Klein's Plan B in the Early Teaching of Analysis: Two Theoretical Cases of Exploring Mathematical Links

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Abstract

We present a theoretical approach to the problem of the transition from Calculus to Analysis within the undergraduate mathematics curriculum. First, we formulate this problem using the anthropological theory of the didactic, in particular the notion of praxeology, along with a possible solution related to Klein's "Plan B" : here, re-linking the theory of Analysis with practical knowledge from Calculus. We explore two cases based on this approach: (1) the contribution of Vector Analysis to the foundations of trigonometric functions, and (2) establishing the ties between the proof of a basic theorem in Fourier Analysis and the computation of elementary infinite series. These two cases, including small-scale trials, illustrate the necessity, importance and possibilities of new didactical approaches aiming to help students to integrate mathematical theories and practices which are otherwise taught separately.

Introduction

An evident tendency of mass university education is what French sociologist Michel Verret (1975, 140) called the *desynchretisation* of knowledge. The word indicates that elements of knowledge which were originally combined and united, for instance in the context of discovery, become separated again as they are taught. At universities, efforts to enhance efficiency and economy of exposition have led to still tighter and shorter modules of teaching, each focusing on a quite narrow and homogenous domain of knowledge. Such modules are considered easier to digest for students and they can also be easily combined with each other in different ways, to cater to different programs or streams. The price of that efficiency may be a loss of connections which are important for the meaning, uses and further development of scientific content. At the extreme, teaching leads students to visit each element of knowledge as a "monument that stands on its own, that students are expected to admire and enjoy, even when they know next to nothing about its *raisons d'être*, now or in the past" (Chevallard, 2012).

For several years, we have worked with a special case of this phenomenon, occurring in the teaching of mathematical analysis at undergraduate level and also observed by other researchers (see e.g. Nardi, Jaworski and Hegedus, 2005). One gross symptom of desynchretisation in this context is the frequent separation, also by name, of courses on *Calculus*, and subsequent courses on various branches of *Analysis.* Calculus courses specialize in mathematical themes indicated by course titles such as "Integral Calculus", "Functions of Several Variables" or "Ordinary Differential Equations". Analysis courses, on the other hand, treat theoretical perspectives on these same mathematical themes, gradually moving from course titles such as “Real Analysis”, “Fourier Analysis” towards more abstract areas such as Functional and Harmonic Analysis. In short, Calculus courses can be roughly characterized as teaching students certain calculation practices related to real and vector valued functions, with little theoretical precision or justification—while Analysis courses tend to present “formal theory with little practice” and consequently little evident need for formalization. Of course, there are practical reasons for the separation: the two types of courses cater to different student populations. While Calculus courses are studied by a large cohort of students in the natural and social sciences, much fewer students study Analysis (mainly students of pure mathematics, theoretical physics and mathematical statistics). For these and other reasons, it may be difficult to change the course structure.

Our aim in this paper is to provide a theoretical approach, supported by two small-scale trials, to the problem of recovering some of the meaningful links with previous courses in post-Calculus teaching on Analysis. A strong motivation for our work is that the transition from Calculus to Analysis is known to present mathematics students with several challenges (for examples, see Winsløw and Grønbæk, 2014). Here is a typical student formulation of some of these (interview with a student of the first author, summer 2016):

> In Calculus courses we learn methods, but usually the why questions are not explained or proved. (...) However, Analysis courses felt as separate. They were more theoretical than applied. I never grasped them as well as Calculus. It was often unclear, what it is leading to. I wish we had a better sense of connection between the theory we covered in pure math courses and the methods shown in applied math courses.

