus rigoureux pour repérer le statut des variables et soit suffisant, l'articulation avec la reformulation des noms amène un travail mathématique intéressant. Cette articulation permet alors de développer des outils de contrôle des résultats (par exemple, la variable *i* ne peut pas apparaître dans le calcul de la somme (d)). Cette articulation n'est pas demandée explicitement dans la consigne de l'exercice. Nous n'avons pas pu observer de déroulement des séances d’exercices, nous n'avons donc pas d'indication sur la manière dont les enseignants corrigent cet exercice.

Dans l'exercice 2, l'articulation syntaxe/sémantique est plus explicitement attendue puisque qu'après avoir demandé une reformulation, qui est aussi l'occasion d'apprendre à manipuler l'écriture en compréhension d'un ensemble, il est demandé d'indiquer les variables libres. Nous faisons l'hypothèse que ce qui est attendu est bien que les étudiants repèrent la variable rendue muette par l'assemblage $\{\ldots|\ldots\}$, les autres variables apparaissant libres dans le nom de départ étant toujours libres dans l'énoncé caractéristique de l'ensemble, et donc dans le nom reformulé.

Ces trois exemples donnés sont représentatifs de l'ensemble de nos analyses qui montrent une entrée syntaxique fortement affirmée, même si elle n'est pas coupée d'une volonté d'articulation avec l'aspect sémantique.

Étude de l'examen et des copies des étudiants

Nous avons analysé plus particulièrement deux exercices de l'examen (parmi 6) proposé à la fin du cours, qui comportent selon nous des possibilités d'articulation des dimensions syntaxique et sémantique. Nous mettons en regard les réponses des étudiants. Étudions tout d'abord le premier exercice de l'examen :

Exercice 1 (4 points)

Soient P, Q, R trois énoncés. On note A l'énoncé :

$$(P \vee (\neg R)) \Rightarrow Q$$

et B l'énoncé :

$$\neg(P \wedge (\neg Q) \wedge R).$$

(On rappelle que $\vee$ est le connecteur logique OU, $\wedge$ est le connecteur logique ET).

(1) Donner les tables de vérité de A et de B.

(2) Démontrer l'énoncé $A \Rightarrow B$.

(3) Les énoncés A et B sont-ils équivalents ?

Figure 2 : Premier exercice de l'examen

La question (1) de cet exercice est un classique des exercices de calcul propositionnel : il s'agit de dresser la table de vérité de deux propositions. Cette tâche s'apparente à un travail syntaxique : une fois connues les règles pour remplir une table de vérité, il n'est pas nécessaire pour les appliquer de se ramener au sens des connecteurs. Les étudiants ont été entraînés sur ce type d'exercices lors des séances de travaux dirigés. Sur 150 copies analysées, 140 étudiants répondent à cette question, et 68 (soit 45% des réponses) dressent une table de vérité correcte, 62 (soit 41% des réponses) dressent une table de vérité incomplète ou comportant plus de 3 erreurs.

La formulation de la question (2) ouvre plus de possibilités de résolution de la tâche. Une formulation du type « Démontrer que l'énoncé $A \Rightarrow B$ est toujours vrai », voire « démontrer que l'énoncé $A \Rightarrow B$ est une tautologie » aurait été plus dans la continuité de la démarche syntaxique attendue pour la question 1, en restant dans le registre du calcul des propositions. Au contraire, une formulation du type « Démontrer que $A \Rightarrow B$ » se situe plus dans le registre du raisonnement déductif, avec des démarches du type « supposons A, montrons B ». La formulation choisie ici est finalement intermédiaire entre ces deux formulations, et effectivement, les étudiants ont mis en œuvre différentes procédures plus ou moins syntaxiques ou sémantiques. La procédure complètement syntaxique consistant à établir la table de vérité de $A \Rightarrow B$ a été mise en œuvre par 44 étudiants de façon correcte, et par 46 étudiants avec des erreurs (soit en tout 76% des 119 étudiants ayant répondu). Une autre procédure s'appuie également sur les tables de vérité, mais fait plus appel à la sémantique de l'implication : il s'agit de vérifier que lorsque A est vrai, B est vrai. Elle a été mise en œuvre par 9 étudiants. Une troisième procédure se situe dans le registre du raisonnement déductif : elle consiste à supposer A vrai, et à en déduire que B est vrai. Construire ainsi le canevas de la démonstration est une démarche syntaxique, dont nous avons vu qu'elle était prônée par les enseignants et dans le polycopié du cours. Cette procédure a été mise en œuvre par 10 étudiants, mais seulement 2 étudiants le font correctement, en listant les distributions de valeurs de vérité qui rendent A vrai, et en étudiant la valeur de vérité de B dans chaque cas, ce qui ramène l'exercice dans le registre du calcul propositionnel, et qui mobilise la sémantique de l'implication. Les 8 autres étudiants cherchent à transformer l'énoncé A ou l'énoncé B, mais de façon non correcte. Une autre démarche, qui n'a pas été mise en œuvre, consiste à transformer A en la disjonction [(NON P ET R) OU Q] et à faire une disjonction des cas. Une quatrième procédure similaire consiste à démontrer la contraposée (NON $B \Rightarrow$NON A), elle n'a pas non plus été mise en œuvre. Notons que 23 étudiants accompagnent leur réponse à cette question de transformations des propositions, essentiellement en utilisant l'équivalence entre $A \Rightarrow B$ et (NON A OU B).

De même, la formulation de la question (3) ouvre différentes possibilités de résolution. Sur les 126 étudiants ayant traité la question, 53 répondent en comparant les tables de vérité, ce qui en fait la procédure syntaxique majoritairement mise en œuvre. Bien que l'énoncé $A \Leftrightarrow B$ n'apparaisse pas explicitement, 27 autres étudiants continuent de mobiliser une démarche syntaxique, en faisant les tables de vérité de $B \Rightarrow A$ ou de $A \Leftrightarrow B$. Notons tout de même que 43% des étudiants qui avaient traité la première question avec des tables de vérité, notamment ceux qui ne les avaient pas mobilisées correctement, ne les utilisent pas ici pour répondre à la question de l'équivalence des énoncés, ce qui montre que ce registre syntaxique des tables de vérité est bien mobilisé dans une tâche où elles sont explicitement convoquées, mais moins bien dans une tâche où cela est plus implicite.

Finalement, environ 35% des étudiants mobilisent pour les 3 questions une résolution syntaxique par les tables de vérité, malgré les formulations des questions (2) et (3) ouvrant d'autres possibilités. Ce traitement purement syntaxique s'explique par l'inscription de la tâche dans le registre du calcul propositionnel, même si elle n'y reste pas strictement.

Regardons maintenant le dernier exercice de l'examen (figure 3). Là encore, la formulation de la tâche oscille entre plusieurs registres : il s'agit de démontrer, donc d'élaborer un raisonnement déductif. Mais l'énoncé à prouver est formulé dans un registre formel du calcul des prédicats, faisant ainsi ressortir sa structure logique, et permettant peut-être plus facilement une approche syntaxique de la preuve. Les quantificateurs, l'équivalence, le « ou » sont notés sous leur forme symbolique, l'étudiant n'a pas à exhiber la structure logique de l'énoncé, ce qui aurait été le cas s'il avait été formulé sous la forme « le produit de deux nombres est nul si et seulement si l'un des deux nombres est nul ».

Exercice 6 (3 points)
On rappelle que pour tout nombre réel x non nul, son inverse $\frac{1}{x}$ est bien défini et vérifie $x \times \frac{1}{x} = 1$. Démontrer l'énoncé suivant :

$$\forall a \in \mathbb{R}\ \forall b \in \mathbb{R} \quad ((a = 0 \vee b = 0) \iff a \times b = 0).$$

Figure 3 : Dernier exercice de l'examen

De fait, concernant la structure des démonstrations produites, 84 étudiants, soit 72% des 117 réponses, cherchent bien à montrer les deux implications. Et 55 d'entre eux concluent leur raisonnement en reprenant l'énoncé initial bien quantifié. Par contre, seuls 12 étudiants introduisent explicitement les variables a et b manipulées dans la preuve (un effet sans doute du fait qu'elles sont déjà présentes dans l'énoncé, même si elles sont muettes).

Pour ce qui est du contenu des démonstrations produites, l'implication $\forall a \forall b$ $((a=0 \vee b=0) \Rightarrow ab=0)$ est beaucoup mieux réussie que sa réciproque. 68 étudiants, soit 58% des réponses, produisent pour cela un raisonnement par disjonction des cas, et 22 étudiants produisent un raisonnement incorrect. Dans la copie ci-dessous, l'étudiant exhibe tous les éléments de structuration de la démonstration : (1) introduire a, b des réels, (2) supposer $a=0 \vee b=0$, (3) faire une disjonction des cas, (4) dire que dans chaque cas on a $ab=0$, et (5) conclure que l'implication est vraie.

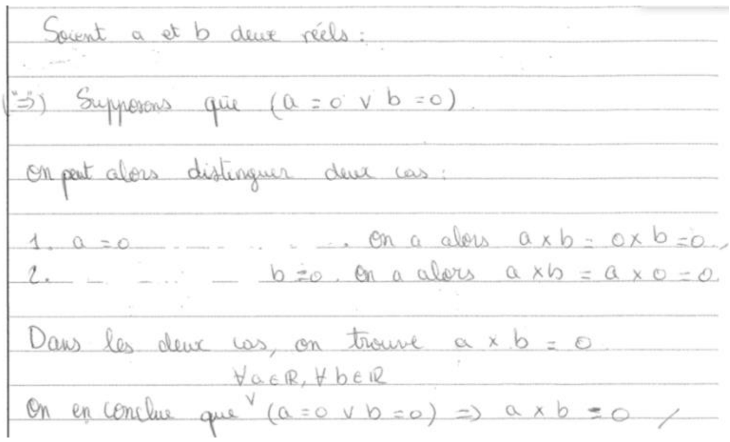
Soient a et b deux réels :
(⇒) Supposons que $(a = 0 \vee b = 0)$.
On peut alors distinguer deux cas :
1. $a = 0$. On a alors $a \times b = 0 \times b = 0$.
2. $b = 0$. On a alors $a \times b = a \times 0 = 0$.
Dans les deux cas, on trouve $a \times b = 0$
On en conclue que $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}$ $(a = 0 \vee b = 0) \Rightarrow a \times b = 0$

Figure 4 : Un extrait d'une réponse au dernier exercice

Peu d'étudiants ont une rédaction dans laquelle nous retrouvons tous ces éléments de structure, notamment les éléments (1) et (4) sont peu présents, les éléments (2) et (5) le sont dans un peu plus de 50% des copies. Cette différence reflète ce qui se fait dans les pratiques usuelles de rédactions. Pour la réciproque, seuls 25 étudiants produisent un raisonnement correct (qui correspondent, à 1 étudiant près, aux 21%, qui ont finalement résolu correctement la tâche), contre 43 qui produisent un raisonnement incorrect.

L'analyse de cet exercice montre, là aussi, que la plupart des étudiants se sont approprié le traitement syntaxique de la preuve d'une équivalence consistant à démontrer une double implication. Cette réussite est peut-être renforcée par l'aspect formel de l'énoncé à prouver. Cependant, nous voyons aussi que structurer correctement la preuve n'est bien sûr pas suffisant, puisque l'une des implications met une grande partie des étudiants en échec. Nous relions le choix de cet exercice, dans lequel l'énoncé à prouver est un résultat connu depuis plusieurs années par les étudiants, aux propos des enseignants sur le fait qu'ils cherchent des démonstrations pour lesquelles il n'y a pas besoin d'avoir d'idées mathématiques difficiles à trouver. Cependant, nous voyons que les étudiants sont familiers du traitement syntaxique consistant à transformer l'équivalence en une double implication, mais pas du traitement syntaxique consistant à transformer $A \Rightarrow (B \vee C)$ en $(A \wedge \text{NON } B) \Rightarrow C$, transformation qui permettrait une résolution plus simple de la deuxième implication.

CONCLUSION

Les analyses montrent que pour le cours étudié, l'entrée choisie est fortement ancrée dans l'aspect syntaxique. Cependant, les réponses des étudiants à l'examen montrent une appropriation relative de ce cadre syntaxique. En effet, dans le cas d'une tâche où il est explicitement demandé de mobiliser une approche syntaxique dans le registre du calcul des propositions en dressant une table de vérité, seulement 45% des étudiants répondent correctement. Et sur des tâches où le recours à des tables de vérité n'est pas explicitement demandé, et où d'autres procédures éventuellement moins syntaxiques sont possibles, il y a encore moins de réussite. Par ailleurs, nous avons repéré en analysant les entretiens avec les enseignants et le polycopié du cours une volonté de donner aux étudiants des moyens de produire des preuves en fonction de la forme de l'énoncé. Dans le cas d'une preuve simple, la majorité des étudiants qui produisent un raisonnement correct exhibent effectivement des éléments de structuration de la preuve.

Après cette première étude de cas, et ce premier choix d'une analyse selon les dimensions syntaxique et sémantique, nous souhaitons élargir notre analyse à la prise en compte d'autres articulations telles que fonction d'explication/fonction de validation (Hanna, 2000), aspect processus/aspect produit (Gandit, 2008), logique naturelle/logique mathématique (Deloustal-Jorrand, 2004). Dans le cours présenté ici, l'entrée syntaxique est fortement liée à la fonction de validation et à l'aspect produit : la forme de l'énoncé guide l'écriture de la preuve, et permet de garantir sa validité. L'articulation logique naturelle/logique mathématique n'apparaît pas dans le cours ou l'examen, mais elle transparaît bien dans les entretiens, et est l'objet de certains exercices. Nous envisageons également de reconduire ces analyses sur d'autres cours, et pour mener à bien cette analyse comparative, nous aurons besoin de construire un test indépendant du cours étudié, permettant d'évaluer l'appropriation par les étudiants des différents aspects de la preuve.

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PROCESS CONCEPTION OF LINEAR TRANSFORMATION FROM A FUNCTIONAL PERSPECTIVE

Diana Villabona[1], Gisela Camacho[2], Rita Vázquez[3], Osiel Ramírez[4] and Asuman Oktaç[5]

[1]Centro de Investigación y de Estudios Avanzados del IPN, México, diana.villabona@cinvestav.mx; [2]Centro de Investigación y de Estudios Avanzados del IPN, México, gcamacho@cinvestav.mx; [3]Universidad Autónoma de la Ciudad de México, México, rita.vazquez@uacm.edu.mx; [4]Universidad Autónoma de Ciudad Juárez, México, osiel.ramirez@uacj.mx; [5]Centro de Investigación y de Estudios Avanzados del IPN, México, oktac@cinvestav.mx

In this proposal we discuss from an APOS (Action-Process-Object-Schema) viewpoint student conceptions involved in the construction of conceptions about domain, image and inverse image of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ as well as the relations between these notions. We present the design of a set of tasks which allows exploring different facets of the above concepts, evidenced by the analysis of the production of a student. Thanks to the design of the instrument it was possible to highlight some conceptions that may not be evident in typical teaching situations. Some suggestions related to teaching strategies are included.

Keywords: teaching and learning of linear and abstract algebra, teaching and learning of specific topics in university mathematics, linear transformation, task design, APOS theory

INTRODUCTION AND RESEARCH OBJECTIVES

The process of learning new concepts builds on previously constructed concepts, particularly in advanced mathematics. Domain, image and inverse image are among such previous concepts for the understanding of linear transformations. These concepts play an important role within linear algebra as well as in connection with other subjects such as calculus and analysis. In the teaching of linear transformations, very often the algebraic and algorithmic nature related to the linearity properties is favored, as opposed to more functional aspects. Zandieh et al. (2017) explore the relationship between a high school function conception and a university linear transformation conception. These authors advocate a unified function-transformation concept; according to APOS Theory this happens when the vector space Object gets assimilated into the function Schema as a possible domain (Roa-Fuentes & Oktaç, 2010). The relationship between these two concepts was also studied by Bagley et al. (2015) in the context of inverse, composition and identity.

Reports about students' difficulties are abundant in linear algebra education research (Dorier & Sierpinska, 2001). Some of these difficulties might be related to the lack of previous concepts that need to be constructed adequately and coordinated. We have not found any literature about the role that the concepts of domain, image and inverse image play in the construction of the linear transformation concept, although the latter has been widely studied (Sierpinska, 2000; Andrews-Larson et al., 2017; Oktaç, 2018).

Given the context explained in the previous paragraphs, the aim of this study is to evidence student conceptions about domain, image, inverse image and linear transformation in $\mathbb{R}^2$including the relations between them. Also of interest is to investigate the role that the former notions play in the construction of the linear transformation concept.

Given the importance of linear algebra as a support for developing advanced topics in mathematics, as well as their applications in different university programs and in real life, we consider it necessary to investigate these relations in order to contribute to existing strategies for an adequate understanding of these concepts.

THEORETICAL AND METHODOLOGICAL CONSIDERATIONS

In our research group we have studied the linear transformation concept from various angles including algebraic, geometrical and matrix representations (Ramírez-Sandoval et al., 2014); dynamical exploration (Camacho-Espinoza & Oktaç, 2018) and connections between research and teaching (Oktaç et al., 2019). The present study is a continuation of this general project that has as its objective to study the understanding of linear transformations.

In this study APOS Theory is adopted as a framework, as it provides a cognitive approach applied in the context of the understanding of advanced mathematical topics. The basic elements of this theory are known as *mental structures*, *stages* or *conceptions*, that are constructed by means of mental mechanisms. *Actions* are driven externally, where the individual can transform previously constructed Objects. *Processes* are developed when an internal stimulus replaces the external algorithms or rules via the mechanism *interiorization*. When Processes are *encapsulated* they become *Objects* to which Actions or Processes related to other concepts can be applied. Finally all these structures and their relationships can come together to form part of a *Schema*.

The research cycle related to APOS Theory has three components: Theoretical analysis (called a *genetic decomposition*); design and application of instruction; and collection of empirical data and their analysis. The construction of the linear transformation concept was studied theoretically and empirically from an APOS viewpoint (see Arnon et al., 2014). In previous research studies that we conducted, some student difficulties that were detected in the construction of this concept gave the impression that they might be related to the lack of construction of previous notions. In order to delve into the role that these concepts play, we designed a research study about student conceptions on these topics, some of whose results we report here.

We designed an instrument comprising of four questions that was applied to a total of 31 university students from three different institutions in Mexico who were enrolled in an introductory linear algebra course. In the design of the problems special care was taken to involve different facets of the concepts in question. Usually in textbooks and in classroom activities only some of these aspects are offered to students, who get used to working on them from certain angles. When this angle is changed, and this is new to the student, it becomes easier to identify where the difficulties lie. In the case of linear transformations, the favored aspect can be the linearity properties over the functional characteristics.

In the design of the research instrument, algebraic, geometrical as well as functional aspects of a linear transformation were used, integrating in this way different facets of the concept. By a functional aspect we mean considering the linear transformation as a correspondence that associates the vectors of a domain space to the vectors of an image space; this is related to a Process conception because of its generality. In what follows we go into more depth about this issue.

A Process conception of domain, coordinated with a Process conception of image, is required for the Process conception of linear transformation. Conceptions that students have about domain and image, and even about systems of linear equations and their solutions, can intervene in their conceptions about linear transformation and inverse image. All these considerations were taken into account in the design of the questions that made up the instrument, with the aim of investigating the relationships between the different elements related to the linear transformation Process, and hence offering suggestions for its construction and enrichment.

ANALYSIS OF DATA

We now present two of the questions from the instrument, their analysis as well as the results obtained from a student. With the application of the instrument we have realized that even if a student gives the appearance of having constructed a conception that allows the solution of a problem with success, when the problem is changed to one that explores different facets of the same concept, the conceptions that are evidenced by the new situation can be quite different from the ones detected initially. We now analyze the production of a student (E) who handles very efficiently algorithmic procedures, when working on questions 3 and 4 of the instrument.

Question 3 was stated as follows:

> 3.- Consider the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ associated to the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.
>
> a) Determine its domain.
> b) Determine its image.
> c) ¿Does the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ belong to the image of the transformation? If the answer is YES find the corresponding inverse image and graph it. If the answer is NO, justify.
> d) Graph the domain of T.
> e) Graph the image of T.
> f) ¿Does the vector $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$ belong to the image of the transformation? If the answer is YES find the corresponding inverse image and graph it. If the answer is NO, justify.

Different parts of Question 3 of the instrument allow an investigation into student conceptions on domain, image and inverse image of a linear transformation via its matrix. Action and Process conceptions about these concepts can be detected in the context of algebraic and geometric representations, as well as the way students relate them to the linear transformation concept.

Before we start analyzing E's responses and arguments, we would like to comment on the cognitive structures that are involved here.

Construction of a Process conception of domain allows the individual to accept that the transformation acts on all the vectors of the domain. This conception might be revealed with the use of generic vectors that represent *any vector* in the domain. However applying the linear transformation on general vectors does not necessarily imply a Process conception of the domain and hence, does not always lead to the construction of a Process conception of image.

A Process conception of image permits the individual determine how the domain and image vectors are related to each other, what conditions a vector should satisfy in order to belong to the image, how that set is represented geometrically and in general terms, how the transformation acts on the domain vectors.

Constructing a Process conception of inverse image requires the individual to identify the set of vectors in the domain that are related to a specific vector in the image through the linear transformation. It also requires determining the characteristics that vectors in the domain should satisfy in order to form part of the inverse image set. Included in this Process conception is the identification of its geometric representation.

In what follows we present empirical evidence that shows that even if an individual can work quite efficiently with algorithms and procedures in solving a problem, this does not imply that solid conceptions about the concepts in questions have been developed.

Items a) and d) of Question 3 aim at exploring the conceptions that a student might have about the domain of a linear transformation. Student E gives the impression that he has constructed a Process conception of domain, since he identifies the domain of the given transformation as $\mathbb{R}^2$ and graphs it shading the whole plane as shown in Figure 1.

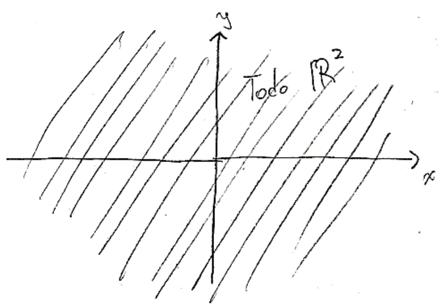

Figure 1. Domain of the linear transformation according to E

In order to determine the image of the transformation, E uses algebraic algorithms applied to generic vectors as shown in Figure 2.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \end{pmatrix}$$

Figure 2. Production of E to determine the image of the transformation

This might give the impression that the student has constructed a Process conception of image starting form a Process conception of domain, evidenced by the use of a general vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ on which the transformation acts through its matrix. However, this does not necessarily imply that the student has understood how the transformation acts on all the vectors in the domain, what the nature of this process is, and what properties it fulfills. One evidence that shows that E has not yet quite understood the nature of this image Process is that even though he determines the image set algebraically, he cannot associate it to the identity line and cannot graph it (see Figure 3).

$$Im(T) = \left\{ \begin{pmatrix} x+y \\ x+y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x, y \in \mathbb{R} \right\}$$

e)

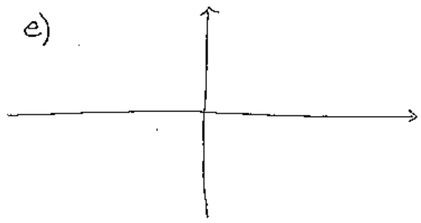

Figure 3. Image of the transformation and its graphical representation according to E

In this case the condition that the vectors of the image satisfy is that the x and y components should be equal. However, the algorithm used by E in order to find the image does not provide enough tools for him to realize this property; in parts c) and f) he does not make use of it to determine if the vectors $(3,2)$ and $(5,5)$ belong to the image set, as we shall see below.

In order to determine whether a vector belongs to the image set or not, E resorts to the Gaussian elimination technique applied to the augmented matrix of a linear system of equations. In the case of the vector $(3,2)$ this system is inconsistent and for $(5,5)$ it has infinitely many solutions (Figure 4).

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & -1 \end{array}\right)$$, que es un sistema inconsistente

$\therefore \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ no está en la imagen.

$$\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right) \therefore \begin{array}{l} x = 5 - t \\ y = t \end{array}, \text{ con } t \in \mathbb{R}$$

Figure 4. Gaussian reduction technique for vectors $(3,2)$ and $(5,5)$

Setting up a system of equations related to a transformation in order to determine whether a vector belongs to the image set or not, might require a coordination between a linear transformation Process and a system of linear equations Process. This allows the individual to recognize that $y = T(x)$ if and only if the augmented system $(A{:}\,y)$ has a solution, where A is the matrix associated to T. Despite the skill in algorithmic manipulation of systems of linear equations, lack of Process conception might lead to the need to repeat this procedure every time regardless of the number of vectors to be determined as belonging to the image set or not. This is related to an Action conception.

In summary, somebody who can perform Actions on a generic vector of the form (x, y) may or may not have constructed a Process conception of image; this is related to understanding the way a linear transformation acts on its domain.

Finding the inverse image of a given vector can be conceivably more complex than establishing the image of a linear transformation. However a solid Process conception of system of linear equations concept helps E to determine the inverse image set of the vector $(5{,}5)$, using a parametric form and graphing the corresponding line as seen in Figure 5.

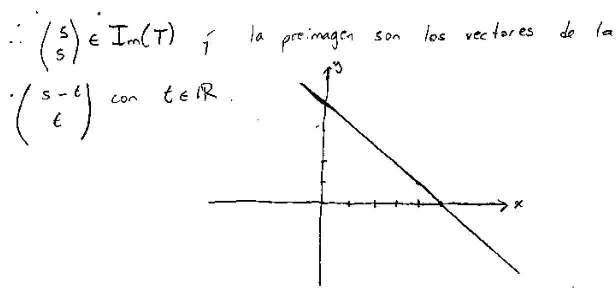

Figure 5. Parametric form of the inverse image of (5,5) and its graph

On the other hand, although apparently E might have constructed a Process conception of the domain concept, there are three pieces of evidence that manifest that he has not constructed a Process conception of image concept: (i) he cannot graph the image, (ii) he cannot determine whether a vector belongs to the image without performing a specific action and (iii) he does not have it clear how the transformation acts on the vectors of the domain. Furthermore, although he was able to determine the inverse image of a particular vector, this is thanks to his solid conceptions about systems of linear equations.

As we mentioned before, a coordination between the Processes of domain and image concepts is necessary so that a Process conception of linear transformation concept can be constructed. If any of these Processes is absent, it is very probable that the conceptions about linear transformations would be limited.

In order to consolidate our interpretations about these conceptions, we now turn into the work that E has produced when working on Question 4 of the instrument, hence examining his conceptions in a different context.

Question 4 was stated as follows:

> Find the rule of a linear transformation under which the image of $\binom{1}{0}$ is $\binom{0}{-4}$ and where the transformation is the combination of a rotation followed by a dilation. Determine the matrices of rotation and dilatation for the transformations used. Justify your response.

This question was designed so that it would be accessible to students with Action, Process or Object conceptions who would employ different kinds of strategies. Figure 6 shows the work of E.

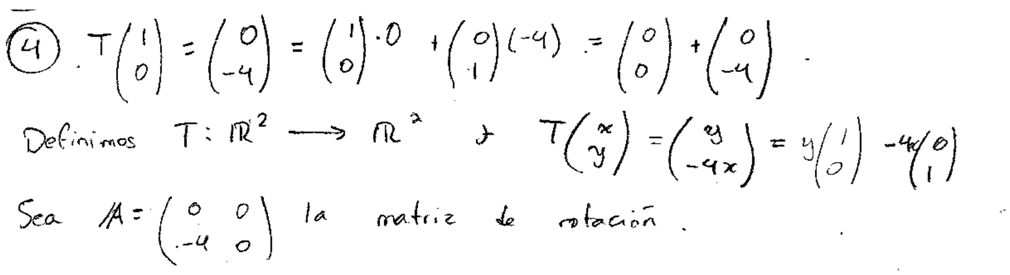

Figure 6. E's solution to Question 4

Here E shows an Action conception about image, by choosing to work with a single vector and its image, where $T(1,0) = (0, -4)$. As can be seen from his strategy, he employs a generic vector (x, y), which might indicate a Process conception, however he only tries to make sure that the conditions of the problem are satisfied by these two vectors and does not take into account the rest of the domain vectors. We should note that the image vector denoted by $(y, -4x)$ does not correspond to rotation and dilation transformations. It is possible that he is mixing up a rotation with a reflection about the line $y = x$. In order to find the image of (x, y) he interchanges the coordinates $(x, y) \rightarrow (y, x)$ and then multiplies the second one by -4, that is $(x, y) \rightarrow (y, x) \rightarrow (y, -4x)$. The linear transformation that he finds complies with the condition $T(1,0) = (0, -4)$. However the rest of the domain vectors do not actually configure into the solution. This confirms the lack of a Process conception of linear transformation concept, because of a lack of a Process conception of image concept, and probably lack of a Process conception of domain concept as well.

CONCLUSIONS

The design of the instrument enabled us to look into the conceptions related to the concepts of domain, image and inverse image of a linear transformation from different angles and how these conceptions intervene in the construction of the linear transformation concept. Question 3 allows the exploration of conceptions about notions that until now have not been prioritized in research as key in the construction of the concept of linear transformation.

Question 4 was designed to explore Action, Process and Object conceptions of the concepts mentioned before. Even though its complete solution requires an Object conception of linear transformation, students can tackle the problem partially evoking Action and Process conceptions. The analysis of the production of a student when confronted with these questions allowed us to investigate and demonstrate the structures involved and those that are to be constructed for the learning of the linear transformation concept.

When we talk about exploring different facets of concepts and conceptions we refer to the different roles that are played by them in different situations. For example in Question 3 the student is asked directly to find the domain, the image and the inverse image set. The student could not graph the image set, which indicated the absence of a Process conception. On the other hand, he was able to graph the domain, which could give the impression that he has constructed a Process conception for this concept. However Question 4 involves the same concept, namely domain, in a different way; the student has to use it in order to solve the problem, even though it is not even mentioned in the statement. Presenting situations that involve the concept in question from different angles allow us to identify the real conceptions that the student has constructed.

In teaching situations usually algorithmic and algebraic aspects of the linear transformation concept are favored, leading to Action-based strategies on the part of the students. The construction of a Process conception requires the interiorization of these Actions, leading to an internal control over them, including the ability to work with general situations. The functional aspect of a linear transformation is precisely related to a Process conception, in which the concepts of domain and image play an important role, which should be taken into consideration when establishing teaching strategies.

As for the limitations of the study, we mention that since it was only conducted in the context of $\mathbb{R}^2$, care should be taken when generalizing into other contexts.

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TWG4: Students' and teachers' practices

TWG4: Students' and teachers' practices: Group report

Irene Biza[1] and Imène Ghedamsi[2]

[1]School of Education and Lifelong Learning, University of East Anglia, UK, i.biza@uea.ac.uk

[2]Department of Mathematics, University of Tunis, Tunisia, ighedamsi@yahoo.fr

INTRODUCTION

The fourth Topic Working Group (TWG4) of the third conference of the International Network for Didactic Research in University Mathematics (INDRUM2020) was dedicated to students' and teachers' practices in the teaching and learning of mathematics at university level. Eleven papers and three posters were proposed and discussed in two thematic groups: *digital or other resources and the use of technology* and *teachers' practices and innovations*. In this report, we present a synthesis of the papers and the posters in each thematic group. Also, we present a summary of our discussion on emerging issues related to the recent Covid-19 outbreak, especially in relation to the shift to online or blended modes of teaching. We conclude with a reflection on the studies presented on the TWG4 and propositions for future research.

DIGITAL OR OTHER RESOURCES AND THE USE OF TECHNOLOGY

Five papers and three posters addressed topics related to resources, specifically digital resources, and use of technology in the teaching and learning of university mathematics. Specifically, Fleischmann, Mai, and Biehler proposed the design and the evaluation frame of a four-week bridging course. The course employed a blended learning design that included face-to-face lectures incorporated together with self-regulated e-learning with multimedia learning materials outside the lectures. The paper proposed a methodological approach for the evaluation of the course that connected the teaching design with student responses to a questionnaire. Results suggest that students appreciated the integration of interactive activities to the face-to-face part of the course. Regarding assessment, Hadjerrouit discussed student engagement with a computer-based assessment that provided formative feedback. The study employed Gibson's affordance theory (1977) on the physical properties of an object and the user-object interactions, to analyse 15 teacher students', who engaged with the computer-based assessment, responses to a questionnaire. Findings indicate that student interactions with the assessment through the formative feedback created affordances for learning at the technological, mathematical, and assessment level.

Gueudet, Buteau, Muller, Mgombelo, and Sacristàn employed the instrumental approach (Rabardel 1995) to analyse the engagement of university students with programming in the context of "authentic" mathematical investigations. The study considered programming as an artefact that develops to an instrument incorporating a structure of schemes that have mathematical only (m-schemes), programming only (p-schemes) or both mathematical and programming (p+m-schemes) goals. The structure of schemes above was illustrated in the case of the participation of one undergraduate student. This participation is elaborated further in the Buteau, et al. poster in which the development of student's engagement with programming is visualised in a diagram that incorporates the complex structure of m-, p-, and p+m-schemes developed by the student. The instrumental approach was used in the Topphol poster as well to analyse university students' participation in task-based interviews, where mathematical tasks are seen as instruments in the development of mathematical competences.

The Heinrich, Hattermann, Salle, and Schumacher paper explains the relationship between the interactivity of 63 pairs of students working in different instructional media on a descriptive statistics activity and their learning gain after participating to this activity. The paper proposes a theoretical instrument based on the ICAP-framework on the engagement activities between learners or between a learner and learning materials (Chi & Menekse, 2015) to analyse students' interactivity. In addition, learning gain through student responses to pre- and post- test was measured. Findings indicate a significant link between students' communicational behaviour and their learning gain. Collaboration is also related to the Glassmeyer poster that analysed the affordances offered by portfolio, which use peer feedback within an online graduate course on problem solving, for mathematics teachers' practices.

Finally, Sabra employed the documentational approach (Gueudet, 2017) to study the relation between research and teaching practices of three university teachers who are also active researchers. The study focused on the interactions between resources and teachers by analysing those teachers' research activities and teaching practices together. Analysis of audio-recorded interviews proposed three forms of use of research resources in teaching practices: use research resource in teaching instantiation processes; research resource to scaffold the learning of a given content; and, no relation of resources.

In the discussion session of the group, we had the chance to address emerging issues from the recent Covid-19 outbreak, especially in relation to the shift to online or blended modes of teaching. Specifically, we dealt with two questions: *How would the research knowledge we have been accumulating all these years in research on the use of technology and resources in the teaching and learning of mathematics at university level help us to address emerging situations due to Covid-19?* and *What new research can emerge from the impact of the Covid-19 outbreak on the teaching and learning of mathematics at university level?*

Our discussion highlighted that studies on students' participation and communication with peers and teachers could inform studies on students' collaboration online. However, when online is the central medium of communication, other factors should be considered as well, such as technological affordances and availability (or lack of) and changes in the visual mediation (e.g. gestures, body language, etc.). In addition, there are potential methodological consequences. For example, studies that were possible before the pandemic (e.g., classroom face-to-face observations or activities that involve students' physical interaction), probably will not be possible in an online mode. For example, a study that involves classroom observations should change radically in the midst of the pandemic where remote is the dominant mode of teaching.

A concern that emerged from our discussion is that current online teaching may privilege direct instruction led by the teacher with less opportunities for student engagement. A potential approach might be alternating between short pieces of direct instruction (before learners go into *cinema-mode*, quiet and passive attendance) and invitations for student interaction and contributions between such short pieces of direct instruction. E-assessment might be an issue as well; studies in this area are gaining more significance in the current circumstances.