We have explored this perceived lack of “connection” in earlier papers (Kondratieva, 2011, 2015; Winsløw, 2007, 2016). In this paper, we first present a theoretical framework for the study of this “connection problem”, which also guides our efforts to design “new connections” to be made in the context of undergraduate Analysis teaching. We then present two cases in which students are invited to explore different types of links between formal Analysis and students’ practical knowledge of Calculus: Case 1 on a link between Vector Analysis and the meaning of trigonometry and angle, and Case 2 on establishing the ties between the proof of a basic theorem in Fourier Analysis and the computation of infinite number series such as $\sum 1/n^2$. In Case 1, we consider an Analysis course in which various elements could have led students towards the mentioned link, while an interview with an advanced student, who has even worked as a tutor in the course, shows that this link was not effectively established. In case 2 we trial a set of exercises, specifically designed to establish a link between Calculus and Fourier Analysis. This activity turns out quite successful, as demonstrated in the recorded experiences of a small sample of students.

Our intention with these two cases is neither to provide solid or ‘generic’ empirical results nor to present a ready-made, teacher-proof set of teaching materials. Certainly, these cases indicate some challenges and potentialities for reflective teachers of Analysis, as well as some concrete ideas they might adapt to their institutional conditions. But our main point with this paper is theoretical: to present and model a research problem of the utmost importance to the undergraduate teaching of Analysis, and to exemplify how mathematician-didacticians could approach it: by designing tasks and activities through which students may recover at least parts of the meanings and links that were lost in decades of didactical desynchretisation.

Theoretical Background and Framework

Klein's Plan A and Plan B

Felix Klein (1908/1932, pp. 77-85) considered that, in the history of mathematics, as well as in the discipline of school mathematics, we may identify two possible "Plans" (one might also say, visions or strategies) for developing a subject; and, he used the case of classical Analysis to illustrate these. What Klein calls *Plan A* is a *compartmentalized approach* to mathematics, very close to what Verret called "desynchretisation", which favors precise and purified work within certain small "areas" of mathematics, which are hardly related with each other:

> Plan A is based upon a more particularistic conception of science which divides the total field into a series of mutually separated parts and attempts to develop each part for itself, with a minimum of resources and with all possible avoidance of borrowing from neighbouring fields (ibid., p. 78).

The most famous examples of "Plan A" in the history of mathematics itself include, of course, Euclid's Elements, with its strict separation of Geometry and Arithmetic. Plan A refers more generally to an economy of exposition which prefers minimal paths of deduction towards key results in each domain of mathematics, with little affection for alternative (e.g. historical) routes and connections.

Plan B, by contrast, involves a more holistic approach which emphasizes and exploits connections between different domains:

> … the supporter of Plan B lays the chief stress upon the organic combination of the partial fields, and upon the stimulation which these exert one upon another. He prefers, therefore, the methods which open for him an understanding of several fields under a uniform point of view. His ideal is the comprehension of the sum total of mathematical science as a great connected whole (ibid., p. 78).

Klein observes that, in the history of mathematics, "Plan A" and "Plan B" both appear during fruitful periods of research, in Analysis as well as in other areas. For instance, the initial developments of the Calculus took place much according to Plan B, led by Leibniz and Newton; later, a progressive move towards Plan A occurred, as Cauchy and others gave classical Analysis the solid foundations we know today.

Klein strongly recommends using "Plan B" as a strategy for presenting mathematics to students, and laments the exclusive use of Plan A. On this point, little has changed except that perhaps the prevalence of Plan A is even stronger in present-day university programs on pure mathematics than in Klein's days, as we have pointed out in the introduction.

In Klein's terms, our aim in this paper is to revitalize the idea of "Plan B" in the context of undergraduate Analysis, and to exemplify how it could materialise in specific situations, with concrete links as learning objectives. To do so, we now introduce a few elements from the anthropological theory of the didactic (ATD) (Chevallard, 2006) which serve as the theoretical underpinnings of our discussion of, and design for, Plan B.

Praxeological analysis as guideline to design for Plan B

Chevallard (2006) defines a *praxeology* as a pair (P, L) consisting of a *praxis block P* and a *logos block L*. A praxeology is a minimal element of human knowledge, P representing the practical part—the "know how" —and L the intellectual part, the "thinking and explaining". The two are interdependent:

> …no human action can exist without being, at least partially, "explained", made "intelligible", "justified", "accounted for", in whatever style of "reasoning" such an explanation or justification may be cast. Praxis thus entails logos which in turn backs up praxis. For praxis needs support – just because, in the long run, no human doing goes unquestioned. (Chevallard, 2006, p. 23).