It might be too early to study and experiment on online teaching. However, one observation is that after Covid-19 outbreak there is a substantial attention to teaching. This increase of attention might be an opportunity for enhancement of the teaching provision at university level overall.

TEACHERS' PRACTICES AND INNOVATIONS

Six papers of our group were related to teachers' practices and innovation in university mathematics. Specifically, Gascón and Nicolás drew on the anthropological theory of the didactic (Chevallard, 1999) to analyse the transition of future teachers from the institution of tertiary mathematics (as students) to an institution of secondary education (as teachers). Their study put forward the necessity for future teachers to undertake deep changes in the institutional "teaching ends" of mathematics and to look for a missing epistemological model. Still on transition, Ghedamsi and Fattoum investigated the possibility to reduce differences in the learning expectations of calculus in the transition between high school and university by engaging high school teachers in reforming their actions and making a connection between the two levels. They deployed a collaborative method founded on guided reflection (Husu, Toom & Patrikainen, 2008) to support teacher reflection on his/her actions by taking into account transitional issues.

In a different transition, this time from mathematics to mathematics education, is the work of Biza and Nardi who presented examples of activities and their assessment frame for mathematics undergraduate students' introduction to mathematics education research. The proposed activities follow task design principles that contextualise the use of theory and the mathematical content to specific learning situations (MathTASK). Students' responses to these activities are assessed in relation to clarity; coherence; consistency; specificity; use of terms and constructs from mathematics education theory; and, use of terms and processes from mathematical theory. The application of these activities and the assessment frame is exemplified through the responses from one student.

Drawing on literature results about students' difficulties and affordance for the teaching of limit notion, Chorlay and Mesnil analysed and compared three lectures by focusing on the use of definitions and examples. Post lecture interviews were used towards a further analysis of lecturers' actions and an investigation of the possibilities for lecturers to discuss alternatives to their actions. The results show that all three lecturers identified possibilities to consider potential changes on their own actions. Still on the teaching of calculus at university level, Karavi, Potari, and Zachariades analysed the characteristics of proof teaching in an introductory mathematical analysis lecture and lecturer's rationale underlying this teaching. Findings show a link between the pattern of proof teaching and the development of proof image for students as well as the impact of lecturer's experience on the building of this pattern.

Finally, Martinez, Gehrtz, Rasmussen, LaTona-Tequida, and Vroom explored what guides course coordinators' actions towards the goal of improving students' learning. They draw on Philipp's (2007) review of mathematics teachers' beliefs and affect to shape what they call "orientation toward coordination". The analysis of interviews with coordinators resulted in the identification of two main orientations: humanistic-growth orientation and knowledge-managerial orientation. Raising awareness to such orientations provides coordinators with materials to reflect on how they can act on the available drivers for change at their institutions.

REFLECTION AND WAYS FORWARD

In reflection on the studies presented and discussed in the group, it would appear that teaching interventions were at the heart of our group also in relation to the use of resources and digital technology. We were introduced to design principles and evaluation approaches that can facilitate the design and assess the effectiveness of such interventions. In addition, evidence was shared on how and what type of collaborative and participatory approaches in learning university mathematics may generate substantial learning gains. Furthermore, the role of digital curriculum resources and educational technology, for example programming, in both teaching and learning at university level, was a significant part of the works presented in the group.

Transition was a recurring theme into research on studies on students' and teachers' practices. We discussed studies addressing issues related to the transition from secondary education to university and, also, studies related to the transition from university to school level, especially in relation to teacher preparation. The importance of double discontinuity raised by Klein (1908/1932) was highlighted as essential in research that goes beyond the description of the problem. Such research proposed interventions that can prepare students for the transition while they are at secondary education or interventions that can prepare teachers before embarking for a teaching profession. Also, we discussed the transition for mathematical to mathematics education practices, in which undergraduate mathematics students are introduced to the theory (and the practices) of mathematics education.

At a more general level, the role of theory in university mathematics research was central in our discussions with the expansion of the use of well-established theoretical perspective to address new research questions. Some examples are the use of the instrumentational approach in the case of programming; the use of the documentational approach in the analysis of teachers' research practices; the use of the anthropological theory of the didactic in the transition of teachers to secondary education.

We dedicated the final session on potential open questions and research areas related to students' and teachers' practices that deserve more attention for the years to come. Topics that emerged from our discussion regard a range of areas. For example, *inclusive mathematical experiences*, was one of these areas, especially in relation to the challenges for teachers in the current (and the post-) pandemic era of serving students with special needs. Another emerging area was the *equity* in university mathematics education in relation to student opportunities for access to tertiary education. Furthermore, more research is needed on challenges and opportunities in *e-learning* and *e-teaching*, such as blended approaches to teaching, e-assessment or e-collaboration. In addition, we discussed the need for more opportunities for *collaboration* between mathematics education researchers, mathematics educators, mathematicians and mathematics teachers. Finally, we would like to investigate further new *methodological* and *theoretical* approaches with potencies in research on e-teaching and e-learning. We look forward to the next INDRUM conference and the new advances in research on teachers' and students' practices.

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FROM MATHEMATICS TO MATHEMATICS EDUCATION: TRIGGERING AND ASSESSING MATHEMATICS STUDENTS' MATHEMATICAL AND PEDAGOGICAL DISCOURSES

Irene Biza and Elena Nardi

University of East Anglia, School of Education and Lifelong Learning, Norwich, UK, i.biza@uea.ac.uk, e.nardi@uea.ac.uk

In this paper, we present examples of activities and their assessment frame for mathematics undergraduate students' introduction to mathematics education research. The activities are inspired by studies that have identified and addressed differences between discursive practices in mathematics and in mathematics education. The proposed set of activities uses task design principles that contextualise mathematical content and the use of mathematics education theory to specific learning situations. Students' responses to these activities are assessed in relation to: clarity; coherence; consistency; specificity; use of terms and constructs from mathematics education theory; and, use of terms and processes from mathematical theory. We exemplify the application of these activities through responses from one student.

Keywords: Novel approaches to teaching, teachers' and students' practices at university level, mathematical discourse, mathematics education discourse, MathTASK.

INTRODUCTION

Some institutions have introduced courses on mathematics education in mathematics undergraduate programmes. The motivation for such courses is to introduce mathematics students to the field of mathematics education research or/and to prepare them for mathematics teaching. Very often, these courses familiarise students not only with the new content of the social science of education but also with the new, to them, practices of educational research, which is a very different enterprise from research in mathematics (Schoenfeld, 2000). For example, in mathematics education, in comparison to mathematics, the perspective is less absolutist, more contextually bounded and more focus on the reasons behind a student's error. Approaches are more relativist on what constitutes knowledge (Nardi, 2015) and evidence is not in the form of proof, but rather more "cumulative, moving towards conclusions that can be considered to be beyond a reasonable doubt" (Schoenfeld, 2000, p. 649). Thus, findings are rarely definitive and are more suggestive. Such epistemological differences affect the experiences of those who, although familiar with mathematics research and practices, are newcomers to mathematics education. Boaler, Ball and Even (2003) analysed the challenges of mathematics graduates when they embark on postgraduate studies in mathematics education. They describe the epistemological shift these students experience in their transition from systematic enquiry in mathematics to systematic enquiry in mathematics education. Nardi (2015) addresses challenges with such epistemological shifts in the context of a postgraduate programme in mathematics education that enrols mathematics graduates and with a focus on the programme's activities "designed to facilitate incoming students' engagement with the mathematics education research literature" (ibid, p. 135).

In this paper, we draw on studies that have observed and addressed such shifts at a postgraduate level to discuss a course that introduces mathematics education to undergraduate mathematics students. Specifically, we propose course activities and an assessment frame for students' engagement with both mathematics and mathematics education discourses. Mathematical discourse is related to the mathematical content seen at upper secondary and first year university level, whereas mathematics education discourse is related to theories on the teaching and learning of mathematics and key findings from mathematics education research.

In the next sections, we describe the theoretical underpinnings of this proposal and the teaching context in which these activities are implemented. Then, we offer an outline of the course and its learning objectives before presenting the assessment and the marking criteria with examples of activities. Finally, we exemplify data collected from one student, Emily, as well as analysis of this data in which we apply the proposed assessment frame to evaluate her responses. Our goal is to investigate whether and how the proposed activities and their assessment frame can generate insight into mathematics students' engagement with both mathematical content and mathematics education theory. We conclude with a brief discussion of the potentialities of such activities in undergraduate students' introduction to mathematics education research.

CONTEXTUALISING MATHEMATICS EDUCATION DISCOURSE

The theoretical perspective of this work is discursive and is inspired by the commognitive framework proposed by Sfard (2008) that sees mathematics and mathematics education as distinctive discourses and learning of mathematics and mathematics education as a communication act within these discourses. We are interested in discursive differences – and potential conflicts – between mathematics and mathematics education and we aim towards a balanced engagement with both. Specifically, we are interested in how students transform what they know about mathematics from their mathematical studies and about mathematics education theory they are introduced to during aforementioned courses into discursive objects that can be used to describe teaching and learning. This transformation is the productive discursive activity of *reification* proposed by Sfard (2008, p. 118). For example, the reification of the theoretical construct of *sociomathematical norms* (Cobb & Yackel, 1996) can describe a situation in which students negotiate different approaches in solving a problem with integrals, while the reification of integration processes can describe the mathematical choices, and the accuracy of such choices.

Nardi (2015) proposed a set of activities for Masters and doctoral level students for their introduction to mathematics education research. In these activities, students are asked to engage with literature from mathematics education research and to produce accounts of their readings. In addition, students are asked to produce accounts of instances in “their personal and professional experiences that can be narrated in the language of the theoretical perspective” (ibid, p. 151) featured in those readings. These accounts of students’ experiences are called *Data Samples*. Engagement with literature together with the production of Data Samples has supported students situating their readings in their own experiences and their engagement with the discourse of mathematics education research. From the analysis of student interviews and written productions, emerged four themes regarding students’ transition from studies in mathematics to studies in mathematics education: *learning how to identify appropriate mathematics education literature*; *reading increasingly more complex writings in mathematics education*; *coping with the complexity of literate mathematics education discourse*; and, *working towards a contextualised understanding of literate mathematics education discourse* (ibid). The contextualisation of the mathematics education discourse triggered by the Data Samples and described by the fourth theme are the inspiration for the activities we outline in this paper.

Another inspiration was from our work with pre- and in- service mathematics teachers in the *MathTASK*[1] programme in which we engage teachers with fictional but realistic classroom situations, which we call *mathtasks* (Biza, Nardi & Zachariades, 2007). Mathtasks are presented to teachers as short narratives that comprise a classroom situation where a teacher and students deal with a mathematical problem and a conundrum that may arise from the different responses to the problem put forward by different students. The mathematical problem, the student responses and the teacher reactions are all inspired by the vast array of issues that typically emerge in the complexity of the mathematics classroom and what prior research has highlighted as seminal. Teachers are invited to engage with these tasks through reflecting, responding in writing and discussing. At the heart of MathTASK is the claim that, theoretical discussion related to the teaching and learning of mathematics is not productive unless it becomes focused on particular elements of mathematics and its teaching embedded in classroom situations that are likely to occur in actual practice (Speer, 2005). The MathTASK design was followed in the activities we outline in this paper.

Recently, we analysed the responses to mathtasks of mathematics teachers who attended a master’s level course in mathematics education (Biza, Nardi & Zachariades, 2018). Our analysis focused on teachers’ engagement with mathematics and mathematics education research discourses – particularly in relation to mathematics education theories they had been introduced to during the course. A typology of four interrelated characteristics emerged from this analysis of the teachers’ responses and used later in the analysis of trainee teachers’ engagement with mathtasks (Biza & Nardi, 2019). An adaptation of this typology became the frame we deployed to assess students’ engagement with the course activities:

[1] We use MathTASK (https://www.uea.ac.uk/groups-and-centres/a-z/mathtask) when we refer to the programme and its principles, whereas we use *mathtask* to refer to specific tasks designed with the principles of the MathTASK.

Consistency: how consistent is a response in the way it conveys the link between the respondent's stated pedagogical priorities and their intended practice? For example, do those who prioritise student participation in class propose a response to a classroom situation that involves such participation of students? Or, does their proposed response involve only telling students the expected answer to a mathematical problem?

Specificity: how contextualised and specific is a response to the teaching situation under consideration? For example, do those who write generally about valuing the use of vivid, visual imagery in mathematics teaching, propose a response to a classroom situation that involves specific examples of such imagery? Or, does the response include only a general statement of their preference?

Reification of pedagogical discourse: how reified is the pedagogical discourse that respondents have become familiar with during the course? For example, how productively are terms such as "relational understanding" (Skemp, 1976) or "sociomathematical norms" (Cobb and Yackel, 1996) used in the responses?

Reification of mathematical discourse: how reified is the mathematical discourse that respondents have become familiar with during prior mathematical studies? For example, how productively does prior familiarity with natural, integer, rational and real numbers inform a respondent's discussion about fractions in a primary classroom situation?

Before presenting how the typology was used in the assessment of students' responses to the activities, we first describe the context of the course and its learning objectives.

THE COURSE: CONTEXT, OBJECTIVES, STRUCTURE

The mathematics education course we discuss in this paper is offered as optional to final year mathematics undergraduate students in a research-intensive university in the UK. The aim of the course is to introduce students to the study of the teaching and learning of mathematics typically included in the secondary and post compulsory curriculum. The learning objectives of the course include: to become familiar with learning theories in mathematics education; to be able to critically appraise research papers in mathematics education; to be able to compose arguments regarding the learning and teaching of mathematics by appraising and synthesising recent literature; to become familiar with the requirements of teaching mathematics – mathematical knowledge for teaching; to become familiar with key findings in research into the learning and teaching of mathematics; and, to practise reading, writing, problem solving and presentation skills with a particular focus on texts of theoretical content, yet embedded in key issues in mathematics education research.

Teaching activities include four hours per week (two for lectures and two for seminars). In the lectures, led by the first author, the theoretical content is introduced while in the seminars, led by the first author and teaching assistants, students present and discuss their work that involves preparing presentations of papers they have read, identifying examples from their experience (data samples, as per Nardi, 2015), solving problems and reflecting on their solution; and, responding to mathtasks (Biza et al., 2007).

Opportunities for feedback are offered during the seminars and in formative and summative pieces of writing. We now exemplify how mathtasks are used in the course and how the typology of the four characteristics (Biza et al., 2018) shaped the frame we deployed to assess student engagement with said tasks.

SUPPORTING AND ASSESSING STUDENTS' ENGAGEMENT WITH MATHEMATICS EDUCATION AND MATHEMATICS DISCOURSES

We now present an example from the summative assessment that was taken by the students in the middle of the term. This assessment had two parts. In Part I, students were asked to solve a mathematical problem and reflect on their solution by using the mathematics education terms they had been introduced up to that point. In Part II (Figure 1), which is our focus in this paper and was inspired by the MathTASK design, students are asked to choose and discuss *one set* of mathematics education theoretical constructs from a list of four that had been discussed in the sessions up to that point and, then, to use these constructs to respond to *one* of two proposed mathtasks.

In discussing the theoretical constructs, the students were also expected to give examples of (1) how these constructs have been used in research, and, (2) how these constructs can be used to describe their own experiences. (1) was aiming to assess students' skills to identify relevant literature and (2) to contextualise the use of these theoretical constructs in their own experiences (as in Nardi's (2015) Data Samples).

Mathtask A (Differential Equation) is in Figure 1 (left) and mathtask B (Reasoning) is in Figure 1 (right). Students' use of the theoretical constructs in their responses to these mathtasks, together with their aforementioned Data Samples, provide evidence of how mathematics education and mathematics discourses have been reified in the students' communication about teaching and learning issues.

For the purpose of this paper, we analysed students' written responses according to the marking criteria: *clarity; coherence; consistency; specificity; use of terms and constructs from mathematics education theory; and, use of terms and processes from mathematical theory* (Figure 2) based on the four characteristics proposed by Biza et al. (2018): *consistency*, *specificity*, *reification of pedagogical discourse* and *reification of mathematical discourse*, where "reification of the pedagogical and the mathematical discourses" have been replaced by the "use of terms and processes from mathematical theory" and "use of terms and processes from mathematical theory", respectively.

Our aim is to investigate mathematics students' engagement with both mathematical content (mathematical discourse) and mathematics education theory (mathematics education research). We now present excerpts from the responses of Emily (pseudonym), one of the students who attended the course and consented to the use of her responses as data for our study. Emily's responses were chosen for presentation in this paper as their articulation and subtlety allows us to illustrate how we used the assessment frame consisting of the aforementioned six marking criteria.

In Part II (2,000 words), you will discuss mathematics education theoretical constructs we have seen thus far and use these constructs to discuss learning incidents. Specifically, for this part of your assignment, you will choose **one** of the options below:

- Relational and instrumental understanding (Skemp, 1976)
- Procepts and reification (Gray & Tall, 1994)
- Social and sociomathematical norms (Cobb & Yackel, 1996)
- Semantic and syntactic proof (Weber & Alcock, 2004)

and **one** of the learning incidents below:

A: Differential Equation

In a Year 13 class, students are asked to find the general solution of the differential equation $\frac{dy}{dx}=y^2$. One student proposes the following.

Student: This is a separable equation, so, I need to separate the variables:

$$\frac{1}{y^2}dy=dx$$

Now, I integrate both sides:

$$\int y^{-2}dy=\int dx$$

Which gives:

$$-y^{-1}=x$$

$$y=-\frac{1}{x}$$

The problem asks for the general solution, I need to add the constant. So, the general solution should be:

$$y=-\frac{1}{x}+C$$

Let me check what the answer says at the back of the book … [checks] Mmmm, it says:

$$y=-\frac{1}{x+C}$$

No, this cannot be correct, it must be a typo and it won't be the first time!

B: Reasoning

In a Year 7 lesson, students are asked to solve the following problem:

"Can you make the two columns of numbers below add up to the same total by swapping just two numbers between the columns? Explain why or why not."

1	7
3	2
8	4
5	9

The following conversation between students A and B takes place:

Student A: If I add up the columns at the moment, I get totals of, umm…17 and 22. So we need to make these the same.

Student B: How about we just try swapping some numbers and see what happens!

Student A: Okay, let's try the top two numbers first…If we swap 1 and 7, we get new totals of 23 and 16. That's worse than before!

Student B: Let's try some others…what about swapping 5 and 7?

Student A: No, that gives 19 and 20.

Student B: We're getting closer, though!

Student A: What about if we swap two numbers that are close together, like 2 and 3?

Student B: Ummm…that gives 16 and 23, that can't be right.

Student A: We could be here doing this forever!

Student C joins the conversation.

Student C: Maybe it can't be done and we have to show why not.

Student A: How would we do that then? We can't try every single possible swap…that would take too long!

You will structure your work on Part II as follows:

Discussion of theoretical constructs [1000 words, 25 marks]: You will present the theoretical constructs of your choice through: discussing their meaning; describing their relationship with learning theories we have seen so far; giving examples (from research papers) on how these constructs have been used to analyse students' responses or behaviour in the classroom; and, giving an example from your own experience.

Discussion of the learning incident [1000 words, 25 marks]: You will discuss the incident of your choice by using the language of the theoretical constructs you have chosen in the first section. It will help you to choose a theoretical construct that can explain the issues you have identified in the incident of your choice. In this section: you will solve the mathematical problem of the incident; you will identify what the issues are in students' responses; and, you will describe your interpretation of why the student(s) have responded in such way.

Figure 1: Assessment activity inspired by the MathTASK design

Clarity: How clear, justified and transparent the arguments are.

Coherence: How logically connected the arguments are.

Consistency: How consistent the arguments are across the text.

Specificity: How contextualised and specific the arguments are in the used examples and the discussed situation.

Use of terms and constructs from mathematics education theory: How precise and accurate the arguments are in relation to the used mathematics education constructs and terms.

Use of terms and processes from mathematical theory: How precise and accurate the arguments are in relation to the used mathematical concepts and process, such as definitions and proof.

Figure 2: The six marking criteria.

EMILY'S ENGAGEMENT WITH THE ACTIVITY

Emily chose the theoretical constructs of *instrumental and relational understanding* (Skemp, 1976) and mathtask A (Differential Equation). In her response, Emily summarises the constructs well (*use of terms and constructs from mathematics education theory*) and draws on a range of research literature that uses these constructs. Also, she reflects on her experiences with high *specificity*, by attributing students' approaches to their schooling experience (e.g., teaching practices, assessment, etc.) and by recognising that relational understanding "has never been required":

> It is clear that achieving a relational understanding is ideal, however, it does have its drawbacks and isn't always necessarily the optimal form of understanding. In lower levels of a student's mathematical education, topics do not need to be understood at a relational level [Skemp, 1976]. Throughout our schooling, when certain topics are met, pupils are often told that they do not need to understand how something works and just simply how to apply it. In my experience of first dealing with quadratic equations at GCSE, I did not know how the formula found the roots of the equation and was told that I did not need to know at that level. As I have progressed throughout my mathematical education there has never been a stage where it is thought necessary to gain a relational understanding as it is not required and is unknown by the majority of people. This lack of relational understanding is not due to a lack of disinterest or ability to understand but is purely due to the fact that such knowledge has never been required.

Later in her response to mathtask A, her approach takes a distance from the school influence and attributes students' approaches to their idiosyncratic characteristic as "instrumental" and "relational learners".

> In the classroom, pupils that understand in an instrumental way exhibit different characteristics to those who relationally understand. One of the main differences between the two types of pupil is not only how they answer questions they are asked, but also in the questions they ask and the answers they expect. A pupil who desires to achieve a relational understanding will eventually come up with an answer to a variety of questions even if it takes an extended period of time, whereas an instrumental learner can only answer an immediate answer to particular questions. […] This leads to the relational learner continuing to try until they gain an answer, unlike the instrumental learner who when they can no longer make any progress, often give up.

This characterisation of learners (as instrumental or relational) contradicts (*consistency*) her earlier view of approaches embedded in institutional practices. Although subtle, this *inconsistency* in Emily's response is a great opportunity for discussion around the simplistic lens of individual learning styles versus the actual complexity of institutional influences on learning processes.

Later in her response, she attempts to combine instrumental and relational understanding:

> Perhaps instrumental understanding should be viewed as a stage within the relational understanding and so students should be taught the skills required for both understanding. Merging the two states of understanding could result in being more powerful than either one alone thanks to the speed and ease of instrumental understanding alongside the profound knowledge gained through relational. Undoubtedly both understandings create a foundation on which new knowledge can develop which is key in mathematical education.

We note that, during class discussions, avoiding the dichotomy between instrumental and relational understanding had been repeatedly emphasised (*use of mathematics education terms and constructs*). This discussion has been assimilated in Emily's attempt to describe instrumental understanding as a "stage within" relational understanding.

In her response to mathtask A, Emily solved the problem correctly and spotted the mathematical error of the student in the incident (*use of terms and processes from mathematical theory*). In her explanation, she uses the relational/instrumental understanding language with precision:

> In the learning incident, it can be argued that the child in focus has an instrumental understanding of integration. Upon first reading the incident, this becomes evident due to the misunderstanding of where to place the constant of integration, c, as the pupil shows that they know they must include a constant when solving an indefinite integral. The student has displayed a common mistake of adding the constant once the equation had been rearranged to make y the subject.

However, her response does not explain the purpose of using the constant "c" in the integration. She thus misses the opportunity to demonstrate the mathematical explanation of *why* this is the correct integration (*specificity, use of terms and processes from mathematical theory*).

Overall, Emily's response demonstrates high *specificity* in the examples she provides and in her discussion of the incident. Her arguments are *clear* and *coherent*, although they are not always *consistent*, especially in relation to her views on institutional vs individual factors influencing students' approaches to learning. The use of mathematics education terms and constructs is precise and accurate (*use of terms and constructs from mathematics education theory*), while the *use of terms and processes from mathematical theory*, although without errors, does not demonstrate the precision and the mathematical detail we expect in the discussion about integration.

CONCLUSIONS

In this paper, we presented examples of mathtasks and their assessment frame used in a mathematics education course for mathematics undergraduates. The course activities are inspired by studies that have identified the epistemological differences between practices in mathematics and mathematics education (Boaler et al. 2003; Nardi, 2015; Schoenfeld, 2000) and have addressed these differences in the learning of postgraduate students (Nardi, 2015). The outlined set of activities uses task design principles that contextualise the use of mathematics education theory and mathematical content in specific learning situations (MathTASK design, Biza et al. 2007). Students' responses to these activities are assessed in relation to: clarity; coherence; consistency; specificity; use of terms and constructs from mathematics education theory; and, use of terms and processes from mathematical theory inspired by the four characteristics proposed by Biza et al. (2018). We see the potency of these activities in the introduction of mathematics students to mathematics education research as they invite students to engage both with mathematics and mathematics education discourses and to contextualise learning about mathematics education theories in their own learning experiences. Finally, we see these activities as affording opportunities for nuanced and concrete formative feedback.

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From an analysis of content to an analysis of ordinary lecturing practices: A case-study in mathematical analysis

Renaud Chorlay[1] and Zoé Mesnil[2]

[1]Sorbonne Université (Paris) & LDAR (U. de Paris, Artois, Cergy-Pontoise, Paris-Est Créteil, Rouen), renaud.chorlay@inspe-paris.fr,

and [2]Université de Paris & LDAR, zoe.mesnil@univ-paris-diderot.fr

This paper bears on teaching practices in lecture courses on analysis in the first year of tertiary education. This case-study shows how the knowledge accumulated by the didactical community on the challenges in the teaching of a specific notion – the formal definition of limits of sequences – allows for a fine-grained analysis of teachers' practices. On the basis of this knowledge, three lecture courses were analysed and compared. Post-teaching interviews were used to test our hypotheses as to the didactical knowledge, choices and repertoire of the lecturers. This progress reports aims to sketch the research rationale and to discuss a small sample of results.

Keywords: Teachers' and students' practices at university level, teaching and learning of analysis and calculus.

INTRODUCTION

This paper reports on a research project developed in the framework of activity theory (Hache and Robert, 2013). It aims to contribute to its adaptation to the study of the teaching practices in lecture-courses (Bridoux, Grenier-Boley, Hache and Robert, 2016), with a specific emphasis on the challenges of higher education (as in (Grenier-Boley, Bridoux and Hache, 2016)). In this context, the focus is on "teaching", regarded as a professional activity. We hypothesize that teaching practices can be described, analysed, (to some extent) accounted for, and (possibly) altered.

Within this larger context, this paper is of a methodological nature. We aim to study to what extent the analysis of the *relief* of the mathematical content at stake in a given teaching context – i.e. an analysis combining mathematical, epistemological, didactical, and institutional aspects (Bridoux et al., 2016) – allows the researcher to carry out an analysis of the empirical data based on the identification of *observables*. These observables will be denoted as "control points" in this paper. In particular, this approach provides means to overcome two common difficulties in the analysis of lecture-courses: The lack of information on actual student activity, on the one hand, and the difficulty to objectively identify what is *not* done, on the other hand. Moreover, the list of control points supports the construction of questionnaires which enable to researchers to test their hypotheses trough post-lectures interviews with the lecturer.

Beyond its contribution to this research program on teaching practices in higher education (with a focus on lecture-courses), this case studies suggests avenues for the professional development of teachers in higher education (Lison, 2013), (Rogalski & Robert, 2015). This aspect can only be touched upon in this short progress report.

THE *RELIEF* OF THE FORMAL DEFINITION OF LIMITS IN THE FIRST YEAR OF TERTIARY EDUCATION

The challenges of the transition from an intuitive notion of limit and the associated techniques (algebra of limits, connections with inequalities) to a formal definition used to prove theorems – in the larger context of the proof system of analysis – has been a continuous focus of didactical investigation since the 1970s (Artigue, 2016). Numerous works have identified the epistemological, didactical and cognitive challenges students face in this transition. On this basis, several didactical interventions have sought – occasionally with some success – to identify and put to the trial teaching paths which can prove conducive to the formal definition (see (Chorlay, 2019) for a recent survey). Moreover, this vast literature forms a fairly coherent whole, since, in spite of the variety of theoretical frameworks, the same key-phenomena are generally identified.

Consequently, we drew on this solid body of didactical knowledge to devise a large list of *control points* on the basis of which actual teaching practices can be described. The fact that we chose to derive our grid of analysis from a part of the didactical literature that is subject-specific – i.e. which bears on the formal definition of limits of sequences – comes at a price. Indeed, the results could be too subject-specific, thus making transfer to other contexts more difficult. Even so, we deemed the price worth paying, for two reasons. First, we wanted to see to what extent a body of knowledge bearing of mainly on students (their misconceptions, their documented behaviour in specific milieus whose didactic variables can be finely-tuned) could provide tools for the analysis of "ordinary" teaching practices, i.e. teaching practices which are not based on this research literature, by lecturers whose professional identity (de Hosson & al., 2018) and trajectory are independent from the didactical community. Second, in terms of professional development (see (Lison, 2013), (Rogalski & Robert, 2015)), studies suggest that professionals are keener to engage in a reflective practice when the focus is on very specific issues rather than on fairly general challenges.

In the two tables below, we list a series of control points which we identified in the literature on the teaching of limits. This is only a sample – albeit a significant sample – and, for lack of space, each of the points is described only sketchily and without systematic references to the literature. A wide-ranging list of references can be found in (Chorlay, 2019). The first table lists negative control points, bearing mainly on cognitive difficulties and misconceptions. The second table mentions more positive control points, since the literature allows for the identification of teaching strategies and moves which can prove conducive to the concept of limit, or, more generally, to advanced mathematical concepts.

The concept image of limits usually encompasses "primitive" models which are non-congruent with the formal definition. In particular, primitive models are usually "x-first"or "covariant", and resort to temporal, dynamic and causal imageries (the values of the variable *n goes* to infinity, and, as a consequence, the corresponding values of the sequence does this or that), whereas the

definition is "y-first" or "contravariant" (the condition on the values of the function or sequence determines conditions on the values of the variable) and string of quantifiers does not reflect or capture any notion of temporal evolution or causality.

The concept image of limits usually encompasses misconceptions, erroneous beliefs, or in-act-theorems such as:

- Every convergent sequence is monotonic (at least as from a certain rank).
- Every sequence which tends to $+\infty$ is monotonic increasing (at least from a certain rank).
- If a sequence converges, then the distance to the limit is monotonic decreasing.
- If the distance between a sequence and a given number L is monotonic decreasing, then the sequence tends to L.

The nested quantifiers can be interpreted incorrectly in several ways:

- The order of quantifiers *does* matter, and a $\forall\exists$ sentence is not equivalent to the corresponding $\exists\forall$ sentence.
- In a $\forall\exists$ sentence, the second variable depends on the first, but this dependence is not of a functional nature (more explicitly: in "$\forall\varepsilon>0\ \exists\ N \in \mathbb{N}$, N can be written N_ε or $N(\varepsilon)$, but N is not actually determined uniquely by the value of ε).
- The presence of the third quantifier is typically not regarded as necessary by students, who, for instance, tend to regard "$\forall A>0\ \exists\ N \in \mathbb{N}\ \ u_N>A$" as an acceptable definition of the infinite limit. This incomplete definition echoes standard informal formulations such as "the values become arbitrarily large".
- More generally, in the absence of the third quantifier, distinct mathematical concepts conflate: finite limit and that of subsequential limit; positive infinite limit and non-boundedness.

The transition between several formulations which are mathematically equivalent can prove challenging for students. For instance:

- For finite limits:
  $|u_n - L| < \varepsilon \quad \Leftrightarrow \quad L - \varepsilon < u_n < L + \varepsilon \quad \Leftrightarrow \quad u_n \in]L - \varepsilon, L + \varepsilon[$
- For infinite limits:
  $\forall\, M \in \mathbb{R}\ \ \exists\, n_M \in \mathbb{N}\ \ \ \forall\, n \in \mathbb{N}\ \ \ n \geq n_M \implies u_n \geq M$

$$\Leftrightarrow$$

$\forall\, M \in \mathbb{R}\ \ u_n \geq M$ except for (at most) a finite number of terms.

The definition of a convergent sequence can be used in two different contexts, and a non-trivial shift of viewpoints is necessary to use it in a relevant way.

• Context 1 (C1): In order to *prove* that a given sequence tends to a given number, any positive ε has to be taken into account, and the existence of at least one corresponding N value has to be proved. • Context 2 (C2): If some property is to be derived for sequences which are *known* to converge, then one is at liberty to select and use one specific value for ε, and the existence of N is warranted.

Table 1: Challenges in the teaching of the formal definition of limits of sequences

There are diagrams, such as the ε-strip diagram, which can be conducive from a change of perspective, from an *x*-first to a *y*-first perspective.
Discussing and comparing several *correct* definitions can have a positive effect in terms of conceptual understanding.
Some non-standard yet correct definitions can more likely be in the ZPD of students that the standard definition. This has been documented for the "non-implicative" form of the definition (here for infinite limits): $$\forall M \in \mathbb{R} \ \exists n_M \in \mathbb{N} \ \ \forall n \in \mathbb{N} \quad u_{n_M+n} \geq M$$
Discussing and assessing *incorrect* definitions can have a positive effect in terms of conceptual understanding, in particular to help student distinguish between neighbouring concepts (e.g. +∞ limit and non-boundedness).
Examples can come in several types, and can be used to serve a variety of purposes, all of which with a positive effect in terms of conceptual understanding: • In addition to *examples* (of a given concept) and *counter-examples* (which invalidate incorrect universal statements), *non-examples* can help students get a better grasp of the scope of a concept. • *Boundary-examples*, i.e. examples of a given concept which are not typically part of the students' concept image, can help students get a better grasp of the scope of a concept. Standard instances are: Constant sequences are convergent sequences; straight line graphs are curves which coincide with their tangents etc. • In *situations of definition construction*, examples can be used in several ways, among which: o The situation can rest on an *example space* o Examples can be used to put a candidate-definition to the test. In these situations, it is made clear from the start that the sought-for definition should be such that some objects should be examples, while some others should be non-examples.

Table 2: Affordances for the teaching of the limit concept and other advanced concepts

TERRAIN AND METHOD

Three lectures were video-recorded in the first semester of the 2018-2019 academic year at Paris Diderot University. The three lectures were delivered by three experienced lecturers (C, F, and B), two research mathematicians, and a secondary-school teacher with a university teaching position. The lectures were part of the same, 1^{rst} year, introductory course to mathematical analysis, covering, in this order: functions, continuity (without proofs), properties of real numbers; a final period was dedicated to the formal notion of limits (finite and infinite) for sequences and to proving in analysis. All the recorded sessions covered: formal definition of converging sequences, proof of the uniqueness of the limit for a converging sequence, proof that converging sequences are bounded. Lecturer B also proved that a subsequence of a converging sequence does converge to the same limit. Lecturer F also covered infinite limits and a few properties (such as: tends to $+\infty \Rightarrow$ not bounded above, while the converse is invalid).

We hypothesize that the lecture-courses under study are "ordinary", or "standard", for several reasons: They took place in standard institutional contexts; they were delivered by experienced lecturers with stable practices, with no involvement with didactical research, and with no claim to pedagogical innovation (as was checked in preliminary interviews); they were not designed specifically for this study, and we have no indication that they might have been affected by the fact that they would be recorded for research purposes.