As for mathematical praxeologies taught and learnt at university, it is obvious that *praxis* (e.g. deciding if some infinite series converges or not) is intimately connected to various forms of *logos*—from *ad hoc* explanations of standard techniques to theories involving general definitions, theorems and proofs. We note here that ATD offers more detailed tools to analyse praxeologies than the simple pairs of praxis and logos considered here, but we have deliberately limited the introduction of theoretical tools to what is strictly needed to present our discussion of Klein's Plan B in the context of Calculus and Analysis.

There is no shortage of logos in the typical Analysis course. The problem is more the opposite: pursuit of intricate logos without praxis—for instance, students are presented with extensive logos about the assumptions under which an infinite sum of continuous functions is again continuous, but not with serious praxis where such logos is needed. Naturally, almost all course teaching involves *some* praxis—but it is often exercises which provide a superficial experience of "applying" the theory to simple, concrete cases—for instance, to "show" that a given infinite sum of functions is continuous on a given interval. With that added praxis, one thus gets a *complete* praxeology, even if the completion is achieved in a minimal and artificial way.

To contrast such "artificial student praxeologies" (constructed for didactic purposes) with the praxeologies of present-day mathematicians, we shall indicate the first using Roman letters (P, L) and the last with Greek letters (Π, Λ). As a rough model of the current situation is, then, that the praxeologies taught and learnt in Calculus courses are of the form (Π_i, L_i) : the *praxis blocks,* including computational techniques, are identical to those used (for tasks of the same type) by professional mathematicians, while the logos blocks L_i are limited to informal explanations of a smaller collection of praxis blocks (such as the various techniques for determining whether a series is convergent or not). On the other hand, Analysis courses aim to teach the scientific form of logos blocks. The taught and learnt praxeologies in such courses are therefore of the form (P_i, Λ_i) where each Λ_i constitutes a logos block consistent with that of present-day mathematicians while the praxis blocks P_i are didactic "supplements" constructed to consolidate the acquisition of Λ_i. As mentioned in the introduction, such teaching practices often fail to motivate students for Λ_i and to provide them with a coherent, autonomous relationship with (Π_i, Λ_i). Part of our research focuses on how this issue can be addressed.

Taken together, Calculus and Analysis courses *in principle* could provide students with complete praxeologies (Π_i, Λ_i) that are close to the standards of present-day mathematics. For instance, convergence tests used in Calculus praxis on series are now supplied with a theory involving precise definitions and proofs of the "criteria" for convergence. However, because the number and technical complexity of these praxeologies is quite high, and the praxis blocks Π_i were taught in other courses, considerable effort and support might be needed for students to "assemble" isolated praxeologies (Π_i, Λ_i); auxiliary praxis blocks P_i are often introduced in attempts to achieve this. Working along these lines corresponds to establishing complete praxeologies which are mutually connected only within different small areas of mathematics, such as "infinite series". This is indeed "Plan A". By contrast, in Plan B, the connections are much more extensive and go beyond the minimal needs for logical coherence and praxeological completeness.

In the two cases below, we tread beyond assembling isolated praxeologies. Namely, we aim to investigate the potential links which could be formed by students between logos blocks Λ_i from Analysis and seemingly distant praxis blocks Π_j, known to them from Calculus, or topics that they learnt at earlier educational levels (such as trigonometry and basic plane geometry). Our overall strategy for realizing Klein's Plan B is thus to design and observe situations in which students experience how the new logos block of Analysis build on—or, conversely inform—praxis blocks which are (or should be) familiar to them.

Case 1: Vector Analysis theorizes pre-Calculus and Calculus

In Calculus courses, students have built extensive praxis blocks Π_i based on standard tasks with functions in closed form. In this section, we explore the special case of trigonometric functions: what students supposedly learn about them up to (and including) Calculus, and how one can revisit this topic, along with the more elementary notion of angle measure, in a first course on rigorous Vector Analysis.

Angles and trigonometric functions in pre-Calculus

It is well known that the functions sine and cosine are often introduced progressively, in three distinct and quite different contexts (e.g. Demir & Heck, 2013, p. 120). To save breath we speak only of sine, given that everything is similar for cosine:

- First, sine appears as a "tool" for solving triangle problems in plane geometry. It is defined as a certain ratio of sides in a right triangle, accompanied with something like Figure 1; notice how the ratio is ascribed, through the notation, to an angle.

Subsequently, calculators are used to compute this "angle-related number" and occasionally also to find an angle with a given value of sine.

- Then, in the setting of analytic geometry, sine is defined as the second coordinate of the intersection of a ray through the unit circle, accompanied with something like Figure 2; again, we notice the link to an angle, indicated in the figure.
- Finally, sine emerges as a function through tables and graphs like Figure 3, with a discussion of function properties such as domain, range, zeros, period etc. Here, there is no explicit link to angles and in fact, the passage to functions defined on all of $\mathbb{R}$ probably loosens that link for most students.

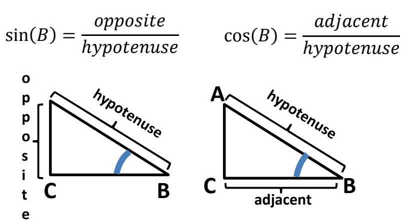

Figure 1: Triangle context.

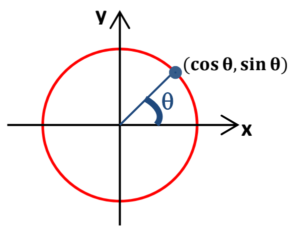

Figure 2: Cartesian context.

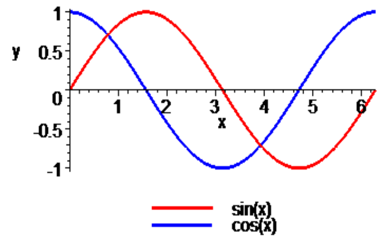

Figure 3: Function context.

Viewed separately, the three contexts can be pursued following Plan A, with different praxis and logos. But in most secondary schools, some efforts are being made to relate both praxis and logos from the three contexts, elements of plan B can be seen in such efforts. Figures 1-3 are really used to support an informal logos block L_{AT} on "angles and trigonometry", to explain and somehow justify a similar (adequate in a scholarly sense) set of practices Π_{AT} for solving tasks related to angles and trigonometry. The postulate character of the graphs in Figure 3 is the main obstacle addressed by Demir and Heck (2013):

> The sine and cosine functions may have been defined, but the graphs of these real functions remain mysterious or merely diagrams produced by a graphing calculator or mathematics software. The complex nature of trigonometry makes it challenging for students to understand the topic deeply and conceptually. (Demir & Heck, 2013, p. 119)

While we do agree with the authors that difficulties remain, a "deep understanding" would need to question that "the sine and cosine functions may have been defined" by the various explanations provided by L_{AT}. In fact, they ultimately rely on the informal notion of *angle* which students met already in primary school: a numerical measure of the space between two crossing line segments (for instance, sides in a triangle). The absence of a firm definition of angle may cause misinterpretations and further student confusion regarding sine and cosine functions. There is a considerable literature on how secondary level students, their teachers, and even university students struggle with making sense of the informal definitions outlined above (see Weber, 2005, and references therein). Naturally, analytic definitions of the functions may be given e.g. in terms of power series, but this is usually unrelated to the previous informal definitions.

Among the mysterious operations which usually accompany the passage from the triangle context to the Cartesian context is the unmotivated change of unit for this measure, as degrees are replaced by radians. But the informal definition of radians gives a clue to solving the mystery of what angle measures are, namely: the length of a specific segment of the unit circle. We now outline how the foundations of angles and trigonometry could be approached, at university, using more advanced elements from Analysis—an approach which seems so far untouched by the mathematics education research literature.