The videos were analysed by the researchers. Both the comparison among lecturers and the comparison between what was done and what could have been done (as identified in the list of control points) provides a wealth of information. Table 3 below shows a sketchy sample of results ("X" stands for "Yes"):

	C	F	B
Uses informal formulations before the formal definition	X	X	X
Comments on the two contexts of use for the definition (C1, C2)			
Mentions alternative correct definitions			
Mentions boundary examples			
Uses an example space to assess a candidate-definition			
Use of *x*-first, dynamic, informal formulations		X	
Explicitly warns students against common misconceptions		X	X
Mentions non-examples		X	X
Uses diagrams to introduce or illustrate the definition of a converging sequence	X		X
Uses the ε-strip diagram for this purpose			X

Mentions incorrect definitions			X

Table 3: Overview of the three lectures (partial)

As mentioned above, the use of the control points allows us to identify things which did *not* happen, even in cases when the comparison among the three lectures would not point to them (lines 2 to 5 of table 3).

The purpose of this paper is not to analyse these data in detail, but to account for how they were used to further the analysis of lecturing practices. The long post-lecture interviews (about 50' each) were not carried out right after the teaching sessions. The videos were first analysed by the researchers. On this basis, the list of topics and the specific questions for the interviews were chosen to allow the researcher to study to what extent the observed practices reflected choices, and on what form of knowledge (*lato sensu*) these choices rested. The topics addressed in the three interviews were the same, but the wording of the questions differed. To promote reflective analysis, the questions were first based, for each lecturer, on her/his own practice, as in a explicitation interview (Vermersch, 1994). Then, alternative elements of practice – preferably based on the other two lectures – were mentioned, for the lecturer to discuss and assess.

More precisely:

- When one of the control points was positive in table 3 (“X”), we attempted to investigate to what extent this reflected a choice (for instance, did the lecturer mentioned alternative moves?) and, if, so, how it was justified: What item of knowledge or what belief did the lecturer but forth – if any – to account for it?
- When one of the control points was negative in table 3 (“ ”), we also attempted to investigate to what extent this reflected a choice (i.e. a choice *not to* do something).
    - If so, how was it justified? What alternatives had been considered and rejected? What item of knowledge or what belief did the lecturer but forth – if any – to account for this choice?
    - If not, we mentioned alternatives so as to study how lecturers made sense of them and assessed them. When their assessment was globally positive, we attempted to spot signs indicating that it could be a lever for professional development: Was the reaction very positive? Did the lecturer readily imagine ways to include this new element of practice in her/his own lecture (extension of the repertoire[1])?

[1] We use “repertoire” to denote the “stock of skills or types of behaviour that a person habitually uses” (Lexico dictionary, https://www.lexico.com/, accessed Jan. 24th, 2020.)

SAMPLE OF RESULTS

Use of *x*-first and/or dynamical informal formulations

In the interviews, the lecturers were first asked to comment on their informal introduction to the definition. In the lecture courses, F was the only one to use contravariant, dynamical formulations; a fact with which he felt completely comfortable in the post-lecture interview, arguing that this formulation carries an intuitive representation of the concept of limit. He mentioned no possible effects in terms of misconceptions:

Int.: All three of you made a similar choice, in so far as you introduced the notion of limit informally, in everyday language, before diving into the formal definition. Let me quote you: "a convergent sequence, a sequence which has a limit, is a sequence which, as it grows, has values which come closer to the limit value". What is the role of this sentence for you? How would you describe this sentence?

F: What matters is for students to retain the meaning [oral emphasis] of this definition. (…) It's a difficult notion, so it seems to me it's important to emphasize the intuitive side of the notion of limit.

By contrast, C et B used only contravariant rhetorical formulations such as "If (u_n) is a complex sequence and L a complex number. One says that sequence (u_n) converges to L when u_n is "as close to L as one might want provided *n* is large enough" " (lecturer C, sentence projected). The selection of this formulation reflects a choice for C and B. In C's case, little justification is provided:

Int.: Maybe you've come across other informal formulations, some which you would rather not use? Some which you find less acceptable?

C: Yes

Int.: Can you think of any one in particular, for which you said to yourself "hmmm, I definitely should not say that"?

C: No, nothing comes to mind straight away. You should not do too much hand-waving. If you go "*bla, bla, bla* comes closer" … it's meaningless!

Relevance of the ε-strip diagram to introduces or illustrate convergence

Neither C nor F used the ε-strip diagram, whereas B used an on-line applet which not only displayed the diagram but allowed the user to change the ε-value and visualize the effect on the horizontal strip and on the associated value of N(ε) (see figure 1 below). B explained that he chose this applet after watching several of them, and accounted for his choice by mentioning students' concept image of the limit in a way which could reflect the *x*-first/*y*-first distinction:

Int. : This on-line applet […] did you sometimes use something else, is that something you chose from among others ?

B : Yes, I chose it, I watched several. What is not intuitive in the definition, compared with the students' representation … students tend to think: there, it's coming closer; while <u>we</u> say that it enters the interval, an interval which becomes ever smaller. It's not exactly the same idea.

Figure 1: The ε-strip applet used in B's lecture[2]

When commenting on the diagram, B mentioned another didactic variable in the milieu provided by the applet to the class: the fact that the distance between the sequence and the limit is not decreasing. But the fact that N is not necessarily the rank of the first value of u which enters the strip was not mentioned, neither to the class, neither during the interview

It turns out that *not* using this type of diagrams was *not* a choice for C or F: They readily admitted they had never come across it, nor thought about it. By contrast, both reacted very positively, which suggests avenues for professional development.

Int.: (…) Some of your colleagues use this diagram to introduce the formal definition. Would you find it interesting, or dangerous?

F: Yes, yes, I find it … I should do it! I love this kind of diagrams. I'm in favour of as many diagrams as possible.

Int.: What qualities or advantages do you think this one has?

F: If we start with a definition in everyday language, and a hard [i.e. formal] definition, with epsilons, this gives a 3rd definition, a visual one. The more definitions we have the better. It's a good training for students, it trains them to reflect on drawings. (…)

In spite of his attempts, the interviewer did not manage to elicit any specific analysis of the qualities (or drawbacks) of this diagram. Lecturer C was equally enthusiast, and, when pressed to comment on the choice of the displayed sequence, managed to spot one of the didactic variables in the milieu provided by the applet: The specific sequence on display is not monotonic, "what's not bad is that is goes above then below, so we

[2] Retrieved from http://gilles.dubois10.free.fr/analyse_reelle/suitesconvergentes.html (accessed oct.9, 2019).

don't have the problem I mentioned earlier" (she had mentioned the belief that converging sequences are monotonic). Other didactic variables went undetected.

Scaffolding the shift of viewpoints when switching between C1 and C2 contexts

As can be seen in the interview below, it took C a little time to identify the issue at stake, but she eventually formulated it in the clearest of ways:

Int.: Maybe it didn't happen in your lecture this time, but we prove the "convergent ⇒ bounded" property, it is not uncommon for students to ask questions. The proof typically begins with "The sequence being convergent to L, I want to show it is bounded. So let's take a neighbourhood of size 1, let's say $\varepsilon = 1$" [C nods, approvingly]. Students sometime say "I don't get it, ε has to be indeterminate, that's the definition!" Did you ever get such questions? How would you respond to "You can't do that, ε has to be indeterminate".

C: [Merely rephrases the definition]

Int: So, later on, if in exercises students study the convergence of sequences by saying: "I say $\varepsilon = 1$" …

C: [laughs] Oh no! Of course not!

Int.: How come?

C: No, but, there's a difference of … of … in one case we know the sequence converges (…) but when I want to prove that it does (…).

The reaction was roughly the same for the other two lecturers. It shows that not mentioning this difficulty in their lectures was not a choice. Also, it suggests that this question is a lever for professional development, since it triggers the realization of a difficulty. However, the reactions of the lecturers were not as enthusiastic as for the ε-strip, and none of them spontaneously mentioned taking this issue into account in their future teaching.

DISCUSSION AND PERSPECTIVES

Our first analyses show that for two of the three teachers, most of the elements observed, then discussed in the interviews, can be justified in some way, but do not reflect choices insofar as no alternatives are clearly mentioned. This conclusion needs to be qualified for the case of B, who mentioned more alternatives and justified choices on the basis of his experience of students' behaviour. The interviews also show a potential for teacher training, as all three teachers adopt a reflective position that leads them to consider changes in their practices. Further analysis is needed to understand why discussing control points and alternatives elicits different reactions.

The point of this admittedly sketchy progress report was to describe the goal and the method of an on-going project. We could not present any results in detail, nor discuss the connections with several theoretical frameworks and issues. From a theoretical viewpoint, this research aims to contribute to collective work of adaptation to the specificities of tertiary education of all-purpose theoretical tools such as the MKT framework or the ergonomic and didactic approach to teaching practices (Hache and Robert, 2013). It should also prove instrumental in the adaptation of tools for the assessment of the quality of instruction (LMTP, 2011), in a context where the correctness of the mathematical content taught is not an issue.

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Design and evaluation of digital innovations for an attendance-based bridging course in mathematics

Yael Fleischmann[1], Tobias Mai[2], and Rolf Biehler[2]

[1]Norwegian University of Science and Technology, Norway, yael.fleischmann@ntnu.no; [2]Paderborn University, Germany

We describe how we extended a bridging course for mathematics at the University of Paderborn, including the lectures and phases of self-regulated learning outside the lectures, with multimedia learning materials. At the example of one lecture day of the course, we illustrate how digital learning materials were integrated into the course concept. In order to evaluate our interventions, we developed a methodological study design with a tight connection to the teaching innovation, supporting high response rates from the students at each conducted survey. We provide data on the students' valuation of different types of digital learning materials. Our results suggest that students appreciate in particular the integration of interactive exercises, but also of more passive didactical elements, in a traditional attendance-based learning environment.

Keywords: Transition to and across university mathematics, digital and other resources in university mathematics education, teaching and learning of mathematics for engineers, bridging course, evaluation design.

INTRODUCTION

During the last two decades at the university of Paderborn, several approaches to support the transition from school to university mathematics for our (prospective) students have been developed. For several years, free pre-university bridging courses of four weeks length have been offered to prospective students, who could choose between a traditional, attendance-based and a mainly e-learning-based course concept. Since 2014, we also developed an online course called studiVEMINT. The studiVEMINT material is designed as an independent online course in mathematics that can be used by any person who wishes to prepare themselves for university mathematics (see Biehler, Fleischmann, Gold, & Mai, 2016; Colberg, Mai, Wilms, & Biehler, 2017; Biehler, Fleischmann, & Gold, 2018 for further description of the contents, design and didactical concept of the course; see go.upb.de/studivemint for the project homepage). In particular, the material is *not* originally designed to be used in the context of an attendance-based bridging course for mathematics at university, but for individual work outside a supervised course.

After finishing the development of this course material in 2016, we came up with the question how the studiVEMINT course material could be used to enrich the didactical concept of our attendance-based preparation courses. The starting point for the study presented in this paper was our desire to create a scenario of blended learning, where our well-tried bridging course concept should be combined with the advantages of additional, digital learning materials. The condition for this integration was that the existing course should not be fundamentally changed or shortened with respect to mathematical contents, but the materials were supposed to be integrated seamlessly into the existent learning environment. The idea was to integrate digital elements into the lecturers talk, a passive use from the students' point of view, as well as into self-reliant learning phases of the students inside the classroom (active use during self-instructing phases that were interspersed into the lecturer's talk) and outside the classroom.

THEORETICAL BACKGROUND

We follow an approach of design research in this study (Bakker & van Eerde, 2015; Nührenbörger, Rösken-Winter, Link, Prediger, & Steinweg, 2019). The integration and evaluation of digital learning materials was first implemented in the course in 2017. Based on the results of the accompanying study, we recreated several interventions for the next implementation in 2018 and repeated this procedure in 2019 (Figure 1). The focus of the subsequent studies was changed moderately in each year, with respect to the particular needs that were identified in the previous analyses. In this paper, we go into details about the implementation and scientific results concerning the study in 2017 and discuss some local results that led us to the changes we implemented in the 2018 and 2019 course designs.

There are many different definitions of blended learning in the literature. Bernard, Borokhovski, Abrami, Schmid and Tamim (2014, p. 91) note the following to this matter: "The issue of blended learning is a complicated one; there has been considerable discussion even of the meaning of the term itself". We consider our interventions within the lectures as a case of blended learning as they are a mix of e-learning and a face to face situation. Despite the problem of its definition, blended learning is of general concern for higher education (Keengwe and Agamba, 2015) and is even subject to meta-studies (Bernard et al., 2014). The case of mathematics in higher education narrows the field in term of its contents a little bit. Systematic evaluations of blended learning approaches usually yield a result that implies its benefits, e.g. as in the work of Lin, Tseng, & Chiang (2017), who conducted a study in a seventh grade math course, Dai and Huang (2015), who systematically compared e-learning, blended learning and traditional instruction, and Kinnari-Korpela (2015), who evaluated the use of short video lectures for engineering students. This glimpse at ongoing research hints at the broad use of blended learning in terms of methods, contents and audience.

As blended learning already includes digital technologies, the use of these for evaluation purposes seems natural. Audience response systems (ARS) can support feedback from learners and provide a means to collect data for research purposes. Ebner, Haintz, Pichler, and Schön (2014) suggest a distinction between front-channel (direct feedback during lectures) and back-channel (asynchronous feedback during and out of the lecture) of those systems. They also distinguish these further into qualitative and quantitative forms of feedback. The subject of this article is an evaluation based on a research method for teaching scenarios in mathematics with blended learning that utilizes the front channel and is, in contrast to many other rather specialized methodological designs, transferable to other teaching contexts in a blended learning setting.

CONTEXT AND RESEARCH INTEREST

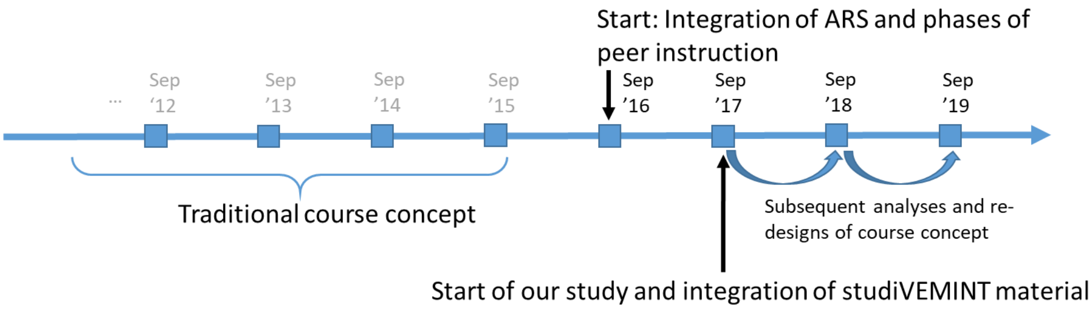

Figure 1: Timeline of the development of the course and our accompanying studies

The course in which we conducted our study was a 4-week-long mathematics bridging course at the University of Paderborn. The target audience are freshmen majoring in mechanical engineering, electrical engineering, industrial engineering, computer engineering and chemistry. The course concept, which was tested multiple times in several preceding years, relies primarily on face-to-face teaching consisting of three hour lectures followed by two hours of tutor group meetings on Monday, Wednesday and Friday of each week. In the tutor groups of up to 25 participants, the students work together on mathematical exercises and under the supervision of a tutor. Furthermore, the students are required to repeat and deepen their understanding of the mathematical contents on the so-called “self-learning days” on each Tuesday and Thursday. The course takes place annually in September in preparation for the following winter term. Participation is voluntary and neither graded nor specifically rewarded in any way regarding the subsequent studies of the students.

In September 2017, between 100 and 150 students regularly attended the course. The interventions which we integrated into the course concept focused on the lecture and the self-learning days. Traditionally, the lecture had a classical concept of knowledge transfer. In 2016, the lecturer had already included some innovative didactical elements in form of short phases during the lecture. These phases included exercises which the students had to solve and discuss with their peers, leading to phases of peer instruction with feedback by a digital audience-response system into the lectures. This didactical innovation was not part of our intervention, but was continued in 2017 during our study. The lecturer reported his positive experiences with this element and was interested in extending the use of digital elements and interaction during the lectures.

One goal of our interventions was to find ways in which the digital learning materials that we had designed for the studiVEMINT online course could also be used as a supplement in general in the context of a “classical” attendance-based bridging course and in particular with this course. The material includes 13 learning units covering mathematical school contents, starting with basic knowledge such as fractional arithmetics up to contents of the final school years such as integral calculus or vector geometry. The course has the particular claim to explain the contents in a way that meets the requirements of correctness at university's standard and rigour, and to present it in a way that closes the gap between school and university mathematics concerning notations and precision of argumentation. The chapters of the course have a consistent structure consisting of several units, starting with an introduction, followed by text units (supplemented by figures and interactive applets) containing explanations and some proofs of the mathematical contents, up to extensive collections of exercises for each contents unit. In most cases, the user can also insert an answer to these exercises into an input box and check the answer for correctness. For all exercises, detailed solutions are presented on demand. Some chapters contain units with additional applications and further complements to the mathematical contents, depending on the subject of the chapter. In the explanation as well as in the exercise part, interactive applets and videos are included in the material to illustrate mathematical contents dynamically.

INTERVENTION DESIGN

The integration of digital learning materials in our intervention had two focuses: Firstly, we wished to enrich the lecture by the inclusion of dynamic illustration, (additional) phases of self-reliant work by the students and other variations of teaching methods into the lecturer's talk. Secondly, we modified the self-learning days by providing the students with digital learning materials that support their independent repetition of the mathematical contents. Because of limitation of space, the procedure and analytical results concerning this second part of our study cannot be considered in detail in this paper. We give an overview about the results and consequences for the following cycles of the study in the conclusion at the end of the paper. Since the tutor group meetings were already well supported by tutors and no need of additional materials to stimulate students' learning in these phases was reported from the previous years, this part of the course was not changed in terms of our intervention.

We developed detailed timetables for the inclusion of digital elements into the lecture together with the lecturer. In particular, our interventions included the use of videos and dynamic applets to illustrate mathematical contents, and we integrated phases into the three hours lectures in which the students were asked to work with parts of the online materials independently. A concrete example for the process of such a lecture is given in Table 1. All lectures were planned in order to provide a balanced mixture of traditional and innovative teaching methods, and to alternate between active and passive phases of the audience.

As mentioned in the description of the context, the ARS had already been used for periods of peer instruction in the previous year. The lecturer first posed an exercise to the students at the blackboard, then collected their answers using an ARS and presented them to the audience. Thereafter, he let the students explain their solutions to each other, followed by a new feedback using the ARS. We distinguish between these exercises (analogue, answer via ARS) and the ones that we integrated in 2017 (digital, answer self-checked via studiVEMINT-course) in terms of our study.

Phase	Activity	Active part	Objective of activity
9:00 – 9:10 Lecture start	Lecturer communicates some organisational matters	lecturer	Course organisation
9:10–9:20 Start mathematics	Students read text in online-material (studiVEMINT)	students	Warm-up phase, revision of the defintion of an angle
9:20–10:15	Blackboard talk	lecturer	Knowledge transfer
10:15–10:45	Break	-	Recovery time
10:45–10:55	Students work on exercises in online-course material (with input and control function for solutions)	students	Repetition and application of knowledge, practising of computation methods
10:55–11:00	Students answer ARS-questions concerning exercises in previous phase	students	The lecturer gets feedback concerning the students' current knowledge and can react to possible problems and questions that occured
11:00–11:55	Blackboad talk, including -presentation of a video -presentation of an applet	lecturer	Knowledge transfer, visualisation of mathematical contents
11:55–12:00, end of lecture	Feedback questions concerning digital elements of the lecture, answered by the students via ARS	students	Scientific evaluation of interventions

Table 1: Schedule of lecture 4 (trigonometry)

RESEARCH QUESTIONS OF THE EVALUATION STUDY

The focus was on the integration of digital learning elements into a classical, attendance-based learning environment in a way that teachers and students accept and students appreciate as a valuable part of their learning process. In order to elaborate a suitable concept for acceptance and appreciation and its measure, the following research questions were in focus of the design of our evaluation study:

Which elements and ways of integration of the digital learning materials included in our online course can be beneficial for the students in an attendance-based learning environment? In particular:

1) Do students appreciate and enjoy the integration of digital learning elements into classical lectures, and do they consider these methods as helpful for their learning process?

2) Do students perceive differences between different medial formats of integrated digital elements such as videos, interactive applets, digital exercises with automatically checked solution entry field or mathematical texts, concerning their acceptance and personal resonance?

DESIGN AND METHOD OF EVALUATION

To approach our research questions in a differentiated way, we needed to design a proper evaluation method that allowed us a detailed scientific analysis of the implementation and provided us with a detailed feedback about our interventions into the course concept. The results produced by many traditional evaluation methods used in attendance-based learning environments do not allow immediate feedback or a fine differentiation between different teaching methods or digital elements used in the same session. In our case, we were interested in concrete and detailed feedback exactly concerning the acceptance and appreciation of the different types of intervention and different digital media formats.

Since the lecturer already reported positive experiences with the use of an ARS during the previous conduction of the course, we came up with the idea to use this channel also to collect feedback concerning our interventions. This ARS became a central part of the design of our study. The students became used to working with it during the lecture and it had proven to be widely accepted by the students in previous year. Hence, we decided to ask the students for feedback concerning the digital elements of the lectures with this system as well. Since it is easily accessible, students could give feedback using their smartphone within a very short period of time. In most lectures, the lecturer asked the students to answer a collection of 4-7 of our research-motivated feedback questions using the ARS after using the last digital element of the day.

We asked the students in particular if they considered the use of the specific digital elements during the lecture of that day helpful for their personal learning processes and whether they enjoyed working (actively or passively) with these digital elements. For the scientific analysis of the collected data, we grouped them into three categories:

- Applets and videos: This category contains all videos and applets that the lecturer either presented during his talk or, in some cases, he instructed the students to work with themselves before he continued with the lecture (therefore, it was mostly not possible to distinguish between active and passive use of these elements in our questionnaires).
- Exercises: Part of the lecture were intervals when the students were asked to work on specific mathematical exercises that are part of the online course, solve them alone or together with their neighbor and check their answers for correctness.
- Texts and figures: These elements were used by the lecturer to integrate a methodic variation into the lecture. The students were asked to work through a section of the explanation part of the online course on their own. This altered the primary teaching method for a limited amount of time (usually 10-20 minutes) and was created with the intention to activate the students during the lecture, allowing them to work on a certain topic at their respective individual learning speed.

Regarding these three categories of digital elements used in the lectures, each time when such an element was used, we asked the students for feedback concerning two aspects:

- Whether the use of the element supported their learning process: “The use of (…) in the lecture was supportive for my understanding”, with possible answers: “fully agreed”, “rather agreed”, “rather not agreed”, “fully not agreed”, “I did not participate”.
- Whether they enjoyed working with this element: “I enjoyed the use of element (…)” with possible answers: “fully agreed”, “rather agreed”, “rather not agreed”, “fully not agreed”, “I did not participate”.

The second type of questions concerning enjoyment were included since we considered the aspect of motivation of the students to be crucial for their success in a course that is, as in our case, completely voluntary and not rewarded in terms of any credits.

In addition to these digital data collections, there was a pen and paper questionnaire on the first and last day of the bridging course, which we used, inter alia, to obtain data concerning the students' overall impression of the interventions. An overall schedule of the course and the data collections is given in Figure 2. All data collected by the ARS and the pen and paper questionnaires was analysed quantitatively.

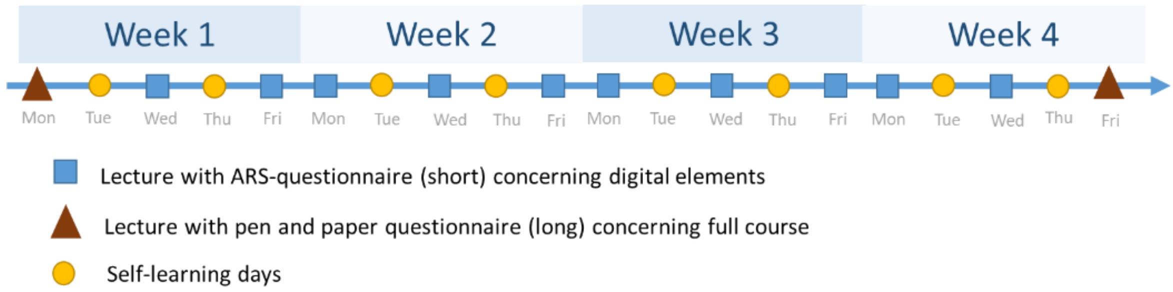

Figure 2: Schedule of the four weeks course concept and scientific evaluation

RESULTS AND LOCAL DISCUSSION

Due to limitations of space, we can report here just a small excerpt of the results that we obtained in our study. We concentrate on the feedback concerning the different types of digital elements during the lecture, which are described in detail in the previous section, the summary of all ARS questionnaires and a general feedback concerning the digital learning material.

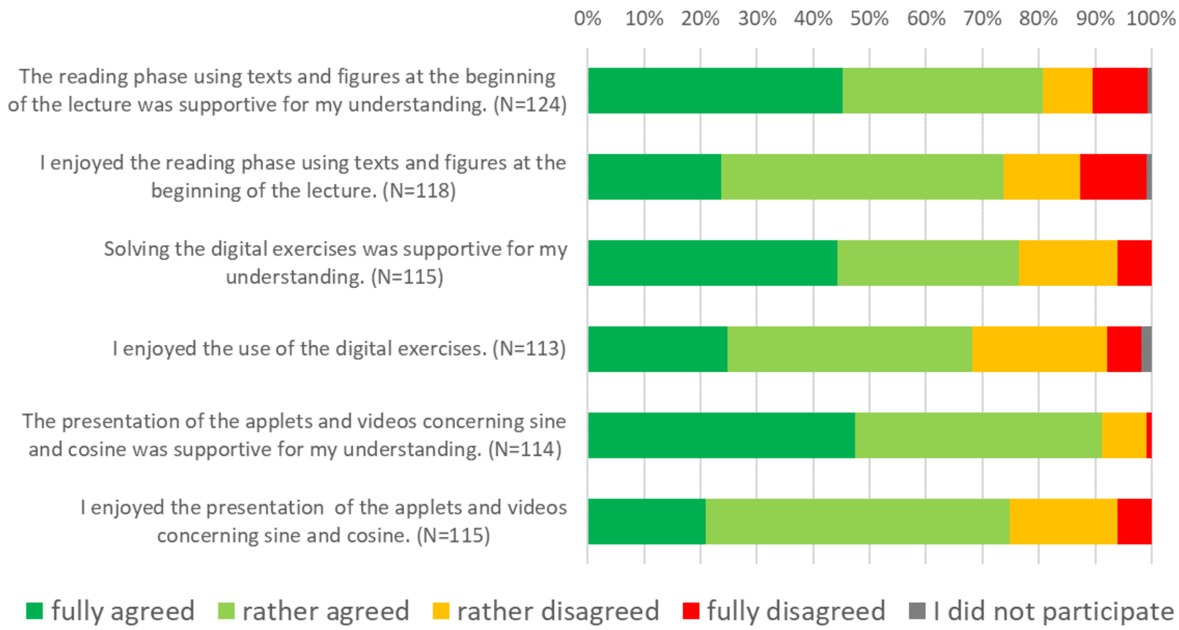

Figure 3: Results from ARS-questionnaire in lecture 4 (trigonometry)

Figure 3 shows students' feedback about the lecture described in Table 1, using the ARS during the same lecture session. We note that overall, students appreciated the integration of digital elements into the lecture, and all elements get comparably similar positive feedback. It is noticeable that the aspect of "support of understanding" is in all cases rated higher than the aspect of "enjoyment". It is also remarkable that in particular the presentation of dynamic applets and videos as a part of the lecturers' talk (so in this case, a completely *passive* element) gets very positive feedback concerning its effect on students' understanding.

category	criteria	
	"The use of (...) in the lecture was supportive for my understanding.", answers: "fully agreed" or "rather agreed"	"I enjoyed the use of element (...).", answers: "fully agreed" or "rather agreed"
applets and videos (N=677)	80%	70%
exercises (N=812)	78%	78%
texts and figures (N=359)	73%	60%

Table 2: Summary of results of all ARS-questionnaires

Over the four weeks of the course, we collected 677 feedbacks, each concerning both aspects (support of learning process, enjoyment) about applets and videos, 812 feedbacks about exercises and 359 feedbacks about texts and figures. A summary of the results is presented in Table 2. Note that this may include several answers from the same student to several singular questions concerning different elements of the same type. The results from the pen and paper questionnaire in the last lecture, in which we collected the answers of 129 students, support the impression that the students appreciated the use of digital elements in the lecture overall. The aspect of support for their understanding was again overall higher valued by the students (74.4% (concerning applets and videos) to 81.4% (exercises) "fully agreed" or "rather agreed") than the aspect of enjoying the use of the elements (where from 68.2% (applets and videos) to 72.7% (exercises) reported "fully agreed" or "rather agreed"). In this questionnaire, we also asked the students whether they considered the digital elements to be a welcomed enrichment of the lectures. About 82% (rather) agreed to this for

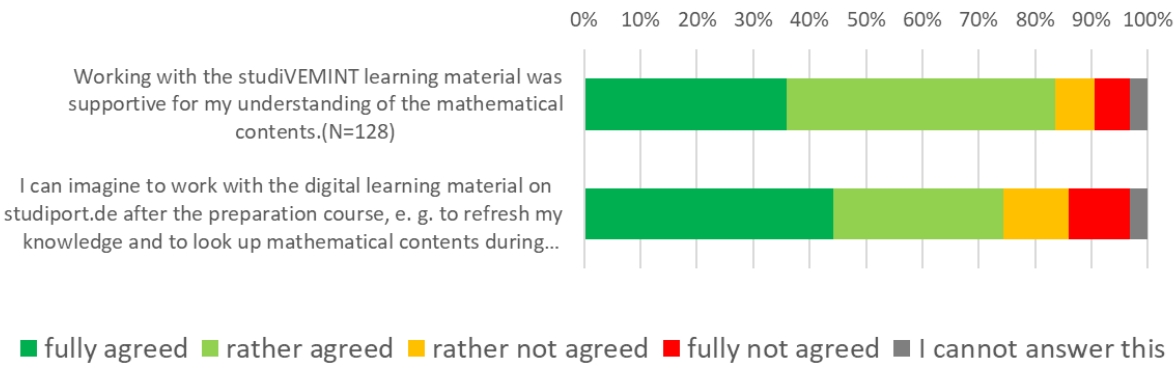

Figure 4: General questions concerning the digital studiVEMINT learning material (pen and paper questionnaire on the last day)

each of the three types of elements. Overall, it can be stated that the use of the digital *exercises* was rated best in all categories in the pen and paper questionnaires. An interesting finding is that students seem to be aware of and able to distinguish between the property of an element of being fun and of being supportive for their learning processes. The overall feedback for our interventions at the end of the course, concerning the support for the learning process by the digital learning material and the chances for future use was quite positive, too (see Figure 4). These data together with constantly high response rates suggest that the double use of a well-accepted system (the ARS) in content- as well as in research-related context is a promising approach.

CONCLUSIONS

Our results, in particular concerning the first research question, support the findings of Bernard et al. (2014) concerning the general benefits of blended learning in higher education for the specific case of mathematical teaching and learning. Ebner et al. (2014) formulate the requirement of scientific evaluation of front-channel feedback systems used in tertiary education; with our study we can present an example of a well-accepted implementation providing relevant feedback. Where Lin et al. (2017) could show positive effects of blended learning on the learning outcome and the attitude of high school students in mathematics, our results on both research questions suggest that these findings also hold for tertiary education.

In the sense of our design research setting, the results of the study were used to further develop the bridging course design in the following cycles, accompanied by scientific evaluation. Many of the innovations designed for the interventions of this study were maintained in the next iterations. We considered this as our first milestone on the way to the development of a design for a blended learning scenario, based on an already existing traditional course concept. In order to increase students' benefit from the digital learning materials in the subsequent cycles of the study, we decided to concentrate on the self-learning days of the course. Students were asked, but sometimes struggled, to work on tasks on their own, using the studiVEMINT course material. Based on the positive results concerning the digital exercises, we designed and refined tasks for the self-learning days. We also decided to put a focus on the aspect of motivation in the following conductions of our study. There, we concentrate on interventions to increase the motivation of students on the self-learning days and collect more detailed data about their engagement on these.

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Méthode pour l'engagement du professeur du lycée dans la question de la transition vers l'université

Imène Ghedamsi[1], et Fatma Fattoum[2]

[1]University of Tunis, Tunisia, ighedamsi@yahoo.fr; [2]Virtual University of Tunis,

Cet article illustre la mise en œuvre d'une technique méthodologique visant l'intégration du professeur de mathématiques du lycée dans un processus de changement de ses actions en faveur d'une meilleure prise en compte des variables didactiques de la transition vers l'université en analyse réelle. La sélection des variables que nous avons jugées pertinentes dans cette étude a constitué l'élément central dans la structuration de l'entretien qui fait suite à l'analyse des séances de classe du professeur qui a participé à l'expérimentation. Les résultats ont montré des possibilités d'adaptation des actions du professeur en regard des variables de la transition tout en restant vigilant au contexte de l'enseignement secondaire.

Keywords: transition to and across university, teaching and learning of specific topics, teaching and learning of calculus, reflective practices.

INTRODUCTION

La question de la transition didactique en mathématiques fait généralement référence à deux types de changements en lien avec l'enseignement et l'apprentissage des mathématiques : passage d'un ordre d'enseignement à un autre (primaire/collège, collège/lycée, etc.), passage d'un ordre d'enseignement à un ordre professionnel (université/enseignement, formation professionnelle/métiers de techniciens, etc.). Les travaux sur cette question abordent particulièrement ces changements à travers des phénomènes épistémologique, cognitif et socio-culturel (Gueudet et al., 2016). Les résultats sont majoritairement décrits en termes de difficultés que génèreraient le basculement d'un ordre d'enseignement à un autre notamment pour les élèves/étudiants, et à un niveau moindre dans la gestion du professeur du niveau supérieur (Ghedamsi, 2015 ; Bressoud et al., 2016 ; Gueudet et al., 2016). Par exemple, dans le cadre de la transition lycée/université en analyse réelle, Bloch et Ghedamsi (2005) identifient et utilisent un modèle de dix variables didactiques pour synthétiser l'ensemble des modifications requises dans le travail des élèves devenus étudiants. Ces variables, considérées comme des paramètres à plusieurs valeurs, traduisent les besoins de flexibilités épistémologique et cognitif qui accompagnent le processus d'acculturation didactique en analyse réelle à l'entrée à l'université. Parmi ces variables, nous en citons essentiellement celles qui sont partagées, en l'occurrence, dans les travaux qui problématisent la transition en analyse réelle : 1) le degré de formalisation et tout particulièrement dans les définitions des notions (Ghedamsi et Lecorre, 2018 ; Oehrtman, 2009) ; 2) le registre de validation notamment fondé, à l'université, sur la preuve de conjecture, la recherche de contre exemples, le raisonnement par l'absurde, etc. (Weber, 2015 ; Ghedamsi, 2015); 3) le statut de la notion, du processus à l'objet d'une théorie à l'université (Tall et al., 1999 ; Sfard, 1991). Dans cette étude, notre questionnement porte sur les possibilités du professeur à réduire, à travers ses connaissances mathématiques, l'écart entre les valeurs de ces variables à partir du lycée tant au niveau de la planification qu'en cours d'enseignement. Nous admettons que les connaissances mathématiques du professeur pour l'enseignement (CME), à un moment donné de sa carrière, sont non seulement déterminées par sa formation pédagogique et sa pratique mais aussi par ses connaissances mathématiques académiques/universitaires (Hill et al., 2004 ; Thompson, 2013). Même s'il est reconnu que la relation entre ces divers aspects de la CME est complexe, ce que nous mettons en avant concerne les significations que le professeur donne à ces connaissances (Thompson, 2015). Dans une situation d'enseignement, ces significations deviennent les causes ou les raisons probables des actions du professeur en classe et dans la préparation de son cours. Dans cette étude, nous mettons en œuvre une technique méthodologique basée sur la collaboration entre le professeur du lycée et le chercheur afin de l'amener à prendre conscience de ces significations et à y intégrer celles qui renvoient aux mathématiques de l'université pour les connecter aux mathématiques du lycée. Contrairement à l'enjeu soulevé par la deuxième discontinuité de Klein (de l'université à l'enseignement) dans la formation des futurs enseignants de mathématiques, nous abordons ici le défi d'alimenter les mathématiques de fin du lycée par un discours universitaire à travers des actions spécifiques du professeur et sous son contrôle. Les difficultés soulevées par la deuxième discontinuité de Klein (Winsløw et al., 2014) pourraient également être associées à des opportunités quand il est question de donner un rôle au professeur dans l'assouplissement des changements conceptuels et la construction d'un pont, en analyse réelle, entre le lycée et l'université.

CADRAGE THEORIQUE ET METHODOLOGIQUE

Les travaux en lien avec le développement professionnel du métier d'enseignant soutiennent généralement l'idée selon laquelle le professeur est un acteur efficace et durable du changement en matière d'éducation (Schön, 1983 ; Sellars, 2012). La qualité du changement est explicitement rattachée à la capacité du professeur en matière de pratique réflexive et plus simplement à la mise en évidence des significations qu'il donne aux connaissances disciplinaires d'enseignement (soit aux CME dans cette étude). Pour Dewey (1933) et Schön (1983), pionniers du paradigme réflexif, cette capacité de réflexion, en cours ou après son action, permet au professeur de faire face à des situations, en l'occurrence de classe, qu'il a jugé problématique d'une manière autonome ou avec l'aide d'un tiers. L'opérationnalisation du changement est à son tour tributaire de la réflexion consciente du professeur sur les causes et les conséquences de ses actions. La planification d'une dialectique action/réflexion, avec pour objectif final la modification de l'action future afin de la rendre plus efficace, requiert l'usage délibéré, régulier et méthodique de dispositifs qui permettraient de modéliser le processus de réflexion (Larrivee, 2000). Plusieurs travaux en éducation se sont intéressés à la modélisation de la démarche réflexive chez le professeur conformément au processus de réflexion sur l'action (De Cock et al., 2006). Dans tous les cas, ces modèles comportent quatre phases qui renvoient successivement à : 1) la prise de conscience d'une situation problématique dans son action ; 2) l'analyse des causes et des conséquences de l'action spécifiée sur les apprentissages ; 3) la structuration de la nouvelle action sur la base de nouvelles CME ; 4) l'expérimentation de la nouvelle action qui amorce à son tour un nouveau cycle de la démarche réflexive. Le processus décrit par ces quatre phases requiert la prise en compte de deux éléments fondamentaux : le mécanisme visant le déclenchement de la réflexivité en écartant toute réduction de ce processus à un ajustement inconscient et continu du professeur au réel (Comment créer le doute dans l'action et en expliciter les implicites -i.e. les significations données aux CME en jeu, donc ce qu'il veut en dire, et leurs aboutissants ?) et la hiérarchisation des niveaux de réflexion (A quoi renvoient ces significations ? Quels sont leurs déterminants ?).

Dans le cadre de notre recherche, nous nous intéressons au processus de réflexion sur l'action, où le professeur opère un retour réflexif a postériori de son action. Notre objectif n'est pas d'analyser le développement professionnel du métier d'enseignant de mathématiques, ni de porter le professeur intervenant dans cette recherche à adopter ce processus de réflexion et d'en étudier les effets. Notre intérêt est fondamentalement méthodologique : comment cadrer théoriquement une discussion entre chercheur et professeur du lycée qui engagerait ce dernier dans la problématique de la transition vers l'université, et l'amènerait de ce fait à opérer un changement dans ses propres actions. Parmi les mécanismes visant la réflexivité du professeur, la réflexion guidée s'inscrit particulièrement dans le contexte de notre recherche et consiste en une interaction entre le professeur et un tiers (donc le chercheur ici) sous forme d'entretien structuré sur la base des résultats de séances de classes assurées par ce praticien et transcrites en verbatim (Husu et al., 2008). L'entrée dans les phases du processus requiert de la part du professeur de la réflexion, notion que nous entendons dans l'acception des travaux sur la pratique réflexive en éducation, c'est-à-dire un processus mental conscient et volontaire qui intervient dans l'étude d'une situation en engageant une prise de distance par rapport à cette situation et l'action qui la génère, débouchant sur la production de nouvelles connaissances et la modification de l'action. L'existence d'une certaine hiérarchisation de la réflexion dans cette littérature, traduite en termes de catégories ou niveaux de réflexion, montre l'impact du contenu de la réflexion sur le processus mental que cette dernière engendre. Bien que les travaux de recherche n'associent pas explicitement les divers niveaux de réflexion avec une ou des phases particulières de la démarche réflexive, ils s'accordent sur l'existence d'au moins trois niveaux globaux (Hatton et Smith, 1995) : 1) Technique, en lien avec le contexte de l'enseignement, ses divers aspects et moyens ; 2) Pratique, lié à l'expérience du professeur de la discipline en tant qu'enseignant, étudiant ou autre ; et 3) Critique ou théorique, qui porte sur les règles reconnues et les éléments théoriques manifestes. Pour ce qui est du professeur de mathématiques, si l'on admet que les questionnements épistémologiques impliquant l'étude des conditions d'adéquation des connaissances mathématiques n'écartent pas les CME de cette étude, alors ces questionnements se retrouvent à tous les niveaux de réflexion. Dans cette recherche, nous avons mis en œuvre le dispositif de réflexion guidée utilisée sous forme de rappel stimulé où le professeur de mathématiques répond aux questions, d'un entretien, posées par le chercheur. L'entretien a été structuré conformément aux trois premières phases de la pratique réflexive et ses questions ont été élaborées sur la base des résultats de séances de classe que ce professeur a préalablement assurées pour introduire les notions de suite et de convergence en fin du lycée. Ces résultats avaient permis de mettre en évidence des situations problématiques dans les actions du professeur en lien avec les trois variables de la transition mentionnées ci-dessus. A travers cet entretien, la collaboration chercheur/professeur a pour objectif de porter l'attention du professeur sur les attentes de la transition et les intégrer dans ses CME pour ne pas les négliger dans ses prises de décisions.

EXPERIMENTATION

Contexte de l'étude

Le professeur, qui a accepté de contribuer à notre recherche en participant à cet entretien, possède une maitrise en mathématiques et quinze années d'expérience dans l'enseignement de fin du lycée. Notre collaboration date de cinq années, et a été initiée par l'enregistrement de trois séances d'une même classe de 35 élèves de fin du lycée, qu'il a assuré et qui ont porté sur les notions de suites et de convergence. Les transcriptions et les résultats de l'analyse de ces séances ont constitué le support de cet entretien et un accompagnement du processus de réflexion sur l'action (pour plus de détails sur cette analyse voir Ghedamsi et Fattoum, 2018). Dans le cadre de la problématique de la transition en analyse réelle, nous avons mis l'accent sur les actions du professeur en lien avec la gestion du formalisme, la gestion de la validation et la gestion du statut de la notion en jeu (expérimentation et objectivation), nous nous limitons ici à la notion de suite sans aborder la question des limites et de convergence. La mise en évidence des CME du professeur qui sous-tendent ces actions et les négociations de leur adaptation en fonction de la problématique de la transition, une problématique d'abord à la charge du chercheur et progressivement partagée par le professeur, constituent l'enjeu majeur de l'entretien.

Analyse a priori des termes de l'entretien

L'entretien a été construit de sorte à confronter le professeur à certaines de ces prises de décision par le biais d'interrogations portant sur les trois formes de gestion indiquées ci-dessus. L'analyse qui suit est organisée suivant ces trois rubriques et sera alimentée, dans la mesure du possible (pour des raisons d'espace), par des séquences illustrant certaines des actions du professeur sur lesquelles porte l'entretien.

- Actions en lien avec la gestion du statut de la notion :

AS1 - Donnée d'exemples, de non exemples, de contre-exemples (Expérimentation) : Les premières interrogations concernent le rôle des exemples et leur diversification pour décrire les différentes facettes de la notion de suite et reconnaître les éléments indispensables pour la caractériser. Fournir aux élèves des exemples typiques et unilatéraux de suites, comme en témoigne les choix du professeur, peut être l'une des raisons pour lesquelles les futurs étudiants peineront à développer des conceptions alignées à sa définition. La production de contre-exemple, quasiment absente aussi bien dans la planification du cours que dans le déroulement des séances de classe, permet à son tour de cerner la nécessité des hypothèses de cette définition.

AS2- Distinction entre suite et fonction (Objectivation) : Dans le cadre des choix du professeur d'introduire les suites comme cas particulier de fonction (et d'exploiter, plus tard conformément au programme officiel, les techniques sur les fonctions afin d'étudier les variations des suites, les minorants/majorants éventuels, et les limites), le processus d'objectivation de la notion de suite est tributaire d'une focalisation sur la distinction entre ces deux notions, ce qui permettrait aussi d'amorcer un raisonnement sur le lien entre le discret et le continu en analyse réelle. Des interventions du professeur telles que "*écris x à la place de n*" en réponse à l'élève qui devait identifier la fonction associée à une suite ou alors "*Non…pourquoi les joindre ? On va juste les placer*" en réponse à l'élève qui confondait la représentation graphique d'une suite et celle de la fonction associée, constituent la base des interrogations sur le sujet : dans quelles mesures est-il possible 1) d'intégrer des exemples de suites, non standards, définies à partir de fonctions qui ne sont pas du programme ? 2) de définir une suite par la liste de ses termes (exemple (1, -1, 1, -1, …)) ? 3) de différencier les propriétés liés aux fonctions, impliquant la topologie de $\mathbb{R}$ et qui n'ont pas de sens pour les suites (graphe, limite en un point, dérivabilité, etc.) et les propriétés des fonctions liées à l'ordre total établi dans $\mathbb{R}$ et $\mathbb{N}$ et qui peuvent être étendues aux suites (monotonie, bornée, etc.).

- Actions en lien avec la gestion du formalisme :

AF1 - Illustration des énoncés quantifiés : Rappelons que la transition vers l'université implique l'usage assez systématique d'énoncés quantifiés. Nous ne rentrons pas dans la manière avec laquelle les futurs étudiants construiraient une signification de ces écritures quantifiées. La réaction en chœur des élèves, "*Quelle est la différence ?",* à propos de l'inversion de l'ordre des quantifications dans la définition d'une suite majorée, sans que le professeur n'en donne une suite, ainsi que le non respect de la nature du quantificateur montrent la nécessité de le sensibiliser à l'intérêt de discuter les règles d'organisation des quantificateurs dans un énoncé mathématique.

AF2 - Enonciation d'expressions informelles : Nous nous centrons en particulier sur l'usage, par le professeur, de formes langagières particulières pour aborder de notions clés de l'analyse réelle : l'infini et le continu. Comme soulignés par Alcock et Simpson (2017) : "*The difference between mathematical and natural categories is commonly understood to cause students errors*".(p.6). Or, le professeur n'hésite pas à employer des formulations portant à plusieurs interprétations sur la base d'exemples de suites prototypés représentées par leurs graphes : "*ils sont proches*", "*On considère des intervalles infiniment petits centrés en 3*", "*ça augmente indéfiniment*", "le nombre d'insectes semble croitre sans dépasser 200", sans s'attarder sur des interventions de futurs étudiants du type "*C'est grand comme intervalle non ?*".

- Actions en lien avec la gestion de la validation - AV1 - Identification de modes de raisonnements mathématiques : Il est reconnu qu'à l'université différents modes de raisonnement sont utilisés pour prouver la validité des résultats mathématiques, il est important qu'une initiation progressive se mette en place à travers l'enseignement des notions de l'analyse réelle. L'appui sur la donnée de contre-exemples offre l'opportunité aux futurs étudiants de pratiquer la négation d'un énoncé universel en illustrant par exemple l'absence d'une propriété telle que la monotonie d'une suite (il suffit de trouver trois termes consécutifs de la suite donc réalisant les conditions imposées dans l'hypothèse sans que soit vérifiée la conclusion). De même pour les raisonnements de types condition nécessaire et condition suffisante qui est au cœur de la question de la réciproque de cette propriété : "*si la fonction est croissante (décroissante, constante, minorée ou majorée), alors il en est de même de la suite qui lui est associée*". L'automatisme qui accompagne l'usage de cette propriété, à travers ce qu'appelle le professeur "*la fonction naturelle*" empêche les élèves de voir qu'à une suite donnée on peut associer plusieurs fonctions et mettre à défaut sa réciproque.

ANALYSE DES RESULTATS DE L'ENTRETIEN

L'entretien s'est déroulé en deux séances, de deux heures chacune, qui ont eu lieu à un jour d'intervalle. Nous avons enregistré un total de 272 interventions lors de cet entretien quasi équilibrées entre le chercheur et le professeur : 144 interventions du chercheur et 128 interventions du professeur. Le tableau 1, suivant décrit la répartition de ces interventions en fonction des trois rubriques d'étude adoptées.

	Gestion du statut	Gestion du formalisme	Gestion de validation
Chercheur	23 (9%)	44 (16%)	77 (27%)
Professeur	20 (9%)	38 (14%)	70 (25%)

Tableau 1: Répartition des interventions par rubrique

Une première étude globale de notre transcription a montré que les trois niveaux de réflexion : technique, pratique et théorique, ont accompagné le contenu du discours du professeur au cours des trois phases de la démarche réflexive. Plus précisément, l'acheminement de la phase de prise de conscience vers la phase de distanciation et d'objectivation des causes et des conséquences de son action, préalablement mise en évidence par le chercheur, passe par une étape où le professeur essaye de la légitimer par une argumentation basée sur le contexte de l'enseignement des mathématiques, et ses exigences, ou sur une pratique qui a fait ses preuves. Durant la phase trois de modification de l'action, le niveau de réflexion du professeur est fondamentalement critique en lien avec les règles mathématiques savantes. Pour chaque rubrique, notre analyse va porter notamment sur les niveaux de réflexion c'est-à-dire sur les CME du professeur qui ont fondé ses actions, et sur le processus de leur régulation à travers les trois phases de la démarche réflexive. Dans toute la suite, l'intervention précédée par P est celle du professeur.

- Actions liées au statut de la notion : La prise de conscience du professeur du rôle des exemples dans la conceptualisation de la notion de suite a été très lente. La remise en question de AS1 a progressivement émergé à travers son expérience des suites non standards de fin du lycée *P :* "Comme *pour le reste de la division euclidienne de 20 par n+1*", mais aussi de suites plus complexes et qui *P : "... ne sont pas décrites par des fonctions d'une manière simple"*. En acceptant de ne pas limiter la cause de son action aux contraintes du programme, le professeur met l'accent sur ses conséquences possibles au-delà du lycée en déclarant simplement qu'il va *P : "... diversifier les exemples*. En outre, la donnée de non exemples de suites où les fonctions ne remplissent pas l'une des conditions de la définition ne fait pas partie de ses CMEs et pour cause, ne pas perturber la construction des connaissances des élèves en prenant le risque de s'écarter des termes du programme *P : "C'est drôle, on leur apprend une chose et juste après on doit leur donner des contre-exemples* (à entendre aussi en termes de non exemples) *où ça ne fonctionne pas !"*. Il a tendance à vouloir "protéger" les élèves contre le doute que peuvent créer les non exemples et les contre-exemples, y compris quand il s'agit de solliciter des suites ne vérifiant pas une propriété donnée (monotonie, bornée) *P : "Et quel est l'intérêt ? Pourquoi on fait ça maintenant ?"*. En confrontant le professeur aux interventions de ses élèves lors des séances de classe *"... une suite est soit croissante, soit décroissante"* ou encore *"... est ce que toutes les suites sont bornées ?"*, le professeur finit par se convaincre de l'importance des non exemples pour mettre en relief les différentes hypothèses d'une définition ou d'une propriété. Conscient des possibilités d'utiliser des graphiques pour donner des non exemples ou des contre-exemples (par exemple de conjectures du type si la suite est croissante et définie par une fonction f, alors la fonction f est croissante), et soucieux des limites de la visualisation pour les besoins de l'abstraction, le professeur demeure sceptique à la donnée d'exemple où le graphique de la suite montre qu'elle *P :"... dépasse toute valeur fixée à l'avance"*. En même temps qu'il semble conscient de l'importance de pouvoir traiter les suites sans un passage obligatoire par les fonctions et des manques au niveau de AS2, le professeur trouve des difficultés à se décharger des termes du programme officiel en gardant la cohérence d'une construction mathématique adaptée à l'aval du lycée. Tout en étant favorable à l'introduction des suites à partir de leurs listes de termes, le professeur semble encore hésitant sur le scénario à adopter dans sa planification future *P : "Ce genre d'exemples, où est-ce qu'on va l'introduire ?"*. De la définition de suite, les élèves retiennent essentiellement qu'il s'agit d'une fonction et omettent le reste des conditions. Ils n'hésitent pas à dériver les suites et étudier la monotonie sur plusieurs intervalles. Au début, le professeur semble désarmer devant cette situation *P : "Je ne vais quand même pas parler des intervalles discrets aux élèves ... je ne vais pas faire de la topologie ??"*. Pour le professeur, la distinction entre le discret/continu est en règle générale très abstraite à ce niveau du cursus et opte pour l'usage des potentialités du graphique pour remédier à la question *P : "Je* *vais insister sur le fait que les points sont discrets et constituent la trace de la fonction sur IN. La trace est la même quelque soit la fonction. Parce que la trace c'est la suite. La suite est la même quelque soit la fonction."* ou alors *P : " ils (les élèves) prennent différentes valeurs de n en commençant par le premier ordre, calculent leurs images et représentent uniquement ces points."*.

- Actions liées au formalisme : Le professeur est d'emblée conscient de sa prise en charge minimaliste des situations problématiques d'inversions des quantificateurs dans les propriétés de suites monotones et de suites bornées. Progressivement, il alimente les échanges par des raisons pour expliquer son action (AF1). D'abord : 1) en déclarant que les futurs étudiants *P : "... ne connaissent pas les règles de la quantification, y compris la négation"* tout en insistant sur le caractère déterminant de l'ordre des quantificateurs dans la signification des énoncés quantifiés ; puis 2) en évoquant la difficulté de trouver un exemple générique pour illustrer les différentes règles. Sur la base d'un débat sur la mise en application d'un énoncé quantifié permettant d'en saisir les différentes composantes, le professeur finit par soumettre les différents exemples à des régulations instantanées afin d'en garder ceux qui sont les mieux adaptés au contexte d'enseignement et aux attentes de futurs étudiants. Contrairement aux énoncés quantifiés, l'usage des expressions informelles en lien avec l'infini et le continu n'apparait pas problématique pour le professeur *P : "il est où le problème ?"*. Il justifie son action (AF2) par les exigences des exercices qu'il déclare ne plus pouvoir faire ! La suite des interactions avec le chercheur a amené méthodiquement le professeur à renverser la situation en faveur de l'usage d'expressions plus neutres telles que *P : "[...] Quand n augmente, quand n devient plus grand. Mais on évite d'utiliser « approche »"*. En outre, il propose d'utiliser le graphique d'une suite particulière pour initier un travail sur les valeurs approchées en prenant différentes valeurs de la précision et en cherchant les rangs n correspondants. L'exemple peut de ce fait être choisi de sorte que *P : "L'approximation se fait de manière alternée par valeurs supérieures et par valeurs inférieures"*.

- Action liée à la validation : Convaincu de l'importance de distinguer une implication de sa réciproque en soulignant que les futurs étudiants ont intérêt à comprendre que la véracité de l'une ne permet pas de confirmer l'autre, le professeur préfère quand même mettre l'accent sur la possibilité de P : "… *de ne pas utiliser les techniques par les fonctions (pour étudier la monotonie ou la majoration/minoration d'une suite)) quand l'occasion se présente"*. En insistant sur les séquences de classe en lien avec les exemples de suites monotones ou bornées, propriétés non partagées par des fonctions qui leur sont associées, le professeur préfère mettre l'accent sur les limites des techniques par les fonctions plutôt d'exploiter ces éventualités pour aborder les questions de condition nécessaire et condition suffisante *P : "C'est pour dire que ces techniques ont des limites"*. Cette dernière intervention, même si elle dénote une certaine distanciation du professeur par rapport au rôle des fonctions dans un travail avec les suites, elle montre qu'il n'est pas prêt à aborder la question des fonctions non "naturellement » associées aux suites au risque de se trouver confronter à des choix qui supplantent le programme et biaisent la rigueur mathématique.

A l’issue de cette analyse trois éléments importants peuvent être dégagés : 1) en aucun cas nous ne pouvons affirmer que nous avons pu déterminer les significations des CME du professeur même si l’entretien a été conçu pour les susciter, le mieux qu’on puisse dire est que nous avons pu en cerner certains aspects nécessaires à l’étude des adaptations potentielles des CME ; 2) quoique les CM du professeur sont suffisamment solides pour lui permettre d’opérer des changements au niveau des CME (l’usage des exemples de suites, la donnée de contre-exemples, l’usage de graphiques, l’expression informelle d’aspects topologiques discrets et continus, l’instanciation des énoncés quantifiés, la modélisation des raisonnements mathématiques) impliquées dans ces actions, l’adoption de nouvelles CME reste tributaire du contexte de l’enseignement et de son expérience d’enseignant ; 3) le niveau critique de la réflexion du professeur renvoie généralement aux normes des mathématiques savantes et fonde son engagement vers la modification de ses actions pour une meilleure prise en compte de la problématique de la transition.

CONCLUSION

La méthode que nous avons mise en place, pour dévoluer au professeur de fin du lycée la question de la transition vers l’université en analyse réelle, est fondamentalement centrée sur ce que peut apporter la démarche réflexive comme changement dans les actions du professeur. La structuration de l’entretien qui a servi de mécanisme pour déclencher la réflexivité du professeur est fondée : 1) du point de vue du découpage et du contenu, sur les aspects spécifiques de la problématique de la transition ; 2) du point de vue plus transversal, sur des interactions actives et constructives axées sur les questionnements et les problématisations pour dénouer les implicites, leurs déterminants et aboutissants. Cette technique est loin d’être finalisée, moins encore facile à réaliser, en raison de la multitude de dimensions qu’elle porte en considération (la didactique du phénomène étudié, la cognition du professeur, la complexité de l’outil entretien, etc.). Elle nous a, en outre, permis d’effectuer des analyses significatives des possibilités d’entrée du professeur de mathématiques du lycée dans le jeu de la transition vers l’université et de commencer à penser, sur une échelle plus large que cette étude, son implication dans la construction d’une passerelle entre le lycée et l’université en analyse réelle. Plusieurs études, en lien avec les connaissances du professeur et sa formation professionnelle, sont nécessaires pour espérer donner de la généricité à cette méthode.

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Programming as an artefact: what do we learn about university students' activity?

Ghislaine Gueudet[1], Chantal Buteau[2], Eric Muller[2], Joyce Mgombelo[2] & Ana Isabel Sacristán[3]

[1]CREAD, ESPE de Bretagne, University of Brest, ghislaine.gueudet@univ-brest.fr; [2]Brock University, Canada; [3]Cinvestav, Mexico

In this paper we discuss how the instrumental approach can contribute to our understanding of the activity of university students using programming in the context of an authentic mathematical investigation. We claim that they develop an instrument from programming considered as an artefact, incorporating a complex structure of schemes. We distinguish between m-schemes, p-schemes and p+m-schemes, for a goal concerning respectively only mathematics, only programming, or both. We illustrate this theoretical construct by studying the case of a student enrolled in a course encompassing programming-based mathematics investigation projects.

Keywords: Teachers' and students' practices at university level, Digital and other resources in university mathematics education, Programming, Instrumental approach, Authentic Mathematical Investigations.

INTRODUCTION AND CONTEXT

In the field of mathematics education, the use of programming for learning has a legacy of half a century that started with the designing of the LOGO programming language for learning (Papert 1972). Studies working in this area have been framed with different perspectives (e.g., see Hoyles & Noss 1992). We present here a study concerning the theoretical contribution of the instrumental approach (Guin, Ruthven & Trouche 2005) –that articulates the mutual shaping of learners and artefacts (e.g., programming) in the learning process–, to analyse the activity of university students using programming in the context of an authentic mathematical investigation.

The instrumental approach has already been used in previous research about university students' use of various technological tools: for example Sketchpad (Ndlovu, Wessels & De Villiers 2011) or CAS (Zeynivandnezhad & Bates 2018). This theoretical framework has also been used in a study about programming by Misfeldt and Ejsing-Duun (2015), but their work concerns the primary and lower secondary levels. As far as we know, the instrumental approach has never been used in a research about programming at university level; we hypothesize that it can enlighten interesting phenomena, specific from this level and from programming.

Our study is part of a five-year naturalistic (i.e., not design-based) research that takes place in the context of a sequence of three university mathematics courses, called *Mathematics Integrated with Computers and Applications* (MICA) I-II-III taught at Brock University since 2001. In these project-based courses, mathematics majors and future mathematics teachers learn to design, program, and use interactive environments to investigate mathematics concepts, theorems, and applications (Buteau & Muller 2010). The research aims at understanding how students learn to use programming for authentic mathematical investigations, if and how their use is sustained over time, and how instructors support that learning.

The research question that we will investigate here can be presented as follows: What do we learn about the activity of students using programming in an authentic mathematical investigation by using the theoretical frame of the instrumental approach, considering programming as an artefact?

In the next section we present the instrumental approach, and how we propose to use it when the artefact is a programming language. Referring to the Theory of Conceptual Fields (Vergnaud 1998), we introduce in particular three different kinds of schemes. Then we briefly present our methods, and illustrate the use of the instrumental approach by analysing the case of a student, Jim, and of his work in a project concerning number theory. Finally we discuss the insights gained from the use of this approach.

INSTRUMENTAL APPROACH, PROGRAMMING AND SCHEMES

The instrumental approach (Rabardel 1995) introduces a distinction between an artefact, which is produced by humans, for a goal-directed human activity, and an instrument, developed by a subject along his/her activity with this artefact for a given goal. The instrument is composed by a part of the artefact and a scheme of use of this artefact (Vergnaud 1998). In mathematics education, the instrumental approach has been used firstly to study learning processes of secondary school students using calculators (Guin et al. 2005). These studies used a detailed definition of schemes, following the work of Vergnaud. A scheme has four components:

- The goal of the activity, subgoals and expectations;
- Rules of action, generating the behaviour according to the features of the situation;
- Operational invariants: concepts-in-action, which are concepts considered as relevant, and theorems-in-action, which are propositions considered as true;
- Possibilities of inferences.

In a given situation, a subject mobilizes a scheme corresponding to the goal of his/her activity. The inferences permit the adaptation of the scheme to the precise features of the situation. Sometimes this adaptation can lead to the emergence of new operational invariants, new rules of action, of even to the emergence of a new scheme. The schemes of use as defined in the instrumental approach come in fact from a more general theory elaborated by Vergnaud in the context of mathematics education: the Theory of Conceptual Fields (TCF). The couple (scheme, situation) is central in this theory to analyse conceptualization processes. We refer also to this more general theory, considering not only schemes of use of “programming”, considered as an artefact, but also mathematical schemes.

In the context of our study, the general goal of the students' activity is to "investigate a complex situation (mathematical or not), combining mathematical knowledge and programming". We claim that, by using programming for this general goal, the students develop an instrument, associating some aspects of programming and schemes of use for specific subgoals (Buteau, Gueudet, Muller, Mgombelo & Sacristán 2019). We also claim that for this general goal, students mobilize different kinds of schemes. Mathematical schemes (noted m-schemes) intervene when the goal (or sub-goal) is the search for a mathematical formulation of the situation, and their interpretation of solutions. Programming schemes (noted p-schemes) intervene when the goal concerns only programming, and could also appear when programming outside of any mathematical context. Combined programming and mathematics schemes (noted p+m-schemes) intervene when the goal concerns both. Along their investigation activity, students develop a complex network of m-schemes, p-schemes, and p+m-schemes. We will illustrate this below by studying the case of Jim.

METHODS

Concerning Jim, we collected and analysed the following data:

Jim was one voluntary student participant (among 6) enrolled in MICA I course (46 students) in the first year of our research. The MICA I course consists of 4 programming-based mathematical investigation projects (which count for 71 % of students' final grades): 3 assigned individual ones, and a fourth one where students select the topic. The course format includes a two-hour weekly lab, where students progressively learn to program in Visual Basic.net (vb.net) in a mathematical context, and two-hour weekly lectures that introduces students to the mathematics needed for their investigation project assignments (Buteau, Muller, & Ralph 2015).

Jim's data included his 4 project assignments (that include each, a computer program and accompanying report) and individual semi-structured interviews after completion of each of his projects. The interview guiding questions were informed by a development process model (referred onwards as 'dp-model'), established in previous works (e.g., Buteau & Muller 2010; Buteau et al. 2019), which describes an individual student's activity in the context of an authentic programming-based mathematical investigation (Figure 1). Jim's data also included weekly post-laboratory session online reflections (with guiding questions) and an initial baseline online questionnaire, followed by an interview, before the beginning of MICA I. We also collected all course material, including lab session and assignment guidelines. For this study, we focused on Jim's baseline questionnaire interview, his first 4 lab reflections, and his first assignment project and interview.

We analysed Jim's interviews by trying to observe in his declarations elements of schemes: goal of the activity; description of how he acted in the situation; reasons for acting this way; and inferences. How he acted can be interpreted as rules of action, if it is described by Jim as a regular practice. If it is described as something new, an original attempt, it can be interpreted as the emergence of something that can later become a rule of action. The reasons for acting regularly in a certain way are interpreted as operational invariants: theorems-in-action, and associated concepts-in-action. We present examples in the next section.

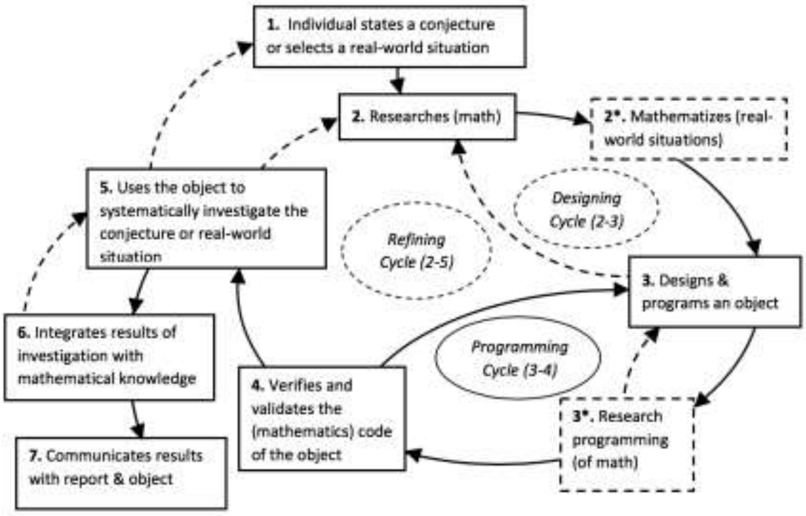

Figure 1: Development process model of a student engaging in programming for an authentic mathematical investigation or application (Buteau et al. 2019).

In this exploratory study, we did not have the possibility to directly observe Jim's work, in order to confront his declarations and his actual activity. Only a part of this activity was accessible through the assignment he produced. This is certainly a limitation of our ongoing naturalistic study, but on the other hand it incorporates all institutional constraints of Jim's activity. We mitigate this limitation by triangulating all the data available on Jim (listed above), and as such, we suggest that our analysis provides significant evidence of Jim's instrumental genesis.

THE CASE OF JIM

The first four weeks of the MICA I course prepare students for their first project assignment. In lectures, students are mainly exposed to prime numbers and hailstone sequences, and to conjecturing about those concepts. In lab sessions, students start learning about basics of programming in vb.net: variables, loops, conditional statements, and create, read from, and print in a graphical user interface (GUI). Starting in lab 3, students are progressively guided to code mathematics; e.g. in lab 3, the code for checking the primality of an integer is given to them for reproducing (and fixing a minor issue) whereas lab 5 guidelines gives a partial pseudo-code for powers in Z_n. The first project directly builds on lab 3 and asks students to state or to select a conjecture about primes, and create a program in vb.net to investigate it.

In this section we present examples of schemes identified in the case of Jim for his first project assignment, chosen to illustrate the three kinds (m-schemes, p-schemes and p+m-schemes). The schemes are presented with general aims: indeed they apply in the context of this assignment, but they are an invariant organization of the activity for all the situations corresponding to this aim. We attempt to give for each scheme its general description, and elements about its application in context, involving precise mathematical and programming contents. The different elements of the schemes are inferred from the description Jim gave of his activity. For each scheme, we firstly present its main elements in a table, and then comment and discuss this table by drawing on excerpts of Jim's interview. There is no inference mentioned in the tables since none were identified for these examples of scheme.

Formulate a conjecture: example of a m-scheme

Rules of action	Investigate the math concept (search on the Internet, take notes); Search for a representation; Search for a pattern
Concepts-in-action	Representation; Pattern
Theorems-in-action	Understanding related concepts helps to formulate a conjecture; An appropriate representation is helpful to find a pattern; I can learn mathematics by exploration

Table 1: Jim's scheme of formulating a conjecture.

The scheme presented in table 1 is a mathematical scheme, since it corresponds to the goal "Formulate a conjecture". According to Jim, he started by trying to understand better the concept of primes.

> Jim: At first I was trying to kind of think of trying to understand more about the nature of primes before I would really do my conjecture (#2)

We interpret this as a rule of action, governed by a theorem-in-action: "to formulate a conjecture, a good understanding of the concepts involved is needed". It is possible to consider the goal: "investigate a mathematical concept" as a sub-goal, for Jim, of the goal "to formulate a conjecture".

After this first step, Jim tried to represent the prime numbers and to observe a pattern.

> Jim: me trying to figure out this conjecture basically where I would plot out the primes and look for any patterns of how they worked (#3)

We interpret this again as a rule of action, probably developed along many problems in mathematics. The concepts of "representation", "pattern" are relevant for Jim in this situation and guide his activity: they can be considered as concepts-in-action (which are explicit here). In the specific case of prime numbers, he started by representing them on a line (we interpret this as a rule-of-action for the sub-aim "formulate a conjecture about primes). He observed that it did not work, and that a two-dimensional representation was more relevant.

Articulate in a programming language a nested process: example of a p-scheme

Rule of action	Code nested loops articulating the nested process; Code them incrementally
Concepts-in-action	Nested system; Nested loops; Loop

Theorem-in-action	A nested system can be processed by programming technology as nested loops; Incremental coding helps to properly structure the nested loops

Table 2: Jim's scheme of articulating in vb.net a nested process.

The scheme presented in table 2 is a p-scheme, since it corresponds to the goal "Articulate in vb.net a nested process". Jim seems to suggest that the MICA course facilitates students to develop a scheme of "articulating, in vb.net, a process involving repetitions"—with a main rule-of-action: "to code loops"—and that through this assignment project, he (and his fellow students) had to then further elaborate it by developing a more general scheme of "articulating, in vb.net, a nested process".