Plan B based on Vector Analysis

In the standard formal approach to curve integrals, the notions of *rectifiable curve* and *natural parametrization* are quite central. We take as available logos for our "Plan B" some main definitions and theorems from a Danish textbook (Eilers, Hansen, & Madsen, 2015), written for a second semester course on Real and Vector Analysis at the University of Copenhagen.

First some notation is introduced: for a continuous curve $\mathbf{r} : [a,b] \to \mathbb{R}^m$ and a partition D of $[a,b]$ consisting of points $a = t_0 < t_1 < \cdots < t_k = b$, we put

$$\ell(D) = \sum_{j=1}^{k} \left\| \mathbf{r}(t_j) - \mathbf{r}(t_{j-1}) \right\|.$$

A figure in the text illustrates how this measures the total length of line segments between points on the curve corresponding to the partition, and how this can be interpreted as a lower bound of what is intuitively the "length" of the curve. Then follows a precise definition of curve length, ultimately in terms of the usual distance in $\mathbb{R}^m$:

> **Definition 7.14.** For a continuous curve $\mathbf{r} : [a,b] \to \mathbb{R}^m$ on a bounded, closed parameter interval $[a,b]$, the curve (or arc) length is given as
>
> $$\ell = \sup\{\ell(D) \mid D \text{ is a finite partition of } [a,b]\}$$
>
> If $\ell < \infty$, the curve is said to be *rectifiable* (ibid., p. 223).

It is then proved that if $\mathbf{r}$ is piecewise C^1, then it is rectifiable, with $\ell = \int_a^b \|\mathbf{r}'(t)\|\, dt$. Under the further assumption that $\mathbf{r}$ is smooth (i.e. $\mathbf{r}'(t) \neq \mathbf{0}$ for all t), it is proved that there is an interval $[c,d]$ and a strictly increasing C^1-function $\varphi : [c,d] \to [a,b]$ such that $\tilde{\mathbf{r}} = \mathbf{r} \circ \varphi$ is a *natural* parametrization defined on $[c,d]$, meaning that $\int_u^v \|\tilde{\mathbf{r}}'(t)\|\, dt = v - u$ whenever $c \leq u \leq v \leq d$. In particular, $d - c = \ell$ where ℓ is the curve length of $\mathbf{r}$. In words: there is a reparametrization of $\mathbf{r}$, such that the length of any curve segment is simply the distance (in $\mathbb{R}$) between the corresponding parameter values. Together, the above definitions and results form a logos block Λ_{CL} on curve length, which is subsequently extended with definitions and results on curve integrals.

The text offers three "examples" (praxis) for Λ_{CL}. One of these shows how to piece together two function graphs to create a smooth C^1-parametrization $\mathbf{r}$ of $S^1(\|\mathbf{r}\| = 1)$, traversing S^1 once from (1,0) to (1,0) in the positive direction, one can use the above to obtain a *natural* reparametrization $\tilde{\mathbf{r}}$ of $\mathbf{r}$. No explanation is given on why this example does not start with the parametrization of S^1 which is most familiar to students: $(\cos t, \sin t)$, $t \in [0,2\pi]$. It is also not mentioned that $\tilde{\mathbf{r}}$ has finite length, that because $\tilde{\mathbf{r}}$ is natural, one might *define* angle measures using the parameter values, or that the coordinate functions of $\tilde{\mathbf{r}}$ may then be used to define cosine and sine on these angles. All of this can be extracted from an appendix in the book on "angle maps", drawing on Chapter 7, but that part of the appendix is not covered in the course. Together, these elements of praxis (Π_{CL}) suffice, in principle, to create a formal logos block Λ_{AT} for the praxis on angles and trigonometry which is already familiar to the students.