Jim: We actually went over how to build this kind of system [involving repetitions] in class. So the only thing new about the project was kind of learning how to nest them, properly structure them, to make this running program. (#18)

According to him, Jim codes nested loops to articulate the nested process. We interpret this as a rule of action, governed by: "a nested system can be processed by vb.net programming technology as nested loops". In addition, Jim seems to indicate coding such nested loops incrementally— a rule of action that we could associate to a theorem-in-action: "Incremental coding helps to properly structure the nested loops".

Jim: to understand this idea of nested kind of system and how to build upon a single system into multiple ones …Like one system inside another and I think that's pretty key but you kind of just have to work with it and hope it works out... It [is] one of those inherent things. (#27)

In this situation, we identify "nested system", "nested loops", and "loops" as explicit concepts-in-action in Jim's activity. Furthermore, this scheme seems to be, for Jim, at the core of programming. As such, this suggests Jim's awareness of mobilizing or developing it further in his future programming-based mathematical investigations.

Articulating a mathematical process in programming: example of a p+m scheme

Rules of action	Organize the math process as a nested system; Decompose the nested system in individual processes before programming; Code individual processes; Start by 'translating' in vb.net what I would do by hand
Concepts-in-action	Mathematics & programming as a nested system; Solving-by-hand method; Decomposition of a system; Individual process
Theorems-in-action	A mathematical process can be seen as a nested system, i.e., made of many parts; To program a nested mathematics process, one can break it down and individually code the smaller parts; A programming language can work in a similar manner as one works by hand; Programming and mathematics as systems have embedded layers

Table 3: Scheme of articulating a mathematical process in the programming language.

Table 3 presents an example of a p+m-scheme: articulating a mathematical process in programming. It is a p+m-scheme because its goal involves both programming and mathematics. In describing how he approached the assignment Jim noted,

> Jim: I basically tried to organize and sort out what needed to be programmed but I kind of realized as I was going, I kind of knew everything that needed to be done. It just required a set of system nested within each other so once I know that I had to figure out how to program each individual system. This one check for prime. This one is a loop … that sort of thing. (#8)

We interpret this description by Jim as indicating many rules of action, such as "Organize the mathematics process as a nested system", which is supported by a concept-in-action, "mathematics and programming as a nested system" and a theorem-in-action, "a mathematical process can be seen as a nested system", i.e., made of many parts. Two other rules-of-action are: "Decompose the nested system in individual processes before programming" and "Code individual processes"; they are supported by two concepts-in-action, "decomposition of a system" and "individual process", and theorems-in-action, "to program a nested mathematics process, one can break it down and individually code the smaller parts" and "Programming and mathematics as systems have embedded layers". For example in this case he identified a part of the program checking primality.

Jim also described his coding by enumerating different processes that seem to align with a by-hand method. We interpreted it as indicating Jim's rule-of-action "Start by 'translating' in vb.net what I would do by hand", governed by a theorem-in-action "A programming language can work in a similar manner as one works by hand", identified in lab3 as potential components since they were not yet put into action.

> Jim: [I] do believe that… I would have…been able to make something resembling it … as the logic of how it searches for primes has already been ... in class. (Lab3)

DISCUSSION AND CONCLUDING REMARKS

The research question studied in this paper was: "What do we learn about the activity of students using programming in an authentic mathematical investigation by using the theoretical frame of the instrumental approach, considering programming as an artefact?". Drawing on the example studied above, we discuss here elements of answer to this question, and indicate directions for future research.

Firstly, we claim that this example of the activity of students using programming in an authentic mathematical investigation illustrated the relevance of the different kinds of schemes: m-schemes, p-schemes, p+m-schemes. Gerianou and Janqvist (2019) argue that the theory of instrumental genesis, and schemes in particular, allow to bridge mathematical competencies and digital competencies. Our study illustrates and confirms this, in the specific case of programming technology. The p+m-schemes can be considered as bridging mathematical and digital competencies; the identification of p+m operational invariants in particular deepens our understanding of how mathematics and programming relate to each other.

This statement could apply to any context of learning programming in a mathematics course. Our second claim concerns specifically the university context, and the type of activity proposed in the MICA course: using programming for an authentic mathematical investigation. The different schemes developed by students along this course are inter-related; they constitute a complex structure. We mentioned above the dp-model describing students' activity in the context of an authentic programming-based mathematical investigation (Figure 1). We claim that the schemes developed by a student are related to the different steps of this model (Buteau et al. 2019). In the example presented above, the m-scheme: "to formulate a conjecture" corresponds to step 1; while the p-scheme: "articulate in a programming language a nested process" and the p+m-scheme: "articulate a mathematical process in the programming language" correspond to step 3 (see Figure 1). In our analysis of the case of Jim, we identified schemes corresponding to all the steps of the model (which cannot be presented here, for the sake of brevity). The steps of the dp-model can be considered as general goals of the students' activity; they correspond to what Rabardel (2005) calls "activity families", gathering situations with a similar aim of the activity.

Another direction for investigating how different schemes are linked concerns the level of generality of the goals. We could consider, for example, that "Designing and programming an object" is a goal (step 3 of the dp-model), and thus associated with a single scheme (a p+m-scheme, in this case). The schemes: "articulate in a programming language a nested process" and "articulate a mathematical process in the programming language" would then appear as sub-schemes (associated with sub-goals). In our study we have made a different choice, since we were interested in looking for precise operational invariants in particular. A study in terms of schemes always needs to choose a "favoured" level of generality; it is then possible to consider more precise goals, and obtain sub-schemes. For example in the case of Jim, we also identified a scheme labelled: "To formulate a conjecture about primes", which is a sub-scheme of the m-scheme presented above.

An important issue requiring further work concerns the complex dialectics between stability and evolution of the schemes developed by the students and of their components. The data that we analyzed for this paper did not include a long-term observation of Jim's activity. This had several consequences on our analyses in terms of schemes. Firstly in most cases (in particular in the examples above) we were not able to describe the "inference" part of the scheme. Indeed the inferences are linked with particular features of the situation, not always described in an interview. Second, some of the rules of action and operational invariants described above can be considered as stable while others are only "potential", since more evidence would be needed to acknowledge their stability; we provide below some examples.

We consider that the students in the MICA course (and Jim in particular) already had stable m-schemes: for example they might have met before situations in mathematics classes where they would have needed to formulate a conjecture. These m-schemes will be adapted to the features of the new situation involving programming, but the organization of the activity will remain stable. In contrast, programming was for them a new activity; the p- and p+m-schemes we identified have most likely been developed during the MICA course.

Nevertheless we claim that some of the rules of action and operational invariants in p- and p+m-schemes are already stable at the end of the MICA course. Indeed the intervention of some of them has been observed on several occasions, and we consider that this acknowledges their stability. For example, we found evidence that, in his second and third MICA projects, Jim seems to utilize his rules of action of the p+m-scheme described above, while also developing additional rules-of-action (e.g. "I ignore coding special cases of the mathematics process that are not needed for the mathematical investigation") and theorems-in-action (e.g. "Special cases in the mathematics code potentially leading to bugs but that don't affect the mathematical investigation, can be ignored").

Some authors have researched the development and evolution of schemes (e.g. Coulet 2011), and consider that along his/her activity, the subject receives feedback which can lead to three kinds of evolutions. Productive loops lead to changes in the rules of action; constructive loops lead to changes in the operational invariants; changes scheme loops can even lead to a new scheme. Studying the stability of potential rules of action and operational invariant requires the study of these loops; this is a perspective for future research.

Finally, another issue requiring further work concerns the nature of the operational invariants. As mentioned earlier, the general goal of the students' activity in the MICA course is to "investigate a complex situation, combining mathematical knowledge and programming". Some of these students, Jim in particular, developed an instrument for this goal from the programming artefact. We hypothesize that these students have developed a theorem-in-action like: "Using programming, I can be creative in mathematics" (and some indices of such a theorem-in-action appear in Jim's interview). This kind of proposition is strongly linked with students' self-confidence and affects. We wonder if such components are involved in schemes.

The links between mathematical and programming competencies are complex and increasingly important at university in several strands: for mathematics majors, but also for future engineers etc. The approach we propose here with the instrumental approach can enlighten these links. Thus we consider important to investigate the future directions evoked above: in particular observe students' activity on a long term, in different contexts, to deepen our knowledge of the schemes they can develop, of their evolutions, and of the complex structure of scheme systems.

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Exploring the affordances of Numbas for mathematical learning: A case study

Said Hadjerrouit

University of Agder, Kristiansand, Norway, Said.Hadjerrouit@uia.no

The computer-based assessment system Numbas offers new learning possibilities in mathematics education by means of formative feedback. The paper proposes a theoretical framework that captures the affordances emerging from students' interactions with Numbas at the technological, mathematical, and assessment level. The aim of the paper is to explore the affordances that arise from these levels using the theoretical framework as a lens. Based on the results, preliminary conclusions and recommendations for future work are proposed.

Keywords: Affordance, feedback, interaction, mathematics learning, Numbas, teacher education.

INTRODUCTION

Although several research studies provide compelling examples of computer-based assessment systems in mathematics education, there has yet to be systematic investigations into how affordances of the systems might support mathematical learning. Numbas is a computer-based assessment system that provides formative feedback to students on whether their answer to a task is correct and reveals solutions to the students' submitted mathematical formulas (Perfect, 2015). Formative feedback allows students to see their own progress and change their mathematical thinking. Students' interactions with Numbas by means of formative feedback create affordances for learning at the technological, mathematical, and assessment level. This article uses Gibson's affordance theory to analyze the affordances that arise at these levels in a course on digital tools in mathematics teaching.

THE COMPUTER-BASED ASSESSMENT SYSTEM NUMBAS

Numbas is a computer-based assessment system with an emphasis on formative feedback. It is developed at the university of Newcastle (UK), and it is used at over 30 institutions around the word (Perfect, 2015). Formative feedback in mathematics education is normally given by a teacher, but it could be viewed as being the result of the student's interaction with any digital or non-digital learning milieu. If the student's action changes the milieu that provides feedback, this very change may cause the student to reconsider her action (Brousseau, 1997). Shute (2008) identified two main functions of formative feedback. Verification of whether an answer is correct, and elaboration to provide relevant cues to guide the learner towards the correct answer. Grounded on the idea of formative feedback, the primary design goal of Numbas is to enable a student to enter a mathematical answer in the form of an algebraic expression and submit the answer. Numbas provides feedback to the student on whether the submitted answer is correct or incorrect and generates information according to the students’ submitted expressions. Numbas user interface offers several options: *Enter an expression* in the input field, “*Submit all part”* of the answer or “*Submit part*” of the answer, “*Show steps*” of the solution, “*Try another question like this one”*, and “*Reveal answers*”, which gives the solution to the question. Numbas has also a marking scheme for the answer, e.g., “*1 mark*” for submitting part of the answer, and “*2 marks*” all parts (Figure 1 and 2).

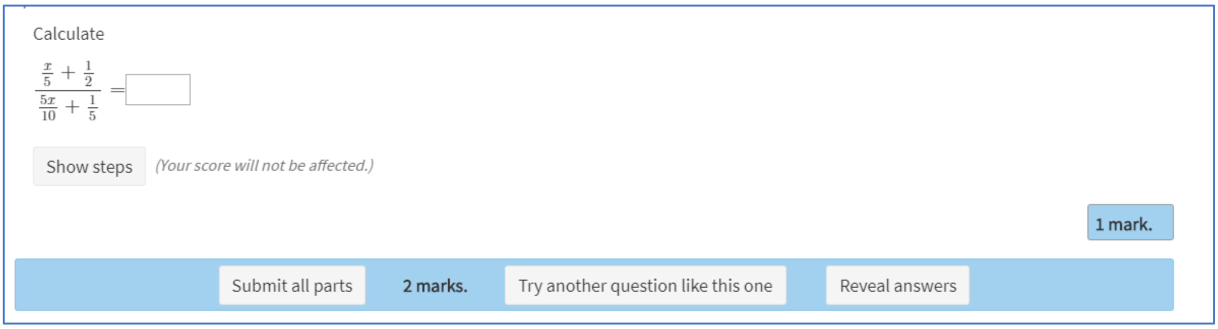

Figure 1: Numbas user interface showing a task at grade 8-13

Like many other computer-based assessment systems with automated feedback, Numbas involves a set of tasks or questions similar to paper-based tests (Van der Kleij et al., 2015). Numbas supports several question types, e.g., number entry, multiple choice/multiple response, and text entry. Questions are randomized according to a set of variables, defined by mathematical expressions (Perfect, 2015). A question consists of a statement, one or more parts, and an advice section, which contains a fully worked solution to the question and is revealed once the student has finished the question.

While many other mathematics assessment systems have a dependency on server-side computer algebra systems, Numbas runs entirely on the client-side, which means that the feedback is either immediate and very fast or with a small delay (Perfect, 2015). Numbas contains a computer algebra system written in JavaScript so that expressions can be interpreted and evaluated entirely on the client-side. Numbas uses a simple syntax similar to the one used on a calculator. Moreover, the student’s expression is rendered in a mathematical notation beside the input field and updated immediately as the student types to make sure that Numbas has interpreted her expression as she intended (Figure 2).

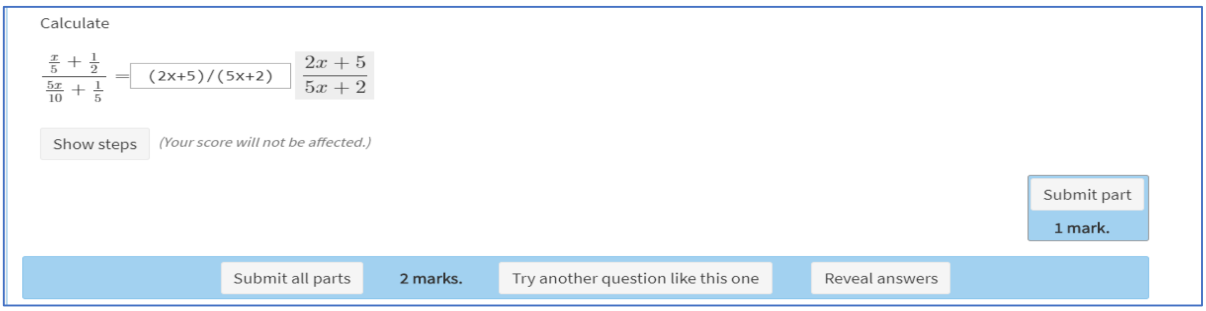

Figure 2: The student’s answer is rendered immediately next to the input field

AFFORDANCE THEORY: A FRAMEWORK FOR ANALYSING STUDENTS' INTERACTIONS WITH NUMBAS

Affordance theory, originally proposed by Gibson (1977), refers to the relationship between an object's physical properties and the characteristics of a user that enables particular interactions between user and object. Affordance theory states that the world is perceived not only in terms of object properties but also in terms of object *action possibilities* offered to an animal by the environment with reference to the animal's action capabilities. A typical example is a tall tree that has the affordance of food for a giraffe because it has a long neck, or a chair that affords a human a possibility of sitting. Hence, affordances are not a set of characteristics that are inherent to the object and independent of the user. Rather, an affordance is ontologically neither an objective nor a subjective property, or in other words, affordances are *relational emergent properties of the user-object interaction*. Likewise, De Landa (2002) emphasizes that affordances are not intrinsic properties of the object or subject. Rather affordances emerge from the interaction between the subject and object and become actualized in a specific context.

Norman (1988) adapted the concept of affordance in the computer world. Accordingly, an affordance is the design aspect of an object which suggests how the object should be used and a visual clue to its function and use. Examples of affordances are user interface elements that directly suggest suitable actions such as clickable geometrical figures, draggable sliders, pressable buttons, selectable menus for figures, etc. Several research studies used the concept of affordance in various educational settings. For example, Turner and Turner (2002) specified a three-layer articulation of affordances: Perceived affordances, ergonomic affordances, and cultural affordances. Likewise, Kirchner et al. (2014) described a three-layer definition of affordances: Technological affordances that cover usability issues, educational affordances to facilitate teaching and learning, and, social affordances to foster social interactions. In mathematics education, Chiappini (2013) applied the notion of perceived, ergonomic, and cultural affordances to Alnuset, a digital tool for high school algebra. Finally, Hadjerrouit (2017) presented a model of affordances in teacher education.

Based on the literature on affordances and the features of Numbas described in the previous section, this work uses Gibson's affordance theory and proposes a set of affordances that may emerge from students' interactions with Numbas at the technological, mathematical, and assessment level. Considering affordances as emergent properties means that Numbas does not have any affordances, except in interaction with students.

Given this background, several technological affordances could emerge as students interact with Numbas. These are: a) Numbas is user-friendly and accessible at any time and place. b) Numbas facilitates accurate and quick completion of mathematical activities. c) Numbas helps to draw graphs and functions, solve equations, construct diagrams, figures and shapes.

Likewise, several affordances could potentially emerge from the mathematical level.

a) Numbas presents the mathematical content in several ways using text, graphs, symbols, interactive diagrams, and dynamic visualizations. b) Numbas facilitates the transformation of mathematical expressions that support conceptual understanding. c) Numbas supports various mathematical activities such as problem solving, multiple choice, and quizzes. d) Numbas mathematics is congruent with textbook mathematics. e) Numbas provides high quality of mathematical content and useful questions that foster higher-level mathematical thinking. f) Numbas displays formulas, functions, graphs, numbers, algebraic expressions, and geometrical figures correctly. g) Numbas simplifies mathematical expressions so they look as they are on paper.

Finally, several affordances could potentially emerge from the assessment level. These are: a) Numbas provides several assessment tests. b) The order and wording of the assessment questions are appropriate. c) Numbas questions are useful and relevant to test mathematical knowledge. d) Numbas gives immediate feedback. e) Numbas provides several types of feedback, e.g. expected answers and advices to the solution. f) Numbas feedback contains useful information to understand the tasks and answer the questions. g) Numbas gives hints in form to problem solving. h) Numbas is flexible to handle a wide range of assessment questions, answer to a question, and whether it is correct or not. i) Numbas provides a summary of the test, statistics on students' answers, what they have done wrong or right, their performances and grading.

THE STUDY

Context, participants and research questions

This study was conducted in a course on digital tools in mathematics teaching at the University of Agder in Kristiansand, Norway. The participants (*N=15*) were three categories of teacher students following teacher education programmes for grade 1-7, 5-10, and 8-13. None of the students had any prior experience with Numbas. The study addresses two main research questions:

- Which affordances emerge from students' interactions with Numbas at the technological, mathematical, and assessment level?
- How do students value the formative feedback provided by Numbas?

Methods

Both quantitative and qualitative methods were used to analyze the affordances that emerge from the students' interactions with Numbas:

- A survey questionnaire with a five-point Likert scale from 1 to 5, where 1 was coded as the highest and 5 as the lowest (1 = "Strongly Agree"; 2 = "Agree"; 3 = "Neither Agree nor Disagree"; 4 = "Disagree"; 5= "Strongly Disagree"). The average score (MEAN) and the standard deviation (Std. Dev) were calculated
- Students' comments in their own words on each of the survey items, and qualitative analysis of the comments

The survey questionnaire was guided by the theoretical framework and the set of potential affordances described in the previous section. The qualitative analysis was based on indicators of affordances from students' comments on the survey items in interaction with the affordances of the theoretical framework and search for meaningful interpretation of the results in the context of the course, the participants and the teacher.

Numbas mathematical tasks

Mathematics tasks at grade 1-7, 5-10, and 8-13 for a period of two weeks were given to the students using Numbas. These are some examples of tasks at grade 1-7:

Answer these mathematical expressions:

a) $3-6$	b) $3-9$	c) $3+6-9$

These are some examples of tasks at grade 5-10:

You may write your expression in the text box. Remember to write for example a^3 to get a^3 and 2a-3b to get $2a-3b$. Next to the text box, you will get an image of how Numbas reads your input. Make sure this is what you intended to answer!

a) Simplify $-2a-3b+4a+7b$	b) Simplify $2a^2+6a+2+7a^2-5a-3$ c) Dissolve the parenthesis and simplify $(6a+4b)-(5a-5b)$

These are some examples of tasks at grade 8-13 (See also figure 1 and 2):

Calculate and give your answer as a fraction or an algebra expression. Use / as the fraction line, $\frac{x-2}{3}$ *is written as* (x-2)/3 *and* $\frac{x+1}{x-4}$ *as* (x+1)/(x-4)

a) Calculate $\dfrac{\frac{x}{2}+\frac{1}{4}}{\frac{3x}{8}+\frac{1}{2}}$	b) Calculate $\frac{x+3}{5x}-\frac{2x-4}{4x}$

RESULTS

Technological affordances

The results (Table 1) show that most students indicated that Numbas has a user-friendly interface and that it is easy to use (Item 1, 2). It has also a ready-made mathematical content, and it is accessible anytime and anyplace (Item 3, 4). This is reflected in many students' comments such as: "*easy and fine design*"; "*easy to find and navigate through the information*"; "*very positive that we get immediate feedback*".

	Mean	Std. Dev
1.Numbas is user friendly	1.80	0.561
2.Numbas is easy to use	1.73	0.704
3.Numbas is accessible anytime and anyplace	1.67	0.724
4.Numbas has a readily available content	1.80	0.775

Table 1: Technological affordances

Mathematical affordances

In terms of mathematical affordances (Table 2), most students agreed that Numbas provides a high quality of mathematical content, and that the questions are well-designed and formulated (Item 1, 2). Likewise, more than the majority of the students found that Numbas displays mathematical notations correctly, and that it simplifies mathematical expressions (Item 7, 8). Many students also think that Numbas provides opportunities to foster mathematical thinking through various entry points, such as "*Submit answer*", "*Try another question like this one*" or "*Reveal answers*" and choosing different questions (Item 6). Furthermore, less than the majority of the students answered positively item 3, 4, and 5, but there is a relatively big variation in their responses (Std. Dev=1.014 and 1.033).

A qualitative analysis of the students' comments reveals two main themes, which are important to deepen some students' responses to the survey (See Table 2, item 1, 3, 5, 7, 8). These are: congruence of Numbas with paper-pencil techniques and textbooks mathematics, and Numbas mathematical tasks and questions. Some relevant comments on congruence are:

> *"It was very good that one could enter the formulas in the fields and calculate the answer here"*
> *"Very good that Numbas writes my answer as you see it on paper even though I write it differently"*
> *"I use paper to figure out the answer in different steps and enter, not just the answer*"

In terms of the questions and tasks provided by Numbas, students think that this issue is to a great degree dependent on the teacher and her knowledge and the way she designed the tasks and questions. Errors in Numbas may also foster reflections. Some selected students' comments are:

> *"Depends on how the questions are asked"*
> *"Again, it really depends on whether the teacher has designed and programmed the questions correctly. On the other hand, errors in the program can also help to stimulate reflection if they try to understand what has gone wrong"*
> *"The degree to which Numbas responds to the questions above, is entirely dependent on the teacher who creates the questions, (...) the program is very flexible in terms of how to create these, as it has advanced features, (...)"*
> *"(...) I think what is very positive is the given response to answers, the possibility of hints and of showing the solution. This information can help students to reinforce their understanding. The teacher has a lot of power to control how Numbas will affect the student."*

It appears from the students' responses that mathematical affordances depend mostly on how the teacher has formulated the tasks, even though an intuitive user interface and the way feedback is designed are pre-requisites for mathematical affordances. The congruence of Numbas with paper-pencil techniques may also facilitate mathematical understanding as mentioned above. Moreover, errors in Numbas mathematics do not automatically hinder student learning if they are able to exploit the limitations of the system to provoke their mathematical thinking.

Mathematical affordances	Mean	Std. Dev
1.Numbas questions are well-designed and formulated	1.87	0.516
2.Numbas has a high quality of mathematical content	2.20	0.775
3.Numbas exercises and questions are useful to foster reflections, metacognition, and higher-level mathematical thinking	2.80	1.014
4.Numbas provides opportunities to exploit the constraints and limitations of the tool to provoke students' mathematical thinking	2.60	0.737
5.Numbas provides opportunities to focus on conceptual understanding of mathematics rather than procedural skills	2.93	1.033
6.Numbas provides opportunities to foster mathematical thinking through various entry points, different questions and exercises	2.27	0.799
7.Numbas is mathematically correct. It displays formulas, graphs, functions, numbers, expressions, and geometric figures correctly	2.07	0.884
8.Numbas simplifies mathematical expressions so they look as if we wrote them on paper	2.29	1.069

Table 2: Mathematical affordances

Affordances emerging from the assessment level

In terms of affordances emerging from the assessment level (Table 3), most students think that the order of the questions is appropriate when trying another one (Item 1). Likewise, the language and wording of the questions are understandable (Item 2), indicating overall satisfaction with the teacher who creates the questions. Most students also think that Numbas gives immediate feedback to the students, and it provides several types of feedback such as expected answers and advices to the solution (Item 4, 5). Likewise, for most students, Numbas provides a summary of the test, students' answers to questions, and what they have done wrong or right (Item 8), and in a lesser degree whether the answer is correct (Item 9), but there is an important variation in their answers (Std. Dev=1.033). Likewise, some students did not find that Numbas feedback contain useful information that helps them understand the tasks and answer the questions (Item 6). Hints in form of videos to problem solving were not always useful (Item 7), but there is a huge variation in their answers (Std. Dev=1.351). Moreover, most students think that Numbas provides a summary of the test (Item 10), and statistics on students' answers to questions and their performances and grading (Item 11). Finally, it should be noted that Numbas does not take the profile and knowledge level of the student into account or serve up appropriate questions (Item 8).

The most important themes that emerged from the qualitative analysis are the quality of the feedback of Numbas and the role of the teacher in formulating the questions and designing the feedback. The following comments show that some students were not satisfied with the quality of the feedback and the way it is implemented to support mathematical understanding.

"That said, the forms of feedback both to the students and to the teachers, as far as I can see, are not very good, and therefore should not be based on such tests alone"

"The fact that the students receive feedback right away is positive, which means that they can make self-assessments, but if the test is difficult, the result will not come from Numbas"

"To some extent, it might have been better that Numbas gives more concrete feedback if I had made an obvious mistake as for example, a wrong sign"

"Numbas measures right / wrong, and has little focus on process of conceptual understanding, even though one can object that if a student gave the correct answer, he might have understood the mathematical concept"

Beyond the feedback, some students emphasized the role of the teacher in providing meaningful mathematical tasks and questions, in line with students' comments on mathematical affordances.

"In Numbas, I feel that the students get a little more control over their own test result as they can choose how much help and support they want themselves. Again, I think this depends much on the design of individual tests that determine the degree to which feedback satisfies the needs of individual students"

"Item 1 and 2 depend on the teacher who creates the questions, but it can be added that Numbas offers the possibility of letting the order of questions be random"

Affordances at the assessment level	**Mean**	**Std. Dev**
1.The order of the questions is appropriate	1.80	0.775
2.The language and wording of the questions are understandable	1.60	0.632
3.The questions are useful and relevant to test my own knowledge	2.20	1.014
4.Numbas gives immediate feedback to students' answers	1.53	0.915
5.Numbas provides several types of feedback such as expected answers and advices to the solution	1.93	0.884
6.Numbas feedback contains useful information that helps me understand the exercises and answer the questions	2.47	0.915
7.If get stuck I can ask Numbas to give a hint in form of videos to problem solving step by step	2.14	1.351
8.Numbas takes the profile and knowledge level of the student into account and serves up appropriate questions	3.13	0.915
9.Numbas shows the answer to a question, and whether it is correct	1.93	1.033
10.Numbas provides a summary of the test, students' answers to questions, and what they have done wrong or right	1.47	0.743
11.Numbas provides statistics on students' answers to questions and their performances and grading	1.77	0.832

Table 3: Affordances at the assessment level

DISCUSSION AND PRELIMINARY CONCLUSIONS

It is acknowledged that this study cannot be generalized due to the limited number of participating students. However, some answers to the research questions and preliminary conclusions can be drawn. First, affordance theory revealed to be a useful framework to analyse affordances that arise from students' interactions with Numbas.

Second, several affordances have emerged from the technological, mathematical, and assessment level, but not all potential affordances were actualized. Another educational context might also open the possibility for the emergence of new affordances. Technological affordances are pre-requisites for a trouble-free interaction with Numbas for mathematical learning and assessment purposes. An intuitive and user-friendly interface is thus an important feature of Numbas in line with the primary design goal of the system. In terms of mathematical affordances, the study shows that Numbas has a high level of mathematical content that is correct, sound and congruent with textbook mathematics. It helps to test a great variety of knowledge from primary to secondary mathematics. At the assessment level, Numbas provides several types of feedback to test students' mathematical knowledge, e.g. immediate feedback on correctness of the answer, advice to the solution, summary of the test, and grading.

Third, the paper shows that the immediate feedback of Numbas provides help and hint to test a broad spectrum of mathematical knowledge. As a result, the participating students benefited from Numbas feedback and associated tasks with varied levels of difficulty across grade 1-7, 5-10, and 8-13. The results are in line with the research literature that shows that Numbas improved student experience, increased student engagement and enjoyment (Caroll et. al, 2017; Perfect, 2015). Through the feedback it gave to students, Numbas proved to be a useful formative assessment tool.

Fourth, Numbas in its current form is suited for use as a large question bank to practice basic mathematical skills. Moreover, Numbas is most suitable in a formative mode rather than summative, which occurs at the end of an activity without the possibility for students to change their actions. In a formative mode, students can receive immediate feedback to their answers with additional and useful information, and each question has a "*Try another question like this one*" button, which generates a new question of the same type. Clearly, formative feedback is an essential component of assessment, and it can take many forms, e.g., immediate feedback to students' actions, advices to solutions that combine various information, and solution step by step.

Some students felt nevertheless that Numbas is limited to wrong/right answers, and it should provide more information that fosters conceptual understanding. Highlighting where a student has gone wrong, and not only right or wrong, giving a fully and detailed working solution to a task with additional information on appropriate strategies would provide more powerful feedback that can make students more confident in their mathematical learning (Clark, 2012; Hattie and Timerley, 2007). However, more creative problem-solving tasks and more powerful feedback are much harder to automate. Hence, work remains to be done to provide tasks that foster metacognition, conceptual understanding, and high-order mathematical thinking.

Finally, the role of the teacher cannot be underestimated for an effective use of Numbas. The teacher is a key stakeholder in formulating meaningful tasks and questions to ensure that students acquire basic skills in mathematics. Clearly, the quality of Numbas feedback depends on the preparatory work and improvements done by the teacher to design, implement and refine purposeful mathematical tasks.

Future work will produce a more elaborated and detailed analysis of the study to ensure more reliability and validity of the results and conclusions in accordance with the theoretical framework. The analysis will include students' responses and comments to open-ended questions to provide a richer empirical basis.

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Learning collaboratively with digital instructional media: Students' communicational behaviour and its influence on learning outcome

Daniel C. Heinrich[1], Mathias Hattermann[1], Alexander Salle[2], and Stefanie Schumacher[2]

[1]Paderborn University, Germany, dch@math.upb.de

[2]Osnabrück University, Germany

In this paper, we present a theoretical instrument based on the interactive-constructive-active-passive framework to gauge the interactivity of dyads' communication processes in collaborative face-to-face learning scenarios. Subsequently, we show the applicability of the instrument to time-sampled video recordings of 63 pairs of students from different fields of study who learn descriptive statistics at tertiary level with different digital instructional media (e.g. video tutorials).

By relating the students' interactivity of their communication processes to their performance in pre- and post-tests we can show that a significant link between their communicational behaviour and the learning benefit exists across different kinds of digital learning materials and different fields of study.

Keywords: (Teachers' and) students' practices at university level, digital and other resources in university mathematics education, communication, collaborative learning, digital instructional media.

THE MAMDIM-PROJECT

During the last years, an increasing number of digital instructional media like video tutorials or commented presentations are used in university level mathematics courses across different fields of study (engineering, psychology, economics, …). At the same time, a lack of research regarding the pedagogical design of digital learning resources and their collaborative aspects is stated (Borba, Askar, Engelbrecht, Gadanidis, Llinares, & Aguilar, 2016). To address this situation, the mamdim-project (*learning mathematics with digital media during the transition from secondary to tertiary education*) explores the usage and benefit of different digital instructional media. In cooperation with four German universities (University of Applied Sciences Pforzheim, Offenburg University of Applied Sciences, Bielefeld University and Brandenburg University of Technology) a total of almost 300 students of different fields of study learned with different instructional media dealing with descriptive statistics in different social forms (in dyads or as single learners). The data used in this research report is taken from the main study which took place in 2016 and 2017. Its design (pre-test | intervention | post-test) had been improved after a pilot study in 2015.

THEORETICAL BACKGROUND AND RESEARCH QUESTIONS

Spoken language and the associated communication processes play an important role in mathematics education research; an overview of the contemporary research can be found in Morgan, Craig, Schuette & Wagner (2014). Steinbring (2015) explains that students' learning and understanding of mathematics is driven by interaction, which especially depends on the use of language and communication processes. Thus, it is not surprising that one of the main benefits of collaborative learning situations is the possibility to share ideas verbally with a learning partner. On the one hand, this advantage has been mentioned and the benefits of collaborative learning situations have been shown in many publications throughout the years (Dillenbourg, Baker, Blaye & O'Malley, 1996; Slavin, 1995).

On the other hand, a non-negligible number of studies show that collaborative learning in small groups does not lead to greater learning outcomes automatically (Barron, 2003). Indeed, our own study finds no significant differences between single learners and dyads in their post-test scores (for details, see Salle, Schumacher & Hattermann, 2019). Even a meta study by Lou, Abrami, Spence, Poulsen, Chambers, & d'Apollonia (1996) finds that in 28 % of published studies on that subject collaborative learning situations have zero or even a negative effect.

While these results might seem contradictory, they can be explained by use of the ICAP-framework by Chi and Wylie (2014) described in the following paragraph. For the origins of the framework, see Menekse, Stump, Krause & Chi (2013). Our instrument to assess communication within dyads is based on this framework.

Engagement activities, the CAP-hypothesis and the ICAP-framework

The ICAP-framework is designed to explain which types of interactions between learners are most effective in small group learning scenarios. To use this framework on a given situation, the overt learning activities of the students must be identified. Those observable interactions between learners or between a learner and the learning material are called *engagement activities* (Chi and Menekse, 2015). These are categorised and rank ordered with respect to their benefit to the students' learning outcome into *passive*, *active* and *constructive* activities (ibid). To give an example in terms of verbal communication, *passive* engagement describes a student who only listens to his learning partner or gives one-syllable responses (like "hm" or "okay"). A student communicating *actively* would be characterised by reading out loud from the given instruction material or repeating what has already been said by her/his learning partner. *Constructive* use of verbal communication will occur if the student elaborates on the learning material, tries to explain what is being said on her/his own or if she or he poses a question (Chi and Wylie, 2014). The CAP-hypothesis claims that with respect to the learning outcome, constructive behaviour is superior to active behaviour which in turn is more beneficial for learning than passive behaviour of learners. For an overview of theoretical considerations and empirical studies that support this hypothesis, see Chi and Wylie (2014). By means of this framework the seemingly contradictory positions in research can be explained. To give an example, a constructively behaving student will not be able to profit from a collaborative situation if the learning partner behaves only passively.

While these categories can be used in single learner scenarios, *only* within collaborative situations a fourth category, namely *interactive engagement*, is introduced. In this category, both partners need to behave *and* communicate constructively in the sense explained above and a sufficient degree of back-and-forth utterances between the students must occur (Chi and Wylie, 2014). For example, if a student raises a question on a specific topic and the other student responds to this question, the dyad behaves interactively. The two constructively learning students will then, in theory, profit from their collaboration because of their interactivity.

This background and the long-term aim to foster students` learning outcome in digital media learning scenarios at tertiary level leads to the following research questions: Can the interactivity of a dyad's face-to-face communication while learning mathematics with a digital instructional medium be linked to their learning outcome in a pre-post-test-scenario? Is this relation applicable for students of different fields of study and across different types of digital instructional media?

STUDY DESIGN AND TEST INSTRUMENT

To answer this question, video recordings of 63 pairs of first year students from different universities and fields of studies have been analysed:

- 42 engineering and economics students, University of Applied Sciences Pforzheim
- 40 psychology students, Bielefeld University
- 44 students becoming primary school teachers, Bielefeld University

During the intervention phase of our study, which lasted at most 70 minutes, each of the three groups used a different instructional medium to learn descriptive statistics (i. e. measures of central tendency and spread). In Pforzheim, a digital script in a moodle-environment containing definitions, formulas, examples, explanations but no further multimedia elements (like audio commentary or video clips) were present (cf. Figure 1). As an interactive element, short multiple-choice questions were incorporated into the material. As an example, following the slides dealing with the harmonic mean of a data set (cf. Figure 1) the following multiple-choice question is stated:

> A garden centre creates a substrate by blending same masses of four different soils. These soils are known to have the following densities: Soil A: 710 kg/m³; Soil B: 920 kg/m³; Soil C: 830 kg/m³ and Soil D: 1000 kg/m³. Calculate the average density of the substrate.

Possible answers were 851 kg/m³, 865 kg/m³ and 857 kg/m³ with the first one being the correct answer (calculated using the harmonic mean). The second (wrong) answer is calculated using the arithmetic mean of the four values and acts as a distractor. After answering the question, students got a direct feedback whether their calculation was correct or not. At Bielefeld University, the psychology students worked with a PowerPoint presentation. Each slide contains formulas and explanations of the topic at hand. Further details are given to the students through an audio narration that can be stopped and rewound at any time. The teacher students at Bielefeld University used a series of animated video tutorials with audio narration to learn descriptive statistics. Each topic is addressed in its own segment. After each segment, the videos are paused automatically giving students time for discussion of the topic at hand before continuing to the next topic. For example, the video explaining the harmonic mean has a running time of three minutes and thirteen seconds (cf. Figure 2).

Especially the latter two instructional media include acoustic multimedia elements (audio narration) which could influence dyads' dialoguing behaviour compared to classical pen-and-paper scenarios. In this regard the moodle-environment used at Pforzheim University can be seen as a benchmark of our methodology because its design is very similar to a classic textbook and lacks audio elements.

Measures of central tendency

The harmonic mean

The harmonic mean of n numbers $x_1, x_2, \ldots, x_n$ is defined by

$$\bar{x}_h = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

The question arises in which situation the harmonic mean has to be used to calculate the average. The answer is given in the following.

If the values in question x_i are given as ratios, e.g. speeds (= length/time), it is crucial whether the data in the data set is referring to the length (nominator) or time (denominator). If the data refer to the time, i.e. the **denominator**, the average speed is calculated using the **arithmetic mean**; but if the data refer to the length, i.e. the **nominator**, the **harmonic mean** has to be used.

This is illustrated in the following example:

Figure 1: Slide from the instructional material used in Pforzheim dealing with the harmonic mean (translation by the authors).

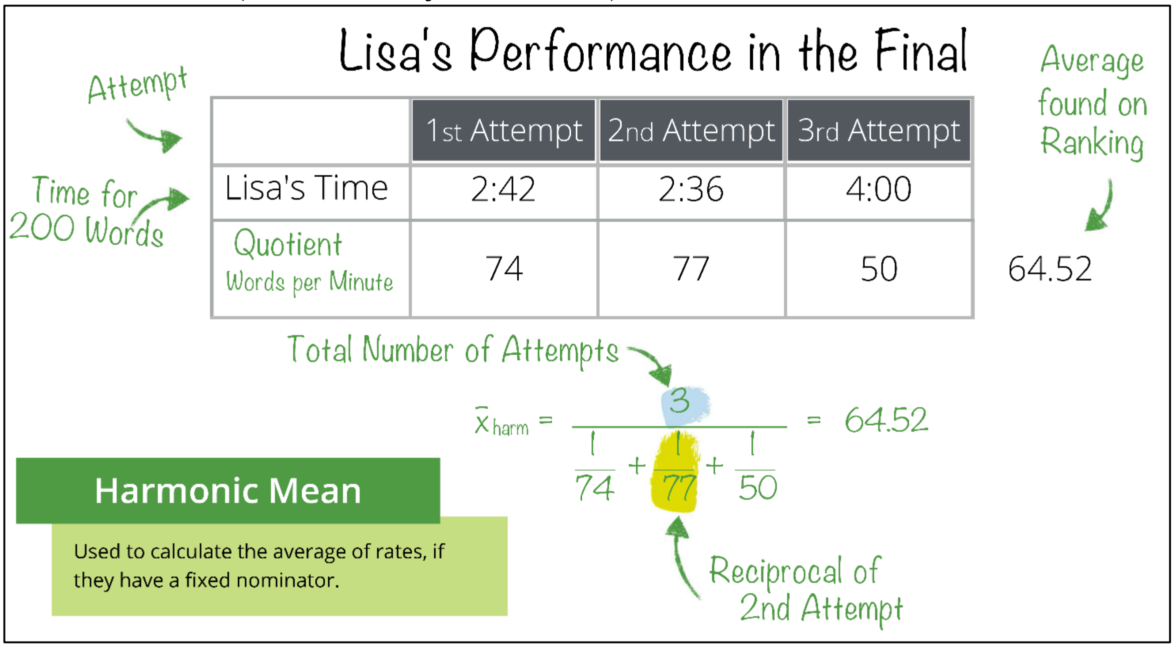

Figure 2: Frame from Bielefeld University (Teacher Students) instructional material dealing with the harmonic mean (translation by the authors).

The media intervention phases of all students have been videotaped and their computer screens were captured. Before and after each media intervention all students had to take the same pre- and post-tests consisting of both multiple-choice items and open questions regarding descriptive statistics. A detailed overview of the test items used can be found in Salle, Schumacher & Hattermann (2019).

Methodology

Based on the ICAP-framework described above and on methods derived from Chi and Menekse (2015), we developed a theoretical instrument to measure a dyad's interactivity based on video recordings, which is explained in the following; see also Hattermann, Heinrich, Salle & Schumacher (2018).

Measuring interactivity: the dialogue pattern score

To analyse the video recordings, the time-sampling method by Bakeman and Gottman (1997) was used. Each video has been organised into intervals of 10 seconds each and those intervals in which students communicate (verbally) for more than 5 seconds about the mathematical topic at hand were identified. This task was carried out by four qualified persons. To establish intercoder reliability, 13 % of the data set was coded by all four persons leading to a Krippendorff's alpha of 0.77 which represents a satisfying value (cf. Lombard, Snyder-Duch & Bracken, 2002).

Based on the ICAP-framework, the interactivity between the learners in each 10-second-segment is then rated on an ordinal scale ranging from 1 to 3 according to the coding scheme illustrated in Table 1 (see Chi and Menekse, 2015; Hattermann et al., 2018).

score	description Student communication in this 10 second segment …
1	… is dominated by one student. *(active-passive or constructive-passive)*
2	… is performed by both partners, but not interactively. *(active-active, constructive-active, constructive-constructive)*
3	… has two constructive partners contributing interactively to it. *(constructive-constructive and interactive)*

Table 1: Coding scheme of communication interactivity.

As an example, the following dialogue is taken from a pair of psychology students working on the slide dealing with the harmonic mean (cf. Figure 2).

S1: One would need … The arithmetic mean would be 200 characters because one would add 100 + 100 + 400 and then one would divide by 3. That would be 200. That's the result of 600 divided by 3.

S2: Mhm.

S1:	That's 200. This would be the arithmetic mean. But this is wrong because in that case he would have had to type faster.
S2:	Mhm.

Student 1 tries to understand the given example of the harmonic mean and elaborates on the material given and thus behaves constructively. His partner, however, responds in one syllable mumblings, which is a passive activity. In terms of their interactivity, the corresponding segments in the video recording have to be coded with score 1.

This analysis has been carried out on all 63 recordings by two coders. To establish intercoder reliability, 7 % of the data set has been coded by both of them yielding a satisfying Krippendorff's alpha value of 0.81.

The interactivity of each dyad is then quantified by the *dialogue pattern score* (*dps*, Chi and Menekse, 2015; Hattermann et al., 2018) calculated as follows: The number of times each score (1, 2 or 3) occurs is counted for every dyad and the dialogue pattern score is calculated by taking the weighted average of the occurrences: For example, suppose the communication of a dyad consists of 19 coded segments in total. 7 of those segments were *score 1* segments, 8 of them reached *score 2* and the remaining 4 of them got a *score* of *3*. From this, we can calculate the dialogue pattern score (*dps*) as

$$\frac{7 \cdot 1 + 8 \cdot 2 + 4 \cdot 3}{19} \approx 1.84.$$

Therefore, the communication of dyads with a dialogue pattern score close to 1 is dominated by single student contributions without interaction between the partners while scores closer to 3 represent a higher level of verbal interaction.

Measuring learning benefit: the normalised gain score

To link interactivity of dyads' communications with their learning outcome, we use the *normalised gain score* g (Hake, 1998) as a measure of learning benefit. This measure relates the pre- (x_{pre}) and post-test (x_{post}) results (as percentages) for each dyad by using the following formula:

$$g := \frac{x_{post} - x_{pre}}{1 - x_{pre}}$$

The post- and pre-test results are averages of both partners' results as percentage figures. This number relates the percentage points a learner actually gained between pre- and post-test to the percentage points he/she could have gained. For example, a student scoring 25 % in the pre-test and 50 % in the post-test achieved a normalised gain score of g = 0.33 (he/she gained 25 percentage points out of 75 percentage points he/she could have gained).

RESULTS

From the complete data set of 63 dyads, seven had to be excluded from further analysis since a lack of communication between the students made it impossible to calculate a reliable dialogue pattern score. For the remaining 56 dyads, we gained information on the interactivity of the particular dyad (as measured by the *dps*) and the learning outcome (as measured by the normalised gain score g). To investigate whether dyads with a higher interactivity score on average show better learning outcomes than less interactive dyads, they are split into two categories: dyads with a high *dps* and dyads with a low *dps*. This is done by calculating the median dialogue pattern score for each of the three cohorts (Pforzheim, Bielefeld (Psychology) and Bielefeld (Teacher Students)) - dyads scoring not more than the median are considered "*low dps*" and dyads with a dialogue pattern score above the median are considered "*high dps*". This analysis has been done for each group of students individually because the cohorts do not form a homogenous group as a whole: For example, they differ in their respective fields of study and the digital instructional media used during intervention and so the distribution of their dialogue pattern scores might vary.

Having constructed groups of *high dps* and *low dps* dyads within all three cohorts, the average normalised gain score for each group is calculated. All results are present within the following chart (Figure 3):

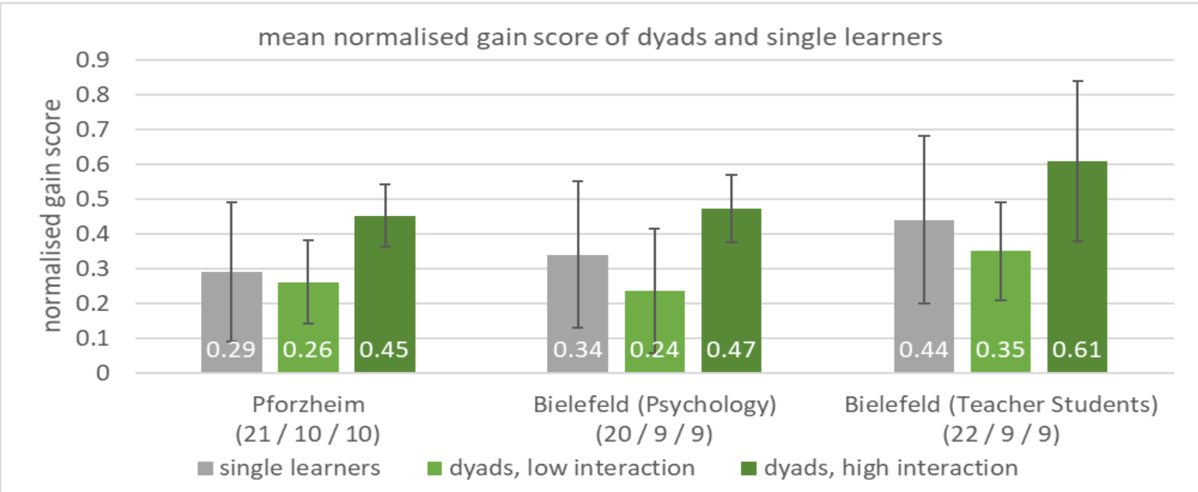

Figure 3: Average normalised gain scores with standard deviation for each group.

Independent t-tests carried out for each cohort show that the difference in means is highly significant for Pforzheim ($t = -3.981$, $p = 0.001$) and Bielefeld (Psychology, $t = -3.507$, $p = 0.004$) and significant for Bielefeld (Teacher Students, $t = -2.886$, $p = 0.013$). This confirms a link between the dialogue pattern and the normalised gain score across different kinds of digital instructional media as well as students of different fields of study.

	Pforzheim	Bielefeld (Psychology)	Bielefeld (Teacher Students)
Correlation	0.595**	0.675**	0.754**

Table 2: Spearman Ranked Correlation coefficients between normalised gain and dialogue pattern score. ** indicates a highly significant correlation of p < 0.01

In order to confirm these findings, we test the two variables *dialogue pattern score* (dps) and *normalised gain score* (g) for correlation using Spearman's ranked correlation coefficient (Spearman's rho) as a non-parametric test. Pearson's correlation coefficient is not applicable here since we cannot assume linearity between the two variables. The correlation coefficients (cf. Table 2) confirm a highly significant correlation ($p < 0.01$) between the two constructs across all three cohorts.

Previous knowledge as a possible factor of influence

It cannot be ruled out that the significant increase in learning benefit between low- and high-interaction dyads can in part be explained by the influence of other variables. One possible factor of influence that is often linked with high achievement is given by the previous knowledge of the students in question. Although the normalized gain score takes the pre-tests results into account when measuring learning benefit, it is possible that students who are high achievers in mathematics can communicate about mathematics more easily and in greater detail – which can lead to deeper interactions with their learning partners. In that scenario a better pre-test score could explain both a high dialogue pattern score and a high learning benefit.

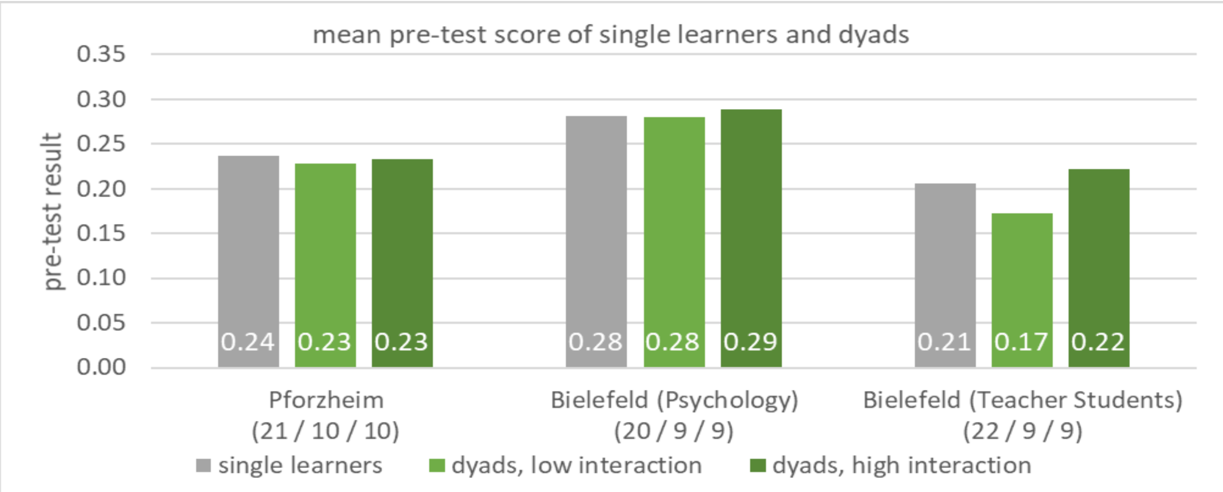

Figure 4: Comparison of mean pre-test results of high- and low-interaction dyads and single learners for each group.

To investigate this effect, we consider the groups of high- and low-interaction dyads as introduced in figure 3 above. For each of these groups we calculate and compare the mean pre-test scores – these can be found in figure 4. Regarding students in Pforzheim and the psychology students in Bielefeld, the mean pre-test scores between low- and high-interaction dyads are (nearly) identical (Pforzheim: $M_{low} = 0.23$, $M_{high} = 0.23$; Bielefeld (Psychology) $M_{low} = 0.28$, $M_{high} = 0.29$) while in the setting of the teacher students in Bielefeld a non-significant difference in means between those groups is present ($M_{low} = 0.17$, $M_{high} = 0.22$, $p = 0.338$).

As the previous knowledge of low- and high-interacting dyads in the domain of statistics is comparable in our study, it can be excluded as a main factor for explaining the higher learning benefit of high-interaction groups.

CONCLUSION AND PERSPECTIVES

Regarding our research questions, a significant link between the interactivity of the dyads' interaction, measured with the dialogue pattern score, and the learning outcome, measured by the normalised gain score, could be found using video recordings of 112 students in total. These results are in line with other research that has been done on this topic in different (non-digital) settings (Chi and Menekse, 2015). Furthermore, by using our framework to analyse students' communicational behaviour while learning with different types of digital instructional media, we were able to show that this link is not dependent on one specific type of media or a particular set of students. It remains an open question for future research whether the strength of this link is influenced by the type of digital instructional media used or the particular cohort of students considered. Indeed, the teacher students in Bielefeld show a higher overall learning benefit compared to other cohorts, but a slightly less significant result when comparing *high dps* and *low dps* dyads and more variance of the normalised gain score within these two subgroups. Furthermore, future research should take into account the possibility that the link between communication and learning benefit is dependent on the chosen mathematical content. To address this, future studies should focus on applying the ICAP framework on leaning materials dealing with domains outside of descriptive statistics.

Additionally, we are in the process to extend our analysis of possible factors of influence on both, the communicational behaviour and the learning outcome beyond the previous knowledge. For example, it is plausible that affective variables like students' interest or their domain specific self-efficacy could influence both, their interest to communicate with their learning partners and their learning benefit.

Using our method of analysing interactivity of dyads, we now aim to identify features of the digital instructional media which promote interaction between learners. This information could then be used to develop instructional material that fosters students' interactivity when learning collaboratively.

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Proof Teaching at the University Level: the case of a lecturer who is mathematician and mathematics educator

Thomais Karavi[1], Despina Potari[2], and Theodossios Zachariades[2]

[1]University of Groningen, Netherlands, t.karavi@rug.nl; [2]National and Kapodistrian University of Athens, Greece

In this paper, we study the teaching of proof in an introductory mathematical analysis course taught by a lecturer who brought experience from not only the fields of teaching at the university and research in mathematics but also from research in mathematics education. The analysis showed that his experiences appeared to affect his teaching as he became aware of students' needs and difficulties, they probably face, making a lesson potentially meaningful for them. In particular, regarding the proving process the lecturer developed a lesson where he filled the gap between informal and formal proving attempting to expand the students' proof image around the theorem, using the semantic approach in proof teaching.

Keywords: Teaching and learning of analysis and calculus; teaching and learning of logic, reasoning and proof; lecture; semantic approach; proof image.

INTRODUCTION

The central role of proof in mathematics is widely accepted and lecturers' attempts to teach proofs to students who study mathematics is an issue that concerns the research (e.g., Pinto & Karsenty, 2018; Weber, 2012). This paper explores proof teaching in an introductory mathematical analysis course taught in a mathematics department. Introductory courses are of significant importance for students' learning as it is the first time the typical proving processes are introduced (Alcock, 2010).

Many studies that tried to shed light in proof teaching were based mainly on interviews with the lecturer(s) of the course (e.g., Alcock, 2010; Lai & Weber, 2014; Weber, 2012). Very few studies combined the lecturers' underlying thoughts on proof and their teaching actions in a lecture with data from both observations and interviews (e.g., Pinto, 2019; Pinto & Karsenty, 2018; Weber, 2004). This study characterizes the teaching of proof in a lecture but goes in depth presenting a pattern that came up from lecture observations and systematic engagement with the data, and connects this pattern with the lecturer's experiences. This pattern is an expression of the semantic style (Weber, 2004) and gives details of the way the lecturer understands the semantic style and develops it in his courses. Also, it connects semantic teaching style with proof image (Kidron and Dreyfus, 2014) in a way potentially meaningful for the students.

At this level the advanced university mathematical courses are usually taught by researcher mathematicians with main experience "on writing proofs for disciplinary, rather than pedagogical purposes" (Lai & Weber, 2014, p. 93). This fact leads them to give more emphasis on the formal aspects of proof (Alcock, 2010) even though they know that the ideas behind the proof are often provided in an informal way (Lai &

Weber, 2014). In their own research, lecturers themselves consider the use of informal ways for proving but pay less attention to these ways during the teaching in university courses. The study we report here is a part of a wider study on university teaching and examines the teaching of proof in an introductory course of mathematical analysis by a lecturer who brings experience from research in mathematics, research in mathematics education but also has many years of teaching experience. With this specific case, we try to understand how these different types of experience blend while he is teaching proofs. In particular, here we ask:

1. Which are the characteristics of proof teaching in an introductory mathematical analysis lecture?
2. What is the lecturer's rationale underlying this teaching in relation to the different sources of experiences?

THEORETICAL BACKGROUND

In this paper, adopting a sociocultural perspective, teaching is considered an activity where the constructive actions and goals of the lecturer are socially and culturally framed and developed. The lecture in this context is an instructional activity, while the activity of teaching concerns the thoughts, decisions and judgments of a lecturer in planning, teaching and reflecting on the lesson (e.g., Petropoulou, et al., 2011).

Regarding mathematical proof teaching in university, Weber (2004) investigated the traditional instruction "definition – theorem – proof", showing that there is not a single paradigm of teaching in the specific instruction. Weber described in detail the teaching styles of the lecturer and offered insights into the reasons he used these specific styles, through lesson observations and interviews with him. Three different styles of proof teaching were identified: the logico – structural, the procedural, and the semantic. The first had to do with the typical mathematical argument. The second was about the emphasis given on the techniques and the general structure of the proof. The last concerned the idea behind the proof while informal representations, such as diagrams, metaphors, generic examples, everyday language, took part during the teaching process. The researcher mentioned that the semantic approach was a style a lecturer followed in order to help students gain rich images of the mathematical concepts. Within this style, the lecturer usually worked with intuitive descriptions of the concepts and focused on the links among them. In contrast, in the first two styles (logico – structural and procedural) informal representations of the mathematical concepts were rarely used. Relevant to the semantic proof production was the concept of proof image described by Kidron and Dreyfus (2014). As they described "if an individual has attempted to understand why a given claim is true, this individual may have a proof image" (Kidron & Dreyfus, 2014, p. 309). During the process of creating a proof, a collection was made of previous constructs, ideas, knowledge and examples that seemed to be useful and fitted to a specific problem. This collection could lead to the proof image and was not necessarily communicable but was complete and provided explanation with certainty. Thus, the individual could use the proof image to move to the formal proof with logical links. The difference between proof image and semantic reasoning was "the entity characteristic of a proof image" which "implies a complete image of the proof rather than specific instantiations of the mathematical object being explored" (Kidron & Dreyfus, 2014, p. 304).

Studies on the teaching of advanced mathematical concepts indicated that lecturers' teaching was affected by multiple factors (Weber, 2004). A combination of knowledge about mathematics and pedagogical concepts, skills and experience, goals for the course and understandings about how students learn and what they have to learn affect teaching decisions and actions in a lecture (Weber, 2004). In order to understand why a lecturer chooses these specific actions we have to get an insight into all these aspects. The aim of this study is to investigate the teaching of proof at the university when the lecturer, a mathematician with research experience in mathematics and with long teaching experience at university mathematics teaching is also aware of pedagogical issues coming from his research on mathematics education. This paper attempts to shed light on an expression of the semantic style. In particular, the emphasis is on the pattern of proof teaching that is based on lecturer's understanding of students' needs and difficulties.

METHODOLOGY: DATA COLLECTION AND ANALYSIS

The study was based on an introductory, proof-oriented, mathematical analysis course, taught in the mathematics department of a central Greek University, for a period of a semester. The content of the course included limits of sequences and functions, theorems about continuous functions, the definition of derivative, and applications of the previous concepts. The course was compulsory and taught in two parallel classes of approximately 100 students in each class. We conducted a case study, focusing on one of the lecturers of the course. The lecturer has a 20 – year experience in teaching this course and he is an exemplary case of a lecturer. Except of teaching experience, he is a researcher both in mathematics and mathematics education and his research concerns the area of mathematical analysis and its teaching. More specific, his research in mathematics is on functional analysis while in mathematics education is on students' learning of advanced mathematical concepts, the role of counter examples in teaching and learning of mathematics, as well as on teaching in undergraduate level and teachers' professional development.

In total, 17 lectures (34 academic hours) were observed during a semester. The lectures were audio – recorded and transcribed while field notes were kept. There were three meetings/ semi-structured interviews with the lecturer. The first meeting took place during the observations of the lectures, the second at the end of the semester, and the third after an initial analysis of the data. The meetings were also audio-recorded and transcribed.

The analysis of the data was done in three stages. In the first stage, we divided each lecture into episodes according to the accomplishment of teaching a theorem. We came up with 52 episodes. Grounded approaches were used for the analysis of the episodes.

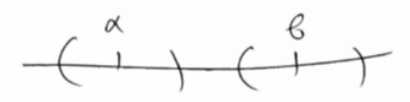
Firstly, in each episode were mentioned codes for the informal representations that the lecturer presented and for his teaching actions. The codes were merged or refined after the continuous engagement with the data and comparison with the current literature (e.g., Petropoulou et al., 2011; Fukawa – Connelly et al., 2017). In the second stage of the analysis, the interaction between teaching actions and informal representations led to the identification of the process of proving the lecturer followed and of emerging patterns. In the last part of the analysis the data from the lectures were connected with the data from the interviews. After the transcription of the discussion meetings, we tried to gain deeper insight into one identified pattern. We explored the reasons why the lecturer followed this pattern while he was teaching proof and we investigated the underlying rationale of each phase of the pattern. We tried make relations with the academic fields he participated (research in mathematics, prior teaching experiences, research in mathematics education). In this stage of the analysis the lecturer played an important role in confirming the interpretations.

RESULTS

Through the analysis of the data we identified a pattern that the lecturer followed during the teaching of a theorem. This pattern consists of four phases: *posing the problem, formulation of the conjecture and informal proving, formal proving* and *reflecting.* The phases will be illustrated using an episode referring to the theorem of the uniqueness of the limit of a sequence. In this episode, the identified teaching actions are typical of the way the lecturer dealt with proof teaching. In the end of the section we give a first insight into the lecturer's rationale of his teaching actions through the analysis of the interviews, highlighting the sources of the different experiences.

This theorem was taught at the 9th lecture of the course. The definition of the limit of the sequence was introduced in the previous lectures and was explained through the use of different representations (e.g., verbal, graphical).

Episode	**Teaching actions**
[1] Let's see something. If we have a convergent sequence, can this sequence converge to more than one numbers?	Posing the problem/ rhetoric question
[2] Do you know a sequence that converges, for example, to both 1 and ½? What do you think? [a student responds that the limit is unique]	Making the problem more specific/ posing question to the students
[3] We have already discussed the definition of the limit of a sequence. We have not proved that there is only one limit. I want you to think intuitively.	Pointing out the need for facing the problem/

	inviting students to think intuitively
[a student gives a general description of the idea of proof] [4] Exactly, so you conjecture that the limit is unique.	Conjecturing
[5] Let's see. We have a sequence that converges to both numbers, α and β. Because these numbers are different we can find an interval around α, an ε here [showing in the area around limit α] and an ε there [showing the area around the second limit β], so these two intervals will be disjoined. 	Explaining the key idea using a diagram
[6] This is an important property of real numbers. We have already used it to prove that $(-1)^n$ is not convergent. For every two different numbers, we can find an ε so these two intervals will be disjoined.	Pointing out the importance of the key idea
[7] If the sequence converges to both numbers then from a point onwards all the terms will be there [showing the area around the first limit α] and from this point onwards all the terms will be there [showing the area around the second limit β], that is a contradiction.	Informal proving/ using a diagram and informal language
[8] Or if you want, if we have these two intervals, outside these intervals we would have finite numbers, so here [showing the area around the second limit β] we would have finite numbers. That is contradiction.	Informal proving using an alternative definition of limit
[9] Let's prove it. *Theorem: if the sequence a_n converges to both α and β, then $\alpha = \beta$.**	Writing the statement of the theorem
[10] The proof is what we described previously. Typical now. We will prove by contradiction. We will assume that a $\alpha \neq \beta$.	Stating the method of proving
[11] *Let's assume that $\alpha \neq \beta$. Let's $\alpha < \beta$.* If $\beta < \alpha$ we would do the same.	Starting the formal proving
[12] [Making a new diagram – as in [5]] which ε we should take to have contradiction? We want an ε so these two intervals would be disjointed.	Posing rhetoric questions related to a specific proving step/

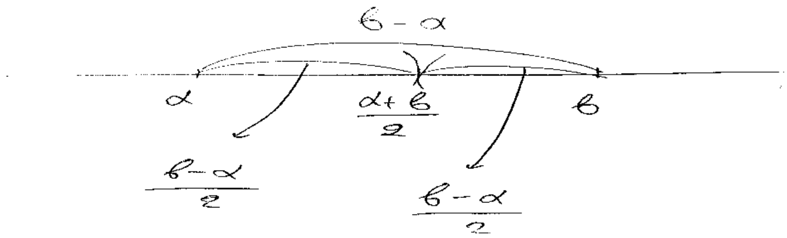
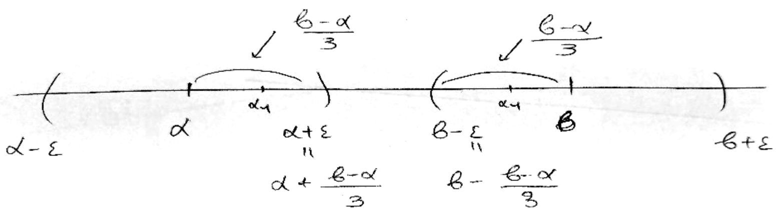
	linking to the informal proof
[13] [Updating the diagram] The distance between α, β is $\beta - \alpha$. Thus the distance of α, β from their mid-point $\frac{\beta+\alpha}{2}$ is $\frac{\beta-\alpha}{2}$.  We can take an ε less than or even than $\frac{\beta-\alpha}{2}$. [Updating the diagram]  	Explaining the idea behind the choice of ε informally, using a diagram
[14] *Let's say* $\varepsilon = \frac{\beta-\alpha}{3}$. *This is positive and* $\alpha + \varepsilon < \beta - \varepsilon$.	Formal proving
[15] What does it mean that α_n converges to α? *There is* $n_1 > 0$, I will write it this way, $\alpha - \varepsilon < \alpha_n < \alpha + \varepsilon \; \forall \, n \geq n_1$. Similarly, because α_n *converge to* β, *there is a* $n_2 > 0$, not necessarily similar with the previous one, $\beta - \varepsilon < \alpha_n < \beta + \varepsilon \; \forall \, n \geq n_2$.	Formal proving/ rhetoric question and comments
[16] This means that if I find a natural number greater than n_1 and n_2 then for this index the corresponded term will be here [adding α_n at the diagram second at 13]. *So, for* $n > max\{n_1, n_2\}$ *we have* $\alpha_n < \alpha + \varepsilon < \beta - \varepsilon < \alpha_n$ *which is a contradiction.*	Formal proving/ using the diagram
[17] There is a property that is crucial in this proof. The key idea of this proof is that we can separate two different real numbers with disjoined intervals. I say this because it will be useful in future courses.	Reflecting on the key idea

Table: teaching episode and analysis (translated from Greek - ****written also on the blackboard*)**

In the episode, the lecturer follows the pattern of *posing the problem, formulation of the conjecture and informal proving*, *formal proving* and *reflecting* phases. During the phase *posing the problem* ([1]-[3]) the lecturer sets the problem by asking the students whether a sequence can converge in more than one numbers [1]. He specifies the question [2] and points out the necessity of facing the problem [3]. The purpose of this phase is to show to the students that any mathematical result originates from a problem.

The *formulation of the conjecture and informal proving phase* follows ([4]-[8]). The lecturer in this phase is trying to develop an inquiry that will lead to the conjecture and to the informal proof of this conjecture. The lecturer uses informal representations like diagrams to support the proving process. Also, the language that he uses seems to be both formal and informal. In the episode, the lecturer uses the student's answer for the formulation of the conjecture [4], draws a diagram and uses informal language and diagrams to explain the key idea [5], [6], [7], [8]. At the end of this phase a proof image has been presented to the students creating a certainty that the conjecture is true.

Then, in the phase of *formal proving* ([9] – [16]) the lecturer translates the informal arguments of the proof to formal ones by using mathematical language. The lecturer in that phase writes on the board the statement of the theorem and the typical proof. During the typical proof, he makes references to the informal proof, translating it step by step, combining different representations, so as to create links between the previous phases of the proving process. In the episode at the beginning, he writes the statement of the theorem [9]. The lecturer makes a new diagram to support his teaching actions. The difference is that this diagram is gradually updated through the proving process. The diagram keeps the structure of the process compact and links the previous phase of the informal proving with the phase of formal proving. There is an interplay between formal and informal proving as the lecturer explains the steps first informally using the diagram or questions [12, [13] and then translates them in a formal way [14], [15].

After the completion of the proof the lecturer takes a few minutes (*reflecting* phase) to reflect on the key idea that arises from the proving process and seems to be useful for the students [17]. Therefore, in the last phase, the lecturer sums up what happened in the previous phase. He focuses on the key ideas of the proof as well as he investigates the necessity of the theorem conditions and its reverse when is needed.

In general, in the phase of formulation of the conjecture and informal proving the teaching has characteristics from the semantic style. The lecturer tries to make students understand what the theorem is about and help them construct meaning of the proving processusing several informal representations. At the end of this phase he has also developed the proof image of the theorem. The steps he makes are explicit and came from what he thinks will be useful for the students in order to be ready for the typical proof. The transition from the developed proof image to the formal proof happens with the identification of the logical links, which in our case is the translation of the previous phase step by step. In the last part of the proving process the lecturer seems to focus on the key ideas of the proof, a characteristic of the semantic teaching style, but also, he separates constructs, ideas etc. that can be used as a proof image for the proof of another theorem.

Lecturer’s rationale underlying proof teaching

During the first interview, the lecturer stated that the main goal of his teaching was to show to the students how mathematics are produced and how they should study and understand mathematics. In his words, “The biggest problem of the students, I realized it latter was that they don’t know how to study mathematics… They study mathematics in the same way as they did in school. This way doesn’t help them now”. He emphasized that he attempts to “gradually introduce the students to the new learning culture that is different from that of school and to the mathematical production that is based on proofs”.

Regarding the emerged pattern, in the second interview the lecturer said that “this is a lesson addressed to future mathematicians, so I try not only to show what the theorem says and solve exercises but to present how our thoughts develop and lead to a conclusion”. Then, when the problem leads to a conjecture, he said that “we don’t know if it is right or not, so we start to think informally to understand what is going on. In the end of this process we have a strong belief that the conjecture is true”. He added that “we sum up what we proved (…) because this is meaningful for the students in order to understand the theorem and get a holistic view of the proof”. The third phase, he stated that “Here I repeat the previous phase, step by step, translating every step in a formal way in order to make links”. In the last phase, he restated the key ideas because “the key ideas will probably help the students in similar situations or in other courses”.