The second author observed (and co-developed, without co-teaching) a course based on this textbook in the spring of 2015. Indeed, the overall emphasis of the course followed a Plan A, laying out theoretical foundations for Real and Vector Analysis, with a strong emphasis on definitions, theorems and proofs. The above material from Chapter 7 (and more) was covered in one lecture during the sixth week. No exercises for students addressed the problem of defining angles directly. However, the students did one exercise on the map $\tilde{\mathbf{r}}$ above, which asked them to show that if we name the coordinate functions $\tilde{\mathbf{r}} = (C, S)$, we can deduce that $\tilde{\mathbf{r}}' = \pm(-S, C)$. Indeed, this follows from:

$$\|\tilde{\mathbf{r}}\| = 1 \Longrightarrow 2CC' + 2SS' = 0 \text{ and } \int_0^T \|\tilde{\mathbf{r}}'(t)\|\, dt = T \Longrightarrow {C'}^2 + {S'}^2 = 1. \ (*)$$

So, the book and its material hold potential for a complementary Plan B to explain and justify the "old" praxis blocks (Π_{AT}) on angles and trigonometric functions by incorporating the new logos block Λ_{CL} from Chapter 7. It links specifically to the logos block corresponding to Fig. 2 above, showing how to define the angle as the inverse of a natural parametrization, and cosine and sine as the coordinate functions of this parametrization.

An informal trial

To gauge if the relations between trigonometry and natural parametrizations had been established by students in the course considered above, we interviewed a masters level student (advanced student, thereafter AS) who had served as a teaching assistant in this course twice. We expected that this student's knowledge would be an upper bound of the new foundations of angles and trigonometry that students in the course could have constructed from the theory surrounding natural parametrizations and the exercises mentioned above. The interview was semi-structured and based on the following questions:

- What is your favorite definition of the sine function?
    - Follow-up questions according to the definition chosen, leading to:
- What is your favorite definition of an angle? How does it relate to sine?

- o Follow-up questions for instance on arc length, if referring to circle arc.
- What mathematical resources does the course (described above) provide to elucidate the previous questions? (textbook at hand, to look up details).

Notice that the tasks were formulated in a very open way, as the intention was to examine the praxeological links which the course in question had led the students to establish.

The interview was audio recorded and transcribed. Below we just outline the main exchanges pertaining to the above points, to conclude with some observations relating to our overall questions.

AS begins answering the first question by tracing the graph of sine (Fig. 4), then enumerates “a lot of properties” of the function (Fig. 5), all from Calculus. AS realizes that they are “certainly not a definition”. The interviewer (IN) insists on AS giving one definition. AS then gives the Cartesian description from pre-Calculus (Fig. 6). After slight confusion, AS identifies x (in Fig. 6) as the appropriate angle in the triangle.

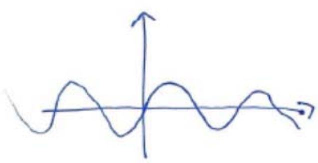

Figure 4: A graph.

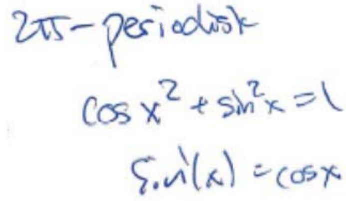

Figure 5: Properties.

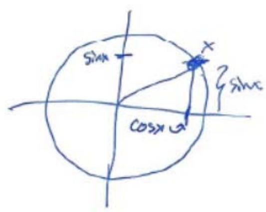

Figure 6: Diagram.

Then follows the dialogue below (some redundancies are left out, marked by //):

> IN: Then, the question arises, the angle, what mathematical object is that? // Can you give like a mathematical definition of that?
>
> AS: It determines like how far two lines are from each other. // [AS draws two crossing lines and says there are four angles, two different, and IN repeats the previous question].
>
> AS: In the old days, you used your compass and your protractor, and then later when you have to compute them, there were, you had these smart things where you get hold of cosine and sine to compute them, the angles…
>
> IN: Yes, but you have just used angles to define sine and cosine…
>
> AS: Oh yes, precisely, then it comes backwards again, so that it not so good…
>
> IN: So, my thousand dollar question, could [*name of the course*] help us with that?