Research in mathematics helped the lecturer adopt this particular teaching approach. Nevertheless, it was the research in mathematics education that gave shape to his teaching by following a specific pattern while he presents proofs. In the last interview, the lecturer made explicit the sources of his teaching decisions and actions:

> The reason to teach a theorem following this pattern came both from my research in mathematics and research in mathematics education. In the research in mathematics the starting point is a problem. By the use of informal and formal tools, a conjecture about the answer of this problem is formulated and then usually informally the researcher develops a general process of the proof of the conjecture. The last phase is to write the formal proof, and then they are sure that the theorem has been proved. I try to follow a similar process in my teaching in order to make explicit to the students how we think when we do mathematics. If I taught only the theorem and the formal proof I would teach the mathematical product and not the thinking that led to this.

The research in mathematics education made him more conscious of the teaching goals he should try to achieve during the lecture:

> My involvement with research in mathematics education helps me to see that common practices in mathematics research, as the use of different representations and the connections between them, are very important in learning and teaching of mathematics.

> So, I try to adapt my teaching with the way that mathematicians work, using ideas and results from the research in mathematics education. I used to emphasize this process in the past, but now I am aware of the importance of this in my teaching. Also the research results from mathematics education on students' difficulties helps me to focus on these difficulties.

DISCUSSION

The present study brings into account the teaching of the proof during a lecture. We studied a case of a lecturer who brings experience from both research in mathematics and mathematics education, and also, has many years of teaching experience at the university level. All the above seemed to affect the lecturer's teaching as he became aware of students' needs and difficulties they probably face. What makes this case particular is that the lecturer is not only a mathematician who teaches at the university level (e.g., Paterson et al., 2011). He also draws in research in mathematics education to make the proving process potential meaningful for the students. A previous research of Petropoulou, Potari and Zachariades (2011) had shown that these experiences of a lecturer affected the way he taught. For instance, in that study, in many incidents sensitivity to students' needs seemed to balance with a challenging mathematical content. In this study, the lecturer focuses on the process of mathematical proof production taking into account students' possible learning needs. In the course, the lecturer tried to develop students' proof image as it is not expected that students would have their own during the first year. This case of a lecturer brings closer the distance between mathematical teaching and mathematical production. The inquiry the lecturer promotes during the proving process is an expression of semantic style and supports the development of students' proof image.

This study highlights the importance of proof and proving at the university level and especially in the teaching of an introductory analysis course for the students of a mathematical department in a lecture format. Although there are studies at the university level focusing on teaching practices and specific teaching actions (Viirman, 2015), our research goes beyond the identification of teaching actions. It offers a global characterization of proof teaching indicating interrelations among different teaching actions. The phases of the pattern appear as a characterization of the semantic style of proof teaching and seems to have potential in developing students' proof image around the proof. The pattern is not linear. For instance, as we described in the episode, in the phase of formal proof, the lecturer used also informal explanation. The process is nested and the pattern appears again in a way similar to the mathematical production.

Further research is needed to study the impact of this kind of proof teaching in students' learning. This would enrich our understanding of the multifaceted process that is the act of proving at the university level.

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Course Coordinator Orientations Toward their Work

Antonio Martinez[1], Jessica Gehrtz[2], Chris Rasmussen[3], Talia LaTona-Tequida, and Kristen Vroom[5]

[1]San Diego State University, United States; [2]University of Georgia, United States; [3]San Diego State University, United States, crasmussen@sdsu.edu; [4]San Diego State University, United States; [5]Portland State University, United States

One of the ways in which university mathematics departments across the United States are making efforts to improve their introductory mathematics courses is by implementing or increasing the level of course coordination. This not only entails creating uniform course elements across different sections but also includes efforts to build a community among the instructors of the course. While many coordinators have the common goal of improving student success, we explore what guides their actions to see this accomplished, what we refer to as their orientation toward coordination. In this paper we introduce and elaborate on two orientations toward coordination that arose from interviews with course coordinators from a variety of institutions across the country. We also discuss the importance of both orientations as they relate to drivers of change.

Keywords: Course coordinators, leadership, teachers' and students' practices at university level, preparation and training of university mathematics teachers

INTRODUCTION

Course coordination for multi-section introductory mathematics courses such as precalculus and calculus is one way in which universities across the country are attempting to improve instruction and the consistency and quality of students' learning experiences (and hence improve student learning outcomes). Because multi-section introductory mathematics courses are often taught by a range of instructors (including graduate students, career line faculty, and ladder rank faculty), course coordination can help mitigate against uneven student experiences that can disadvantage students in different sections of the same course. Such uneven experiences include different content emphasis or coverage, different grading schemes, and different quality enactments of active learning. Active learning as it is used here refers to a wide range of instructional approaches that invite students to engage in challenging mathematics and to share their reasoning with their peers. These differences in learning experiences are potentially problematic because they offer different opportunities for students to learn the intended content, and hence be adequately prepared for subsequent courses. As such, course coordination can be an important contributor to student success.

One of the first studies of course coordination in mathematics departments investigated the *coordination system* at five mathematics departments identified as having relatively more successful Calculus 1 programs (Rasmussen & Ellis, 2015). The phrase coordination system is used to evoke the image of coordination that goes beyond surface features of uniform course components (e.g. syllabus, textbook, homework, exams) to include efforts to build a community of instructors working together to create rigorous courses and high-quality learning experiences for students. In this study the authors identified concrete actions that the course coordinators took to provide both logistical support that promotes greater course uniformity and hence more equitable student experiences as well as just-in-time professional development support for teaching difficult topics, implementing active learning, pacing, etc. Rasmussen and Ellis (2015) liken the role of course coordinator to that of a *choice architect*, which comes from the work of Thaler and Sunstein's (2008) work in behavioral economics. A choice architect is someone who is able to structure choices for others in ways that can "nudge" them to make particular choices while still maintaining the feeling of independence. For example, one of the things that course coordinators at the five mathematics departments, studied by Rasmussen and Ellis, did was to make instructors' lives easier by providing a range of default options, including homework sets, class activities that actively engage students, pacing guides, etc. Instructors had leeway in how they made use of these options and thus maintained pedagogical autonomy. They further argue that this framing of a coordination system is consistent with effective change strategies identified by Henderson, Beach, and Finkelstein (2011).

In ongoing work at a different set of mathematics departments, Rasmussen et al. (2019) conducted five case studies of mathematics departments that have successfully initiated and sustained active learning in their Precalculus to Calculus 2 (P2C2) curricula. These researchers highlight the different ways that coordinators across the five sites make instructors' lives easier and build community among instructors. Williams et al. (2019) further analyzed these five sites to highlight the ways that coordinators can function as change agents by leveraging the following three key drivers for change: providing materials and tools, encouraging collaboration and communication, and encouraging (and providing) professional development. An important contribution of the work by Williams and colleagues is the strong connection between ongoing mathematics department change efforts and the substantive and growing literature focused on change in higher education (e.g., Shadle, Marker, & Earl, 2017).

One thing that is common (and abundantly clear) from this prior work is the critical role of the course coordinator in a coordination system. Hence, a better understanding of what values, beliefs, dispositions, etc. coordinators take toward their toward their role is needed. In conceptualizing these aspects of coordinators, we are inspired by the work of Thompson, Philipp, Thompson, and Boyd (1994), who examined the influence that teachers' conceptions have on their implementation of innovative curricula. In particular, they identified two contrasting orientations toward mathematics teaching: calculational orientation and a conceptual orientation. They illustrated how these different orientations have significant consequences for how teachers interact with students and content and hence offer different opportunities for learning. Similarly, we were curious to better understand coordinators' conceptions toward coordination because such beliefs and understandings profoundly influence how they interact with their colleagues and the consequent opportunities for professional growth. Thus, the research question that drives the analysis presented here is: *What orientations do course coordinators take toward their work?*

The potential contribution of this analysis is both pragmatic and theoretical. Pragmatically, a deeper understanding of the orientations of course coordinators offers mathematics departments a language for thinking about what their goals of coordination are and who, either in their department or new hires, would have the perspective on coordination that is likely to be able to enact their goals. Theoretically, this work contributes to conceptualizing the role of coordinators and coordination systems more generally.

THEORETICAL BACKGROUND

To frame course coordinator orientations we draw on Philipp's (2007) comprehensive review of mathematics teachers' beliefs and affect, where beliefs are described as the "lenses through which one looks when interpreting the world," and affect is thought of as "a disposition or tendency one takes toward some aspect of his or her world" (p. 258). Our use of the term "orientation" encompasses both beliefs and affect as described by Philipp. In his chapter, Philipp attends to the differences and similarities between a teacher's affect, beliefs, belief systems, conceptions, identity, knowledge and values as these terms are inconsistently used in the literature. Each has a unique impact on the way a teacher interacts with their classroom and can provide researchers with new perspectives on how to measure teacher development. While these terms require a localized focus, Philipp also steps back to discuss the existence of a teacher's orientation as it encapsulates a variety of the localized terminology and requires a broader focus from a researcher's perspective to better understand teacher impact in the classroom.

As described in Thompson et al.'s (1994) paper, varying teacher orientations can produce markedly different discussions in the classroom due to what the teacher considers valuable information. For example, a teacher with a calculational orientation will consider a procedural answer to the question, "How did you get that answer?" as all that is needed, whereas a teacher with a conceptual orientation is more interested in how the student is thinking about the quantities that are used and the relationships between them (Philipp, 2007; Thompson et al., 1994). The orientation of a teacher emphasizes the goals and intentions of the teacher as enacted through their actions and discourse in the classroom. We draw a parallel between the orientations of a teacher and the orientations that a coordinator may have, as their goals and intentions for how the course should be run are enacted through their actions and influenced by their beliefs, knowledge and values.

METHODS

This study is part of a larger national study investigating Precalculus through Calculus 2 (P2C2) programs and student supports at the post-secondary level. As part of this larger study a census survey was conducted of all mathematics departments that offer a graduate degree in mathematics (Rasmussen, et al., 2019) and twelve institutions were selected as case study sites based on what the research team viewed as noteworthy or otherwise interesting features of their P2C2 programs. These features included ones previously identified as being associated with successful Calculus 1 programs, one of which being course coordination (Hagman, 2019; Rasmussen, Ellis & Zazkis, 2014). After the project team's initial site visits and data collection, seven sites were identified as leveraging a coordination system that went beyond simply implementing uniform course elements (e.g., syllabus, textbook) to also include intentional efforts to build a community among instructors. In order to answer our research question, we conducted 13 interviews (2018-2019 academic year) with 19 P2C2 coordinators across the seven sites. We conducted 10 individual interviews and three group interviews that included two or more coordinators. Interview questions included probes such as what one likes most (and least) about being part of a coordinated course, level of autonomy, and characteristics of what makes for a "good" coordinator.

Interviews were audio-recorded and transcribed for analysis. We conducted a thematic analysis (Braun & Clark, 2006) to identify orientations coordinators take towards their work. Each researcher open coded the transcripts for three sites, with at least two researchers coding the same site and comparing codes to reach consensus. The research team met to discuss and revise codes and group them by theme, reaching consensus on the grouping and descriptions of the categories. This phase of analysis resulted in 11 categories (henceforth referred to as themes) that shed light on these coordinators' approach to their role. Each theme consists of three or more codes from the first round of coding. The research team then engaged in further axial coding and identified two orientations towards coordination that encapsulated 10 of our 11 themes (with the theme of Personal Qualities not fitting into either orientation).

FINDINGS AND RESULTS

Our analysis of the coordinator interviews resulted in identification of two distinct orientations to coordination. We refer to these two orientations as a Humanistic-Growth Orientation and a Knowledge-Managerial Orientation. We next illustrate each of these orientations, using interview excerpts that were selected to be representative of each respective theme within the orientation.

Humanistic-Growth Orientation

Five themes were identified during analysis that we later grouped to define the broader category that we call Humanistic-Growth Orientation toward coordination. These five themes are: a) intentional instructor support, b) interested in relationships, c) community builder, d) attends to student experience, and e) flexible. Together, these themes describe the orientation of a coordinator that incorporates humanistic values and a belief in the potential for professional growth of the instructors under their purview. For the purposes of this proposal we highlight three of these themes: intentional instructor support, community builder, and flexible.

Intentional instructor support. This theme goes beyond providing resources and materials for the instructors of the course to make their lives easier (which aligns more with a Knowledge-Managerial Orientation). All of the actions categorized under this theme are deliberately made by the coordinator to support instructors' improvement of their teaching. One example of this is exhibited by a coordinator describing their goals and intentions for coordination:

The coordination is to try to get them [instructors] up to speed for thinking about how students learn math, how to help students be successful, how to help students connect to the ideas that are being taught in this specific class, but also for them to think a little bit more carefully about how they present things.

This coordinator is not only attending to student experiences from a content perspective, but is addressing the ways in which they can intentionally help instructors think about how to provide a more thoughtful and enriching experience for the students in the classroom. The following quote describes the level of intentionality of a coordinator that provides this type of support:

> But to the extent that I have been effective as a coordinator... I think it's been as a result of my intentions to influence instruction and influence the instructors' confidence with respect to teaching. I don't think that that view of coordinating is shared amongst others necessarily. I think the others really do view their role as being not only including, but limited to the managerial aspects. And that is very much secondary in my view.

While these quotes describe just two aspects of intentional instructor support, we noticed other actions of the coordinators that reflect this theme such as providing professional development opportunities, observing instructors' classes and giving feedback, supporting instructors to be reflective practitioners, as well as willing to be the "scapegoat" (as opposed to letting the instructor take the heat) when students are upset with how the course is being run.

Community builder. There was evidence of various community building efforts in all 13 interviews. Some of the actions that we identified to build community were: valuing contributions and feedback, getting people to work as a team, getting instructors excited about the course, and generating buy-in for the philosophy of the course. Some of these efforts are characterized well by a coordinator that had the following to say about coordination, "Coordination is not autonomy. It's about a team effort and setting up best practices that everyone follows." Many of the coordinators from our interviews reflected similar beliefs and viewed the coordination practices as a collective effort. A related aspect of community building was an intentional effort by the coordinator to distribute power amongst the instructors of the course. For example, a coordinator at a large research university reflected on their own work as coordinator:

> I do my best to structure those meetings to give the impression, not entirely artificial, that we're kind of engaged in a collective enterprise to improve all of our students' learning. So, I truly try to position myself as a co-participant in that process. Not somebody who's necessarily dictating to everyone else, you know, what to do or how to teach, but, you know, I'll pose particular questions or issues and invite people to offer their own perspectives and that sort of thing. And again, I'm sort of trying to nudge things along in particular directions and buy things in particular ways. But, I want individual instructors to feel like they have some agency over the direction of the course for everybody. And I think that this would result in kind of a sense of, at the very least, sort of codependence amongst the instructors where they are all like, we'll have lunch together, that sort of thing.

The efforts to build community vary from coordinator to coordinator, but the goal to establish a community is central to this theme.

Flexible. Most coordinators lead nonhomogeneous groups of instructors. In many circumstances, the heterogeneity of the instructors exists in the experience that they have teaching the course or teaching in general. As a means to provide the necessary support for the instructors as a collective, we saw that some coordinators would adhere to varying levels of coordination practices as described by a coordinator when asked about instructor autonomy:

> The degree of autonomy that instructors want when they're teaching the course is directly related to how many times they've taught the course or their experience with the course. The [graduate] student that's teaching Calc 2 for the very first time doesn't want any autonomy. They want to come in and they want to talk to me about here's what I'm doing next. 'How do you do this? What are the things that you emphasize?'… so, usually the greener the teacher, the less autonomy they want. Whereas the person that's taught the course over and over again has got- they have a good handle on it and they tend to not [need extensive advice],they just have it down.

By incorporating a flexible approach to coordinating, the coordinator is able to provide a tailored experience for each instructor that has the potential to generate more buy-in from the instructors and foster a collaborative team environment.

Knowledge-Managerial Orientation

The themes from this analysis that shed light on a Knowledge-Managerial Orientation to coordination include the following aspects of coordination: a) course content and curriculum, b) organizing and attending to the details of the course, c) communication, d) knowledge of the course history (including department and university structure), and e) knowledge of teaching the course. While every coordinator described performing actions of one form or another from this orientation, in this proposal we only detail the themes of course historians and communication.

Course historian. Coordinators who discussed their role as a course historian demonstrated a rich knowledge of both the coordination structure and history as well as knowledge of the larger departmental and university system in which coordination is embedded. Notably, coordinators leveraged this knowledge to work towards sustaining and facilitating change because they knew what worked well and what has been met with resistance. For example, one coordinator said:

> We don't give ... a common exam. And I was sort of toying with the idea of maybe we should give a common exam, and I was told ... that would require a departmental vote. Only because it's calculus and people care about what calculus is… Because I will have tenured faculty teaching, often there is … a limit that's been made, not explicit, but implicitly clear to me about like you can't just take total control of this course. … It's not like it [a common exam] would never happen, but it would not be as simple thing that I could just decree that that's going to happen. So, it would take a lot of work.

This excerpt highlights an understanding of some of the departmental barriers to change and includes an understanding of ways to work within the system to facilitate changes for a course. Being a course historian also requires a continuous involvement within the coordination structures so that one's understanding and knowledge remains current and relevant. A participant highlighted this when they said a coordinator must be embedded within the department and ask, "'Hey, how are you? How are things going? Do you want to teach again?' or like, 'What are you doing now?' Like you have to be able to be part of the social network of the department in a way." The coordinator's involvement within the department is integral to their effectiveness. In addition to the importance of having this knowledge of the course history to make content or policy changes, coordinators that demonstrate a Knowledge-Managerial Orientation to coordination also draw on this knowledge when communicating department and university policies to instructors who are likely less familiar with this information.

Communication. The communication aspect of the Knowledge-Managerial Orientation to coordination includes both communicating important content and logistics about the course to instructors and being responsive to student and instructor emails. Some coordinators created a document or a set of examples to communicate important content, saying things like:

> We have these 62-page documents that are the expected learning outcomes for our calculus course that I developed. And it was so I can just be like, 'Hey grad student, this is the course, and it's a lot of high-level things. Students should be able to do blank... all organized in some hierarchical way. And that took a lot of experience to write that thing and now it, it's a lot of detail and it's all organized, and then it's communicated and disseminated.

Other coordinators communicated key content by drawing attention to it during formal or informal meetings/discussions with instructors. One coordinator acknowledged that he likes to allow room for instructors to have agency in the course in addition to clearly communicating important content, saying:

> If there's a certain thing that I really, really want to test students on ... I might, like say to them, 'Hey, try to implement something in your class, try to do something like problem

number 25 on page 381.' Yeah, I might say something like that, but I try not to, I try not to overstep that with other people.

All of the themes encapsulated by the Knowledge-Managerial Orientation to coordination illuminate an approach that allows for the coordination structure to be implemented in an organized way, clearly communicating the coordinated elements and expectations to instructors. Coordinators who embrace this orientation leverage their knowledge of the students and their experience teaching the course to create appropriate resources and coordinated elements. Additionally, this approach allows for a coordination system that is well-informed by the course history, and departmental culture/policy surrounding it.

DISCUSSION

All of the coordinators in our study demonstrated aspects of Knowledge-Managerial Orientation to coordination, highlighting the importance of being familiar with the course they are coordinating as well as creating and sharing resources with instructors teaching the course. This is not surprising since uniform course elements are a key component of coordination. Approximately half of the coordinators also demonstrated a Humanistic-Growth orientation. Moreover, when this subset of coordinators discussed managerial or resource aspects of their work, they tended to frame their actions from a Humanistic-Growth Orientation. For example, providing instructional materials was done in the spirit of supporting instructors to excel in their teaching. It is important to note that while not every coordinator demonstrated a Humanistic-Growth Orientation toward their coordination work, those that did were deliberate and prioritized personal and professional growth to improve the quality and effectiveness of their P2C2 courses.

We see a similar level of intention from the coordinators in the study by Williams et al. (2019) as various coordinators deliberately take action to improve student success by acting on three drivers of change to implement and sustain more active learning in their P2C2 sequences. These drivers, providing materials and tools, encouraging collaboration and communication, and encouraging (and providing) professional development nicely align with the two orientations presented in this proposal. Providing materials and tools is an action taken by coordinators with a Knowledge-Managerial Orientation while encouraging collaboration and professional development are two actions taken by coordinators that approach their work with a Humanistic-Growth Orientation. Thus, by encouraging coordinators to initiate change through an approach to coordination that incorporates both the Humanistic-Growth and Knowledge-Managerial Orientations, mathematics departments across the country could reap the potential benefits of increased active learning in P2C2 classes.

By attending to these drivers and orientations, mathematics departments now have the language and research evidence to support their goals of improving or implementing active learning and coordination. Drawing on the data from a census survey sent to all

Ph.D. and master's granting institutions across the country, we know that there is a need for the improvement of professional development support as well active learning practices in the classroom (Rasmussen, et al., 2019). Math departments reported valuing active learning and professional development, but also reported not being very successful at each. In fact, 44% of mathematics departments saw active learning as very important, 47% saw it as somewhat important and 9% did not see it being important. However, when asked about how successful they were at implementing active learning, only 15% of the 199 mathematics departments reported that their program was very successful. Similarly, with graduate teaching assistant (GTA) professional development, 50% and 32% of the mathematics departments saw it as very and somewhat important (respectively), while only 29% of the respondents reported being very successful at it. Clearly, mathematics departments across the country are looking for ways to improve their active learning and professional development efforts, and effective course coordination is one opportunity to achieve this goal.

Our hope is that by bringing awareness to coordinators' orientation(s) we are not only supporting mathematics departments in search of coordinators but are also encouraging coordinators themselves to reflect on how they approach their role and how they can act on the available drivers for change at their institutions. By providing this perspective towards coordination, we also hope that this empowers mathematics departments across the country to improve their active learning and professional development efforts. The next step in our work surrounding P2C2 coordinators' orientations will be to analyze the instructor and GTA interviews to compare and contrast what is valued in terms of effective coordination. A future study might also analyze the work of coordinators in science and engineering departments and then compare this to the orientations identified here. Such research may lead to even greater significance of our findings as it might identify related or expanded efforts to improve instruction in a range of introductory courses typically required for mathematics, science, and engineering students.

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The step from tertiary to secondary education in mathematics. In search of a shared paradigm

Josep Gascón[1] and Pedro Nicolás[2]

[1]Autonomous University of Barcelona, Faculty of Mathematics, Spain, gascon@mat.uab.cat; [2]University of Murcia, Faculty of Education, Spain, pedronz@um.es

Whereas there is a widely accepted epistemological model of mathematics in the community of mathematicians, there is no such a thing for the teachers of mathematics in secondary education. This makes problematic the transition from tertiary (as students) to secondary education (as teachers). In this work we analyse this didactic phenomenon.

Keywords: teachers' and students' practices at university level; transition to and across university mathematics; epistemological model of mathematics; teaching ends; mathematical praxeology to be taught.

INTRODUCTION

Didactic phenomena concerning the step from secondary education to university level have drawn the interest of researchers for many years. Some thematic working groups of the conferences INDRUM, CERME and ICME, and even some plenary talks, have addressed it in different editions. However, the transition from tertiary to secondary education has been much disregarded. But, first of all, does such a transition exist? Indeed, it is the one experienced by those students at university when they become teachers at secondary education. A possible explanation is that one might naively believe that didactic obstacles only exist when you move forwards in mathematics to meet 'more sophisticated' mathematics, but not when you move backwards, revisiting 'more elementary' mathematics. Another explanation for this lack of attention is that, perhaps, transitions are implicitly assumed not to entail a change of position within the institution, but only a change of institution. It is true that the institutional position changes: they go from the position of students at university to the position of teachers at secondary education. But still, this change of position does not make the transition less problematic and not deserving attention.

Concerning possible obstacles for the transition from the tertiary to the secondary level, there are many differences in the mathematical activity. For instance, in the tertiary level the students typically learn the strongest technique for a given type of task (e.g. Lagrange multipliers for optimisation problems), and this technique cannot be directly translated to the secondary level. In this work we do not try to give a thorough account of all those obstacles. Instead, our goal is to present an analysis of one of the major problems of this transition: the lack of a 'solid' epistemological model of mathematics to support the mathematics to be taught at secondary level. As we will explain below, the epistemological model of mathematics accepted among the community of mathematicians cannot be directly transfered to secondary education. Hence, future teachers at secondary level need an alternative epistemological model of mathematics, which is not provided in a standard way by our society.

THE ANTHROPOLOGICAL THEORY OF THE DIDACTICS

The theoretical framework of this contribution is the anthropological theory of the didactics (ATD). In this first section we will introduce some basic notions to be used later on. For more information about ATD, the reader is invited to read (Chevallard, 1999, 2007) and (Gascón & Nicolás, 2019).

Praxeologies

According to the ATD, didactics of mathematics is devoted to the analysis of the genesis and diffusion of mathematical knowledge, regarded as an output of intentional actions. ATD has the notion of *praxeology* for the simultaneous analysis of the intentional actions and the resulting pieces of knowledge. Notice that, whenever there is an intentional action, there is, by definition, an *agent*, that is to say, someone trying to carry out this action. A *praxeology* is made of two interrelated components: the *praxis* and the *logos*. In turn, the praxis is made of:

- a certain set of *types of tasks* the agent wants to deal with,
- a certain set of *techniques*, which are the ways the agent has in order to deal with those types of tasks.

The *logos* is made of:

- the *technology*, which is devoted to describe the techniques, to show their usefulness, to delimit the scope of validity, and to study their economy (how much effort it takes to use those techniques) and reliability,
- the *theory*, which includes an ontological description of the region of the world involved in the types of tasks, the techniques and the technology (that is, which are the objects or beings under consideration, and which are the relationships between them), but also a normative vision (which should be the goals of my intentional actions, which should be the kind of techniques employed, etc.).

Personal and institutional praxeologies

As we said before, the notion of praxeology helps to describe individual intentional actions. But it is also used to deal with *institutional intentional actions*. Within ATD, the concept of *institution* is understood as a set of constitutive rules that:

- define and determine positions and relationships in a social scheme fixed in a conventional way,
- determine rights and duties, permissions and prohibitions, rewards and penalties.

Some examples of institutions: any regulated game (for instance, chess), matrimony, nationality, procedural law, languages, scientific theories, the teaching of mathematics in a faculty of mathematics, the teaching of mathematics at Secondary Education.

Typically, the members of an institution, due to the fact that they are members of this institution, are the agents of individual intentional actions which share relevant features, and so we can speak of ‘intentional actions’ of the institution. These institutional intentional actions give rise to the so-called *institutional praxeologies*, as opposed to *individual* or personal *praxeologies*. Due to its generality, it is easier to describe institutional praxeologies than personal ones.

Mathematical praxeologies and the epistemological model of mathematics

Mathematical praxeologies are those praxeologies which describe, at once, both the mathematical activity and the output of this activity (the corresponding mathematical works). Since every mathematical praxeology is based on a *theory*, every mathematical praxeology entails (perhaps implicitly) a certain ontological description and a certain normative vision of mathematics. Therefore, we could say that every mathematical praxeology assumes a certain *epistemological model of mathematics*. This terminology is reasonable because, among other things, the *theory* of a mathematical praxeology determines which are the *basics objects* on which mathematics are built, and which are the kind of *accepted arguments* to verify propositions. In other words, the *theory* of a mathematical praxeology provides an account (a *logos*) of how knowledge (*episteme*) is achieved in mathematics.

Mathematical praxeologies to be taught and mathematical praxeologies for teaching

Cirade made in (2006) a distinction between *mathematical praxeologies to be taught* (MPTBT) and *mathematical praxeologies for teaching* (MPFT). Given an educational institution, $\Im$, the *mathematical praxeologies to be taught* in $\Im$ are those mathematical praxeologies that teachers in $\Im$ plan to teach. Of course, those MPTBT are chosen after considering certain questions, getting certain conclusions, etc. This activity devoted to decide what are going to be the MPTBT constitute by itself a different kind of praxeologies, the so-called *mathematical praxeologies for teaching*.

Let now $\Im$ be a faculty of mathematics. Let us consider a possible MPTBT:

- Type of task: given a certain function, $f : [a, b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, and given a real number $y_0 \in \mathbb{R}$, check whether there exists $x_0 \in [a, b]$ such that $f(x_0) = y_0$.
- Technique: If $y_0 = f(a)$ or $y_0 = f(b)$, then we can take $x_0 = a$ or $x_0 = b$, respectively. Otherwise, we can use Bolzano’s theorem. It says that, in the situation above, if f is continuous in $[a, b]$ and y_0 is between $f(a)$ and $f(b)$, then there exists such an x_0.

- Technology: Among other things, technology is responsible for proving the vailidity of the techniques employed. Now we prove Bolzano's theorem, underlying the beginning of the main parts of the proof. First notice that we can assume $f(a) < f(b)$ without loss of generality. <u>Now, we are choosing the candidate for our required x_0</u>. For this, consider the set $X = \{x \in [a, b] \mid f(x) \leq y_0\}$. It is not empty, because $a \in X$. Also, it has an upper bound, for instance b. Then, due to the *least upper bound property* satisfied by the real numbers, there exists a least upper bound x_0 of X. <u>Notice that if we prove that $x_0 \in X$ we are done.</u> Indeed, if $x_0 \in X$ then, by definition of X, we have that $f(x_0) \leq y_0$. In this case, equality is guaranteed. Notice that, if $f(x_0) < y_0$, then there exists a neighbourhood $\mathscr{V}$=$(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ of $f(x_0)$ such that $f(x_0) + \varepsilon < y_0$. By continuity of f in x_0, there exists a neighbourhood $\mathscr{U}$=$(x_0 - \delta, x_0 + \delta)$ of x_0 which is mapped inside $\mathscr{V}$ by f, which, in turn, implies that x_0 is not an upper bound of X, because there are numbers greater than x_0 (for instance $x_0 + \delta/2$) which still are in X (because $f(x_0 + \delta/2) < f(x_0) + \varepsilon < y_0$). <u>Therefore, it only remains to prove that $x_0 \in X$.</u> For this, we will use a *reductio ad absurdum* argument. If $x_0 \notin X$ then $y_0 < f(x_0)$. Then there exists a neighbourhood $\mathscr{V}$ of $f(x_0)$ 'above' y_0. By continuity of f in x_0, there exists a neighbourhood $\mathscr{U}$ of x_0 which is mapped to $\mathscr{V}$ by f, which, in turn, implies that there exists a neighbourhood $\mathscr{U}$ of x_0 such that, for every $x \in \mathscr{U}, f(x) > y_0$. Therefore $\mathscr{U} \cap X = \varnothing$. But it is not difficult to prove that the fact that x_0 is a *least* upper bound implies that every neighbourhood of x_0 has no empty intersection with X. Whence the absurdum.
- Theory: there are a lot of theoretical elements supporting the previous technology. For instance, there are some properties taken for granted, for example the *least upper bound property* of real numbers, which says that a non-empty set of real numbers with an upper bound always has a least upper bound. There are also definitions (that of upper bound, least upper bound, neighbourhood, function, continuity). Notice that, in the definitions, there must be always a clear separation between the logical terms ("for every", "there exists", "if … then …", "and", "or") and the non-logical terms ("real number", "function", "less than or equal"). And, still, the non-logical terms can always be analyse in terms of logical terms and, eventually, only three non-logical terms (i.e., $\in$, = and $\varnothing$). But there is a another deep feature, which is closely related to the aforementioned logical analysis of definitions. Namely, the arguments used in the technology have to be *deductive*.

Later we will take a closer look at the feature of deductive arguments. But first let us sketch some elements of possible mathematical praxeologies for teaching underlying the previous mathematical praxeology to be taught.

- Tasks: Which proof *should* we choose for Bolzano's theorem?
- Technique: One possible way is the one chosen above, which uses the least upper bound property of real numbers. Another possible technique is to make a different

proof by using a different property of real numbers, for instance Cantor nested intervals, or Dedekind cuts, or the convergence of Cauchy sequences. The choice of one option depends on how the teacher wants to deal with the completeness of real numbers. Concerning the definition of continuity of functions, we have chosen the ε-δ approach. But we could have chosen a different one, based on sequences of real numbers and limits of functions, which would have required a different proof.

- Technology: We may have reasons to prefer one formulation of the completeness of real numbers rather than another one. For example, we might prefer to introduce real numbers axiomatically, and so we would prefer the least upper bound property because it is typically included in the standard axiomatic definition of real numbers. But maybe we prefer to construct the real numbers as an enlargement of the set of rational numbers. This can be done, for instance, by using equivalence classes of Cauchy sequences, and in this case it seems appropriate to define completeness in terms of convergence of Cauchy sequences. But we can also enlarge the set of rational numbers by adding Dedekind cuts, and in this case completeness would be regarded from a different perspective.
- Theory: The theory in this mathematical praxeology for teaching is larger than that of the mathematical praxeology to be taught. For instance, here we consider new properties of real numbers such as the convergence of Cauchy sequences, the property about Cantor nested intervals, Dedekind cuts. But there is an important feature we should remark: this theory is larger but of the same nature because, even if it includes more objects, it embraces the same epistemological model of mathematics. This model says that: first, definitions must be expressed, eventually, in logical terms ("for all", "there exists", "no", "and", "or", "implies", etc.) and three non-logical terms ($\in$, = and $\varnothing$); and second, arguments must be deductive.

THE EFFECT OF TEACHING ENDS ON THE EPISTEMOLOGICAL MODEL OF MATHEMATICS

We agree with (Postman, 1996) in pointing out the importance of clarifying and analysing the teaching ends embraced by the different educational institutions. We defended in (Gascón & Nicolás, 2017) that all the scientific activity in didactics of mathematics relies on (typically implicit) assumed teaching, and that only by making those ends explicit rational discusion in didactics of mathematics would be possible.

Here we would like to show that the analysis of teaching ends of educational institutions contributes to explain interesting didactic phenomena. In particular, we would like to explain how the reasons for teaching mathematics assumed by that institution determine to a great extent the epistemological model of mathematics used in that institution. This model, has an influence over the mathematical praxeologies for teaching, which, in turn, shape the mathematical praxeologies to be taught.

Teaching ends of the faculty of mathematics

Let us inspect this idea in more detail with the example of the institution $\mathfrak{I}$ of the teaching of mathematics in a faculty of mathematics. The institution $\mathfrak{I}$ typically embraces, among others (perhaps less taken for granted), the aim of raising future mathematicians. Therefore, faculties of mathematics are typically used to explain to virtual future mathematicians how mathematics is made today, which can be perfectly understandable, and perhaps even desirable. Graduates in mathematics are thus expected to be aware of how propositional knowledge is officially achieved in mathematics nowadays. In particular, graduates should be familiar with the role played by sets (which are typically regarded as the basic objects) and *deductive* arguments (which are the only accepted arguments in official documents) in contemporary mathematics. In other words, graduates should be familiar with the current epistemological model of mathematics prevailing in the community of mathematicians.

Some features of the current epistemological model of mathematics of 'professional' mathematicians

Here we do not aim to give a thorough account of what is this model. We will rather emphasise one key feature concerning the current standards of achievement of knowledge. This feature is about the kind of arguments allowed nowadays in the community of mathematicians. At the end of the nineteenth century and the beginning of the twentieth century, the existence of alarming contradictions and paradoxes led many mathematicians to look for sound foundations for mathematical knowledge (Kline, 1972). Finally, Hilbert's proposal was gradually adopted and it is today a standard commonly accepted (Hintikka, 1996). At the center of this proposal one finds the notion of deductive argument.

First of all, let us see what an *argument* is. It is a speech act with which the speaker attempts to make someone else (or perhaps to herself) agree that a certain statement, the *conclusion*, is supported by a certain set of statements, the *premises*. An argument is *correct* if it really shows that the conclusion does receive support from the premises. Notice that, so far, we have not referred to the idea of truth. Now it is the right moment. An argument is *successful* if it is correct and the truth-value of the premises is justified. In this case, the truth-value of the conclusion would also be justified.

One of the characteristic properties of mathematical arguments is that they are intended to be not only successful, but also *deductive*. An argument is *deductive* if the speaker claims that nobody could believe that the premises are true without believing that the conclusion is also true. A classical example of deductive argument is the following:

- Premises: {All men are mortal, Socrates is a man}.
- Conclusion: Socrates is mortal.

The discipline which studies deductive arguments is *deductive logic*. To get a better understanding of what deductive arguments are, let us review some basic notions of a part of deductive logic called *first order logic*, strongly related to mathematics. All the known mathematics nowadays is virtually expressable in a *first order-language*, that is to say, in a language made of:

- logical symbols: variables, parenthesis, connectives ($\vee$ for the disjunction, $\wedge$ for the conjunction, $\neg$ for the negation, $\rightarrow$ for the implication), quantifiers (the existential $\exists$ and the universal $\forall$).
- non-logical symbols: constants, n-ary predicates (with $n \geq 1$).

For different purposes we use different first-order languages, distinguished one from the other by the non-logical symbols. For instance, in the deductive argument above, we use the constant s for Socrates, the 1-ary predicate H for the property "being a man", and the predicate M for the property "being mortal". In this first-order language, the argument would be as follows:

- Premises: $\{\forall x\ (Hx \rightarrow Mx), Hs\}$
- Conclusion: Ms

To provide the sentences in a first-order logic with a meaning, we need a *model*, which is a way to link the non-logical symbols with parts of the world. In the model underlying the previous argument, the constant s maps to the man Socrates, the predicate H maps to the set of all men, and the predicate M maps to the set of all mortal things.

When a sentence φ, written in a first-order language, is *true* in a model $\mathscr{M}$ of such a language we write $\mathscr{M} \vDash \varphi$. We say that a sentence φ is a *logical consequence* of a set Γ of sentences, written $\Gamma \vDash \varphi$, if we have $\mathscr{M} \vDash \varphi$ for every model $\mathscr{M}$ which satisfies $\mathscr{M} \vDash \Gamma$. Now we can give a more precise definition of *deductive argument*: it is an argument in which the speaker claims that the conclusion is a logical consequence of the premises.

Notice that, not being the idea of deductive argument relative to a precise fixed model, the property of being deductive is independent of any model. In other words, the fact of being deductive relies uniquely in the logical form of the argument, (the syntax), not in the interpretation of the non-logical terma (the semantic).

Apparently, by the very definition, in order to present a deductive argument, we would need to consider all the possible models for our language and to check that they do not make the premises true without making the conclusion true. Fortunately, this is not the case. Instead, we can use a certain collection of *deductive rules* which allow us to derive the conclusion from the premises. One of these rules used in the proof of Bolzano's theorem is the *reductio ad absurdum*: if from a set of premises $\Gamma \cup \{\alpha\}$ one can deduce both β and $\neg\beta$, then from Γ we can deduce $\neg\alpha$. Another rule, also used in the proof above is: if from a set of premises Γ one can deduce both α and β, then one can deduce $\alpha\wedge\beta$. In our proof we have also implicitly used a rule which tells you when it is allowed to deduce a statement involving the universal quantifier. The interested reader can fin more information about first order logic in (Smullyan, 1968).

When we can derive a sentence φ from a set Γ of sentences by using those deductive rules, we write $\Gamma \vdash \varphi$ and we say that φ is *deducible* from Γ. We do not only have that $\Gamma \vdash \varphi$ implies $\Gamma \vDash \varphi$ (which means that the relationship of deducibility is right: that is, if you use the deductive rules you produce a correct deductive argument), but also that $\Gamma \vDash \varphi$ implies $\Gamma \vdash \varphi$ (which means that the relationship of deducibility is complete: every correct deductive argument can be expressed by using the deductive rules).

Of course, we do not claim that the only kind of arguments mathematicians take under consideration in their *everyday* activity are the deductive ones. Concerning this (Thurston, 1994) and (Brousseau, 1995) are interesting reading. Neither do we say that mathematicians point out *explicitly* all the used deductive rules when they deal with deductive arguments. But it would be too naïve not to admit that deductive arguments are essential in the contemporary epistemological model of mathematics. According to this model, the only *institutional* knowledge is the one produced by deductive arguments. Indeed, if someone shows that a theorem has not been proved with a deductive argument, then this theorem will be immediately removed from the realm of the official knowledge. In other words, the community of mathematicians accepts in practice arguments which do not present explicitly the deductive rules used, but only because: first, those arguments are believed to be theoretically expressable in terms of deductive rules, and second, it would be extremely tedious to write down all the deductive rules used.

Evidence of the key role played by deductive arguments is that, for them to be theoretically possible, one needs to use definitions expressed in the first order language. This entails a deep logical analysis of intuitive notions to make them ready to play a role in deductive games. For instance, the notion of continuity of a function, or the very notion of function, in the eighteenth century was used according to an intuitive meaning, rooted in pragmatic considerations related to whether a function could be written down by using a single analytic expression or nor (Kline, 1972). But, as deductive arguments started to become more and more important in the epistemological model of mathematics, a logical analysis of function or continuity was needed. This is how we ended up with the ε-δ definition which expresses continuity in terms of quantifiers, implications, real numbers and inequalities.

It is important to notice that the current epistemological model of mathematics shapes not only the mathematical praxeologies to be taught in the faculties of mathematics, but also the mathematical praxeologies for teaching. Indeed, these last praxeologies consider different forms of reconstructing certain works of mathematics, but they never question the epistemological model.

Teaching ends of mathematics in secondary education and the missing of a corresponding epistemological model of mathematics

Primary education aims to provide the very basics of our culture and our knowledge of the world, for the students to begin their path towards the status of autonomous and suitable citizens and to be able to continue further in their studies. Then, secondary education aims to provide with more specialised culture and knowledge for the students to become entirely autonomous and suitable citizens, and for them to be ready to get involved in some profession or to get into the deep study of some disciplines, for instance mathematics.

But, of course, the institution of the teaching of mathematics in secondary education does not intend to raise future mathematicians, and so there is no point in explaining to the students of secondary education how mathematical knowledge is officially achieved today among professional mathematicians. This entails that teachers in secondary education should not frame their teaching within the current epistemological model of mathematics of the faculties of mathematics. In particular, there is no point in putting the logical analysis of notions and deductive arguments at the center of the teaching. But then, how does one explain those notions? What kind of arguments should be used? For instance, if there is no point in using set theory (ordered pairs, Cartesian products) to define what a funcion is, or the ε-δ statement to define what continuity is, how to explain what a continuous function is? And if we do not use the logical analysis for the definitions, then we cannot use deductive arguments for the theorems. Hence, how to make a non-deductive argument for Bolzano's theorem? Is it possible? Is it needed? These are crucial questions of mathematical praxeologies for teaching mathematics in secondary education.

CONCLUSIONS

The change from the institution of teaching in the faculty of mathematics to the institution of teaching mathematics in secondary education entails a change of teaching ends. This, in turn, forces the future teachers of secondary education to look for a new epistemological model of mathematics. The one transmitted at the faculty of mathematics is not only deeply rooted in those future teachers, but also it is received as if it were the *faithful* account of *real* mathematics. On the contrast, there is no official alternative epistemological model of mathematics at hand.

This problem could be tackled at university, by posgraduate masters' degree on teacher training in secondary education, but it is far from being the case. On the contrary, it seems to be widely assumed that, on the side of mathematics, teachers at secondary education should not find any problem at all, as they already are sufficiently well-informed in this discipline.

The lack of a genuine and coherent alternative epistemological model of mathematics for secondary education means a huge field of open problems for the community of researchers in didactics of mathematics.

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The relation between mathematics research activity and the design of resources for teaching at the university

Hussein Sabra

Cérep EA 4962, University of Reims Champagne Ardenne, France, hussein.sabra@univ-reims.fr

We study the relation between research and teaching practices of teachers-researchers at university. We examine this issue from the documentational approach point of view that focuses on the interactions between resources and mathematicians by considering their research activities and teaching practices. We suggest indeed theoretical and methodological developments to take into account, from the documentational approach to didactics, the interactions with resources during the research activities of the mathematicians. The data collection consists in audio-recorded interviews. We identify three forms of use of research resources in teaching practices.

Keywords: resources in university mathematics education, teachers' practices at university level, relation between research activity and teaching practices.

INTRODUCTION AND CONTEXT OF THE STUDY

The professional activity of a university teacher usually involves teaching activity and research activity. In France, among the university teachers there are teachers-researchers: they do research and have to teach at different levels (Tertiary level and Master's degree). At the early post-bachelor years, some universities offer to teach the basics of classical mathematics. Some of teachers-researchers consider those kinds of courses as important and necessary to students but unfortunately too basic regarding their field of research. The present paper constitutes a part of our research interest that concerns the understanding of the relation between teaching and research activities, and this by highlighting the disciplinary specificities. Indeed, we aim to highlight aspects that might be considered to characterize the factors underpinning it.

We are particularly interested in the study of the relation between research activity and teaching practices through the lens of interaction with resources. As Adler (2000), we give to the "resource" here, a meaning related to the verb "re-source", to source again or differently. We are conducting exploratory studies related to this issue considering different aspects:

- Considering two disciplines mathematics and physics, work that allows us to characterize factors determining the relation between research activity and teaching practices, either related to the epistemology of the discipline or not (Sabra & El Hage, 2018).
- Setting a contemporary field in mathematics (graph theory) and varying the institutions of teaching (Tabchi's PhD work, in progress) (Tabchi, 2018).
- Setting an institutional context – Engineering Education – and considering teachers of mathematics, coming from different research disciplines (physicist,

mathematician, and engineer), study that allows us to characterize factors that enhance the design and use of resources in terms of the personal relationship to mathematics and his/her (researcher) domain of research (Sabra, 2019).

The present paper constitutes a contribution to this research work. We provide a case study of the research activity of three mathematicians through the lens of the interactions with resources. We particularly dwell upon the place of research resources and their impacts on the designing and the use of the resources in and for teaching. Indeed, our general question is: how do the resources coming from research activity are related to the teacher's capacity to re-design them for his/her teaching work?

We present some theoretical and methodological development based on the *Documentational Approach to Didactics* (DAD) (Gueudet, Pepin, & Trouche, 2012).

RELATION BETWEEN RESEARCH ACTIVITY AND TEACHING IN THE RESEARCH LITERATURE

Some researches in science education attempted to find evidence of "positive" or "negative" correlations between research and teaching without taking into account a specific discipline (Elton, 1986; Neumann, 1992). They tried to characterize the relation that may occur between teaching activity and research activity (symbiosis, conflict, tension, etc.). Neumann (1992) presents three aspects of what he calls "nexus" that can exist between teaching and research: 1) the tangible aspects, generally linked to an articulation between content transfer of knowledge from research in teaching; 2) intangible aspects, which relate to the actions of the researcher in the teaching activity and vice versa); 3) the global aspect, which relates to nexus between teaching institution and research institution. In a more recent study, Elton (2001) examined the reasons behind the presence or absence of the relation between teaching and research in the practice of university teachers. In a perspective of transformation of practice, he suggests ways that could reinforce "positive" articulations between the two kinds of activities.

The question of the correlations between the two activities of a university teacher has been studied recently depending on the discipline involved. As an example, Madsen and Winsløw (2009) emphasize that the relation between research and teaching in the case of mathematics significantly differs from the physical geography discipline. In their comparative study between teachers in geography and mathematics, they emphasized the fact that the forms of relation between teaching and research strongly depend on the disciplinary specificities (institutional and epistemological characteristics of the discipline). They also stressed that the relation that can take place between both teaching and research activities depend on the perception of university teachers on the specificities of their disciplines.

Other comparative study based on the interviews with teachers-researchers in physics and mathematics, emphasizes the place of what they called *professional identity* of university professors (Lebrun et al., 2018). They highlight that the professional identity of the teachers-researchers in both disciplines seems to be in tension due to the epistemology of the discipline; interviewed professors from both disciplines highlight the importance to teach following methods derived from research activities (group work, problem solving, modelling, etc.). However, they raise organisational constraints that prohibit applying them, particularly the assessment practices and the limited time.

Therefore, we claim to understand the relation between teaching and research within the mathematics discipline through the lens of interaction with resources. This interaction can take place at different moments of teaching practices in: the design of the classroom sessions, the choice of the contents, the implementation of resources in the classroom, and in the evaluation of learning. In addition, university teachers could use the same resources in their teaching practices and their research activities (Broley, 2016).

DOCUMENTATIONAL WORK IN RESEARCH AND TEACHING INSTITUTIONS

The DAD considers the activity of the teacher as a continuous process. In the DAD, there is a distinction between resources and documents. We define here resources as all the things that could re-source a university teacher activity (research and teaching). The interaction with the resources generates a document, which is the association of resources and a scheme of use of these resources. We can assume that in the case of university teachers the research resources re-source particularly the research activity. However, this dimension is not investigated here. We are interested in how research resources influence the design of resources for teaching.

A scheme is used here as defined by Vergnaud (1998) as the invariant organization of conduct for a set of situations having the same aim. According to Vergnaud (1998), a scheme is a dynamic structure that has four interacting components: aim, rules of actions, operational invariants, and possibilities of inferences. A class of situations includes all the situations having the same aim.

A university teacher develops a professional experience by interacting with the teaching institution and the research institution simultaneously (Madsen & Winsløw, 2009). The interaction with resources in each of the institutions are related on the one hand to the specific classes of situations (research classes of situations, teaching classes of situations) and on the other hand to the specificities of the discipline. The relation between research and teaching could take place as a migration and adaptation of the resources between institutions, or also like a dissemination by a university teacher of the professional knowledge and mode of teaching (the "operational invariants" component of scheme of use resources, Gueudet & Trouche, 2009).

We distinguish between: 1) the *teaching document* (aims related to the teaching class of situation, resources for teaching, rules of action and operational invariants) in the meaning of (Gueudet, 2017); 2) the *research document* (aims related to research classes of situation, resources for research, rules of action and operational invariants). Each kind of document is considered in its institution with corresponding conditions and constraints. Gueudet (2017) notices that university teachers develop a resources system for research in the research institution and a resources system for teaching in the teaching institution. The study of both resources systems and their interaction requires new theoretical and methodological developments. Given the background, we have explored the process of interaction between both systems from the point of view of "pivotal" resources in research activities of the university teacher.

The concept of "pivotal resources" is characterized in the previous studies using DAD since resources that intervene in several classes of situations (Gueudet, 2017). In this paper, the "pivotal resources" are considered in the teaching documentation work. In our contribution, we define a "pivotal resource" as a resource that contributes for a given teacher to the construction of many research documents in an institution. We consider that a pivotal resource is used in several class of research situations. Using frequently a pivotal resource could influence a part of the research activity. For us, if there are relations between research and teaching activities, it will take place in terms of the classes of situations where pivotal resources are mobilized.We hypothesized that there is at least one pivotal resource in the research work of a given mathematician. It could be a software of numerical computation, a founding book in his/her field of research, or others. Consequently, our general question turns out to be as follow: How do the pivotal resources coming from research institution enrich the teacher's capacity to re-design and use them for his teaching work?

CONTEXT AND METHODOLOGY OF THE STUDY

As an exploration of different facets of the issue of relation between research activity and teaching practices, we present here a study based on three interviews with French teachers-researchers (see Table 1. for the profiles). To keep the anonymity we will call them M1, M2 and M3. A university teacher in France must teach at different levels, a variety of subjects and topics ranging from the basic level in a discipline to very specialized courses in her fields of research.

	Research experience	Research domain	Teaching experienc e	Teaching level
M1	16 years	Mathematical modelling of physical phenomena	16 years	Undergraduate degree (Mathematics and computer sciences) and Master degree (applied Mathematics)
M2	6 years	Mathematical modelling of scientific phenomena	6 years	Undergraduate degree (Mathematics)

M3	17 years	Theorist mathematician (number theory)	17 years	Undergraduate (Mathematics) and Master degree (pure Mathematics)

Table 1: The profiles of the three university teachers.

We elaborated the interviews guidelines from two distinct parts: research activity part and teaching activity part. We did not ask direct questions about resources so that the interviewed could express themselves freely about their research and teaching activities. This choice allowed us to identify the resources quoted in their answers that we considered as a pivotal resource. The interviews lasted between an hour and an hour and a half; they were semi-structured; each interview took place in the office of the university teachers. All the interviews were recorded and conducted in French.

The transcripts of the interviews were coded according to the theoretical framework and our development/adaptation in order to build for each interview two tables: the teaching documents table corresponding to the teaching work, and the research documents table corresponding to the research activity (see Table 2). The tables allowed us to consider the list of documents in each of both institutions: research institution and teaching institution.

Research documents tables			
Research aims	Resources	Rules of action	Operational Invariants
Teaching documents tables			
Teaching aims	Resources	Rules of action	Operational Invariants

Table 2: Presentation of the research documents table and the teaching documents table.

To build the teaching documents tables, we proceeded in the same way as (Gueudet, 2017). Actually, we tracked in the transcript of the teaching part of each interview the given answer of the aim of the teaching activity mentioned by the university teacher. For each aim, we added the resources explicitly mentioned in the transcribed declaration. Then, we identified stable elements in the way these resources were used (rules of actions). Concerning stability, we relied on the teacher's declarations (e.g., "for …, we always start by…"). Finally, we noted the operational invariants (this corresponds to statements in the interview such as: "I do this way …. Because I think that …".

We proceeded in the same way for the research part of the interview, which concerns research in order to build the research documents tables. First, we defined a research aim. Then we added resources, we identified rules of actions in the declaration. Finally, we noted the operational invariants.

Once both tables were built, we first identified the pivotal resources in the research documents table (see table 3).

Research documents table			
Aims (A_i)	Resources	Rules of actions (RA)	Operational invariants (OI)
A_1	$\boxed{\text{Resource 1}}$	RA_1	OI_1
A_2	Resource 2	RA_2	OI_2
A_3	Resource 3, $\boxed{\text{Resource 1}}$	RA_3	OI_3
…	…	…	…
A_n	Resource 4, $\boxed{\text{Resource 1}}$	RA_n	OI_n
Teaching documents table			
Aims (A_i)	Resources	Rules of actions (RA)	Operational invariants (OI)
A_1	$\boxed{\text{Resource 1}}$	RA_1	OI_1
A_2	Resource 5	RA_2	OI_2
A_3	Resource 6, $\boxed{\text{Resource 1}}$	RA_3	OI_3
…	…	…	…
A_n	Resource 7, Resource 8	RA_n	OI_n

Table 3: Identifying pivotal resource, which is Resource 1 in this research documents table. Resource 1 appears also in the teaching documents table.

Then we checked whether the pivotal resource in the research documents table (Resource 1 in table 3) was mentioned or not in the teaching documents table. When it was the case, we took into account the teaching document where this resource appears (the table line corresponding to the document). If not, we tried to understand the reason behind the lack of this resource regarding the operational invariant in research institution and/or the consideration of constraints in the teaching institution.

This methodology enables to question the resource mobilization process from research institutions to teaching institutions, by considering a horizontal analysis of each document in each institution.

FORMS OF RELATION BETWEEN RESEARCH AND TEACHING IN TERMS OF RESOURCES

By our analysis, we identified three forms of relation between research and teaching in terms of resources.

First form: research resource in instantiation processes

In the case of M1, we identified seven aims in the research institution, in which the software (Matlab, Maple, etc.) is fundamental in numerical modelling research (6 aims over 7). M1 uses the software to conjecture, validate (a conjecture or a modelling method). The place of the software occupies the main line of his research approach. In the teaching institution, we identified two teaching documents where the software is used. He uses the software with the Master's degree students in order to sensitize students to the characteristics of the software in the activity of mathematical modelling (see table 4 for an example of those documents).

	M1- teaching document
Aims	Sensitizing students to the characteristics of software in the activity of mathematical modelling.
Resources	Software of numerical computing. Resources from previous teaching years that contains problem to solve.
Rules of action (way to use the resources)	Choosing software used in the research. Choosing and adapting a problem solving that permit a manipulation, an observation and the interface of software and experiment with it.
Operational Invariants (reasons for using them this way)	The modelling activity in mathematics is exploratory and experimental.

Table 4: Presentation of a teaching document where the pivotal resource is used.

In this case (table 4), we qualify the use of pivotal research resource in teaching institution as an action of instantiation of it. The instantiation of this resource consists in the mobilization of the research resource from research institution in the teaching institution in, as far as possible, the similar situations and in the similar role in both institutions, but in a more restricted domain of validity.

Second form: research resource to scaffold the learning of a given content

In the case of M2, we identified six aims related to his research activities, in which the software (Matlab, Maple, Scilab, etc.) is fundamental in numerical computation and graphical simulations (3 aims over 6). His research activities using a software particularly consists in analyzing, modelling biological phenomena, validating the experimental results, and communicating results to the biologists he works with. In the teaching institution, the software of numerical simulations appears in two teaching documents. We develop, in the table 5, one of them which corresponds to the aim "designing session to experiment and discover mathematical properties with software".

	M2 – teaching document
Aims	Designing session to experiment and discover mathematical properties with software.
Resources	Software of numerical computation. Resources corresponding to the course in question.
Rules of action (way to use the resources)	Select a phenomenon of stability of differential equation. Show the stability on a graphical representation. Offer the possibility to vary values and parameters in order to lead a discussion about hidden properties.
Operational Invariants (reasons for using them this way)	A software is a tool that gives the results in a visual way and hides the properties. We have to stimulate the spirit of imagination to make links between representations and mathematical properties underpinning.

Table 5: Presentation a teaching document related to the aim "designing session to experiment and discover mathematical properties with software".

M2 assigns the same role to the software in the construction knowledge in both institutions (research and teaching), while the operational invariants show that M2 uses software in teaching institution to scaffold contents, in the design of the resource as well as in the implementation.

Third case: the no relation form in terms of resources

In the case of M3, there is a pivotal resource in the research documents table; however, it is not mentioned in the teaching documents table. This result is strengthened by the words of M3 during the interview acknowledging that there is a gap between mathematics research activity and mathematics teaching activity. From his point of view, if there is a link it will be in the way of teaching (Operational Invariant). He teaches the proof following the same process lived in his research: he makes hypotheses then he determines the properties to be mobilized. There are no resources in common between teaching institution and research institution. He has a perception of "divorce" between the two institutions. He does not place his students in research situations. According to him, to be able to do this, the whole community of the class does not have to know how to solve tasks. The relations that can exist are not tangible (Neumann, 1992). They correspond to for instance, the relations between the way of teaching "to follow the same approach of research" in the treatment of a proof.

We can deduce that there is a relation between teaching and research which could be seen through the process of using the resource in the classroom and not only as a process of migration of resources from research institution to teaching institution. This result meets the ones identified by Tabchi (2018) in the case of teachers-researchers in graph theory. We qualify the interactions between research and teaching institutions as an action of spreading scientific attitude (research process) in teaching practices.

FINDINGS, DISCUSSION AND PERSPECTIVES

It appears mainly that the relation maintained between research and teaching depends closely on the university teachers' perceptions of his/her research resources. We remind that our methodological choice requires to identify the pivotal research resources of university teachers and then study the relation between research and teaching. The analysis results support our hypothesis that the pivotal resources influence an important part of the research activity and thus if there are relations between research and teaching activity, it might take place in terms of the classes of situations where these resources are mobilized.

The documentational approach offers a possibility to characterize *tangible nexus* (Neumann, 1992) between research and teaching (via the kind of interaction with resources), but also *intangible nexus* (Neumann, 1992) related to the interaction links to the specific professional knowledge of the university teachers; the operational invariants resulting from the research activity partly determine teaching practices. Therefore, schemes (Vergnaud, 1998) in the interactions with resources are challenging to infer. One source of complexity of the scheme concept is the component 'operational invariant', which is invisible and not always conscious to the teacher. From a methodological point of view, it is a matter of inferring schemes by cross-referencing data from different tools and sources: interviews, observation of teachers, and so forth.

The study of the relation between the research resources system and the teaching resources system deserves further study or even a long-term study that contains observations. In addition, A teacher may have two different forms of relation between teaching and research depending on the teaching aims (indeed, the associate classes of situation). This is a field to explore in order to understand the interactions between the teaching resources system and the research resources system.

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A Student's Complex Structure of Schemes Development for Authentic Programming-Based Mathematical Investigation Projects

Chantal Buteau[1], Ghislaine Gueudet[2], Kirstin Dreise[1], Eric Muller[1], Joyce Mgombelo[3], & Ana Isabel Sacristán[4]

[1]Department of Mathematics and Statistics, Brock University, cbuteau@brocku.ca; [2]University of Brest, France; [3]Brock University, Canada; [4]Cinvestav, Mexico

Keywords: Teachers' and students' practices at university level, Digital and other resources in university mathematics education, Programming, Instrumental approach.

INTRODUCTION AND RESEARCH QUESTION

In Gueudet et al. (submitted), we discuss how the instrumental approach can contribute to our understanding of the activity of university students using programming in the context of an authentic mathematical investigation. In particular, we distinguish between m-schemes, p-schemes and p+m-schemes, for a goal concerning respectively only mathematics, only programming, or both. Each of these three types of schemes is illustrated in the case of an undergraduate, Jim, as he engaged in the first of four programming-based mathematics investigation project, within a Mathematics Integrated with Computers and Applications (MICA) course at Brock University.

In this poster, we extend the work in Gueudet et al. (submitted) and present a visual summary of the complex structure of m-, p-, and p+m-schemes developed by Jim through his engagement in the 4 course projects. Our guiding research question is:

What do we learn about the activity of students using programming in an authentic mathematical investigation by using the theoretical frame of the instrumental approach, considering programming as an artefact?

THEORETICAL FRAMEWORK

Our work is informed by the instrumental genesis approach (Rabardel, 1995) which provides a lens to describe how a student, in an activity with a math goal, learns to use an artefact (e.g. programming) and learns mathematics at the same time, through the development of schemes. A scheme is a stable organization of the subject's activity for a given goal (Vergnaud, 1998). It comprises four components: i) the goal of the activity; ii) rules-of-action (RoA), generating the behaviour according to the features of the situation; iii) operational invariants: concepts-in-action and theorems-in-action (TiA), which are propositions considered as true; and iv) possibilities of inferences.

METHODOLOGY AND RESULTS

Jim was one voluntary student participant (among 6) enrolled in the MICA I course (46 students) in the first year of our 2017-2022 research study. Data collected for this poster work were generated from: Jim's 4 MICA I projects and 4 semi-structured individual task-based interviews following each project submission; a baseline questionnaire and interview; and 10 online weekly lab reflections. Jim's early data was first analysed for an initial identification and description of schemes, then reorganized in m-, p-, and p+m-scheme types, and ordered according to the development process (dp) model shown in Fig.1. The whole data was thereafter coded and regrouped in themes. Using codes pertaining to perceptions and strategies themes, the initial table of schemes was then refined and chronologically extended to the whole data.

In this poster, we present 9 of Jim's 21 identified schemes as brief bubble descriptions linked to the related 9 steps of the dp model, as partially exemplified in Fig 1.

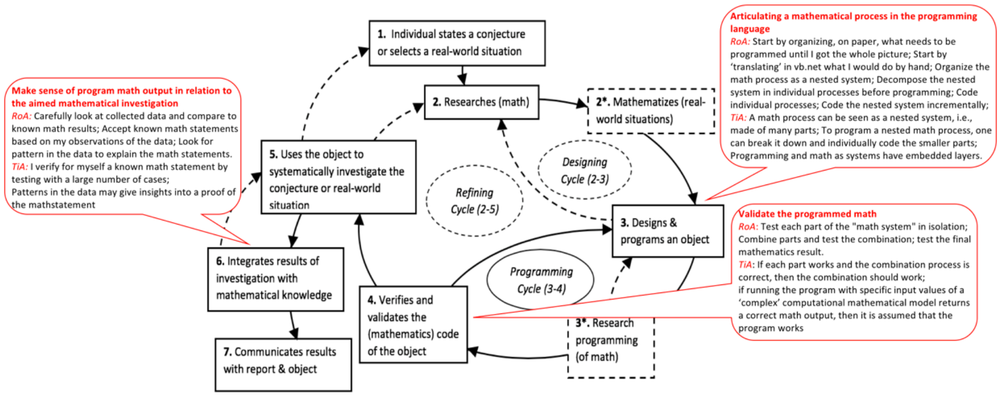

Figure 1: Development process (dp) model of a student engaging in programming for an authentic mathematical investigation or application (Buteau et al. 2019), enhanced with 3 examples of Jim's identified schemes (red bubbles).

IMPLICATIONS

Using the instrumental approach led i) to elaborate the dp model as a composition of goals, highlighting the complex structure of schemes; and ii) to expose how the activity of programming-based mathematical investigations is organized (RoA) and why (TiA).

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Promoting mathematics teacher reflection in online graduate problem solving course through peer feedback and portfolios

David Glassmeyer[1]

[1]Kennesaw State University, United States, dglassme@kennesaw.edu

The content of this poster will include a brief problem statement, research question, theoretical framework, methodology, results, and implications of this study. A portion of the poster will also be visual examples of the process of using peer feedback and portfolios in online settings to promote mathematics teachers to incorporate problem solving in their classroom.

Keywords: mathematics teachers, online, peer feedback, portfolios, problem solving.

RESEARCH TOPIC

Researchers have documented the difficulties associated with online teaching in comparison to face-to-face teaching (Shao & Abaci, 2018). Specifically, "the ability to engage with students and encourage and inspire them to reach their full potential through online teaching is difficult" (Rossi & Luck, 2011, p. 70). Within mathematics teacher education, there is a challenge of providing online graduate education that positively impacts teacher practice.

Promoting mathematics teacher reflection has been linked to productive changes in teacher education. Researchers have documented how portfolios can be a tool for reflection that helps teachers make productive changes to their classroom practice (McIntyre & Dangel, 2009). Furthermore, incorporating peer feedback as an element of portfolios can promote teacher reflection in higher education settings.

Little research exists on how portfolio and peer feedback can be incorporated in online mathematics teacher education, but this knowledge is needed to help mathematics teacher educators promote reflection and productive changes to classroom practice that benefit students. For example, McIntyre and Dangel (2009, p. 82) recommend that "teacher candidates should have the opportunity to orally present their portfolio" but it is unclear how that might occur in online settings.

This study addresses this research need by examining the affordances offered by portfolios that use peer feedback within an online problem solving graduate course for mathematics teachers. The research questions were: (1) In what ways does this online problem solving portfolio promote mathematics teacher reflection?; and (2) How does online peer feedback impact mathematics teacher reflection in their portfolio?

METHODOLOGY

The 23 participants in this study were all in-service U.S. mathematics teachers in an online problem solving course. A course requirement was to create a problem solving portfolio containing six entries. Four of the entries were problems that the teacher had posted on the discussion board, responded to two peers' postings, and revised based on the peer and instructor feedback. The fifth entry was including a problem one of their peers had posted on. The six entry was a reflection that summarized their past, present, and future efforts to implement problem solving in the teaching of mathematics. These portfolios were the study's data source.

The theoretical framework used in this study was Sparks-Langer, Simmons, Pasch, Colton & Starko's (1990) Framework for Reflective Pedagogical Thinking (FRPT). This framework was used to code teacher portfolios based on the seven levels of reflective thinking. This framework has been used in multiple teacher education settings (McIntyre & Dangel, 2009) and provides a basis for which to answer research question 1. Research question 2 was answered using open coding on the portfolios to determine when and how peer feedback contributed to the participant's reflection.

ANTICIPATED RESULTS AND IMPLICATIONS

Analysis of the teachers' portfolios revealed at least one entry of each teacher exhibited a level 5 or higher FRPT coding due to portfolio requirements to reference research-based articles when describing the problem solving entry. Several teachers exhibited a level 7 with at least one entry of the portfolio, in part due to portfolio requirements to include a social justice entry. Peer feedback appears to have been a secondary factor following the course materials, with instructor feedback being a tertiary factor in teachers' reflections. Implications for this work include an example for how teacher educators can structure online portfolios and peer feedback cycles for in-service mathematics teachers.

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A novel application of the instrumental approach in research on mathematical task

Vegard Topphol

Norwegian University of Science and Technology (NTNU),
Department of Mathematical Sciences, Trondheim, vegard.topphol@ntnu.no

Keywords: teacher's and students' practices at university level, transition to university mathematics, tasks, competencies, instrumental approach.

INTRODUCTION

In the Nordic countries, a focus on the nature of mathematical tasks and how they can be used in education has been central (e.g. Bergqvist, 2007). In this poster, I therefore explore a new approach to analysing mathematical tasks, where I see tasks as instruments in the development of mathematical competence.

A link between tasks and transition can also be seen in Bergqvist (2007), where tasks in early calculus courses are analysed, and Roh and Lee talk about tasks designed to "bridge a gap between students' intuition and mathematical rigor" (Roh & Lee, 2016, p. 34).

THEORETICAL FRAMEWORK

By seeing tasks as an instrument, I propose to use the instrumental approach (Trouche, 2004), to describe how a student could be able to develop mathematical competence through working with tasks. In the instrumental approach, a tool is just an artefact as long as a subject has not yet connected any usage patterns with it. But through the process of instrumental genesis, the tool becomes an instrument. The process consists of an instrumentalisation, where the artefact becomes an instrument, as the subject personalises it and appropriates it into the subject's activity, and an instrumentation, where the subject becomes a tool user.

The notion of tasks as mediating artefacts in an activity is not in itself new (e.g. Johnson, Coles & Clarke, 2017), but describing them as tools, according to the instrumental approach, is to my knowledge new. Based on Activity Theory (Leont'ev, 1978), and using Leont'ev's three levels of *activity*, *action* and *operation*, I argue that this use of the instrumental approach is viable. The task can be seen as a tool in the activity, oriented towards the objective of becoming competent in mathematics. Solving tasks are then actions in this activity, and the different operations done to solve the task corresponds to usage patterns.

TASKS

I propose a definition of a formal task. For a task to be a formal task, it must fulfil four criteria. It must have a purpose accessible to and possibly benefiting the one performing the task. This purpose should also be possible to know before the task has been solved, so that it will be possible to choose a task according to the goal of the learning activity, without knowing the solution to the task. It should be possible to finish the task in a meaningful way, that is, the task should have a logical and identifiable conclusion. And lastly, the solution to the task should be possible to arrive at by some form of logical inference. The two last points are connected to the usage patterns. For a task to be an instrument, there needs to be a predictable way of learning to know how to use it. As I see it, both an identifiable conclusion, and a logical way to arrive at this conclusion, adds to this predictability.

METHODOLOGY

This is a qualitative study. More specifically, I use a series of task-based interviews, where I follow a number of university students through a single variable and a multi variable calculus course. They are asked to describe their own thinking process as they solve tasks. Further, I plan to analyse tasks found in calculus textbooks both in upper secondary and in university, in light of the findings from these interviews.

Using the instrumental approach combined with my notion of a formal task, I claim that it is possible to describe and analyse how students are able to use tasks as instruments for developing mathematical competence. This will have implications, not only for how tasks are created and selected, but also for how tasks are presented and used in education.

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