So far, AS has only reproduced logos from pre-university mathematics, essentially corresponding to Figures 2 and 3 above, together with one formula from Calculus. Explicitly asked to draw on the Analysis course, AS immediately recalls this “fantastic exercise” where you had two functions, and it turns out they are cosine and sine. After a few minutes, AS finds it in the section on curve integrals; the exercise begins with “a natural parametrization”

$\mathbf{r}(t) = (C(t), S(t))$ of the unit circle, and calls for (*) to prove that $(C'(t), S'(t)) = \pm(-S(t), C(t))$. However, AS does not notice, at first, the assumption of $\mathbf{r}$ being *natural*, and when IN points it out and asks what it means, AS looks it up in the index of the book. This leads her to the definition mentioned above (Eilers et al., 2015, p. 226). AS reads the definition for a while, and then says:

> AS: the parameter values, they should, if we subtract them from each other // the curve length should be like one minus the other // the curve cannot make like strange crossings, that must create a mess, I think…
>
> IN: // you get such a length preserving map from an interval onto the curve. How could that help us with the question about sine, cosine and angles?
>
> AS: That's a good question.

IN points to the appendix on trigonometric functions. AS recalls that they did do an exercise from there but "otherwise we did not look at it". AS does not recall the part on the "angle map" γ and has no idea where γ comes from. IN points out the reference to the main text and the explicit mention that γ is a natural parametrization. AS returns to the main text and looks at Definition 7.14. After turning a few pages for a bit, AS finds the theorem on the existence of natural parametrizations of smooth C^1-curves. IN asks if AS could verify the conditions for the circle. Supported by a hint ("could you parametrize the circle without using sine and cosine?"), AS comes up with the parametrization $(\sqrt{1-t^2}, t)$, for the circle in the first quadrant. AS says this can be differentiated many times, so it is C^1. AS does not recall the definition of "smooth", but quickly looks it up, and then verifies it for the parametrization above. Going back to the theorem, AS concludes that then we can construct the angle map.

> IN: Using that, can you give a precise definition of what an angle is?
>
> AS: Not immediately…
>
> IN: What should it be, if you look at the unit circle? //
>
> AS: It has something to do with the arc length //

AS tries to find out what the arc length is for γ and writes down the formula $\int_0^x \|\gamma'(t)\| dt$. As seen above, the meaning of "natural parametrization" is not familiar to AS. AS also seems confused that the values of γ are clearly not angles (but points), in spite of the name "angle map". After some dead-end circling around these matters, IN asks AS if we could make "a function from points on the circle to arc length". AS suggests that $\gamma(t)$ should be mapped to t in some sense. As this is very close to a satisfactory answer and as the agreed time is almost up, IN briefly shows how to formalize this last point, and wraps up the conversation.

The above conversation indicates that the angle question is likely to be challenging for many students as answering it requires more than what the course offers: meeting Λ_{CL} along with some "examples" which, together, suffice in principle to create Λ_{AT}. Indeed, AS initially repeats the informal logos L_{AT} and has evidently not realized how Λ_{CL} allows one to replace it with Λ_{AT} (passing through Π_{CL}). We conclude that a more explicit and detailed work with Π_{CL} is needed than what was offered in the course.

Case 2: Calculus praxis supporting Fourier Analysis theory

In Fourier Analysis, trigonometric functions play the role of “basis elements” which can be used to “build” all common functions on an interval. Our second case is about the fundamental result underlying that theory, and the extent to which students can relate it to Calculus. Namely, we consider the important case of Fourier series, defined for a 2π-periodic, piecewise continuous function $f : \mathbb{R} \to \mathbb{C}$, as

$$F[f](x) \stackrel{\text{def}}{=} \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 0,1,2 \ldots.$$

In general the two infinite series may not converge at a point x. In 1829, Dirichlet gave one of the first sufficient conditions for pointwise convergence of a Fourier series. Following common practice, we refer to the statement below as Dirichlet’s theorem, although we don’t use his original formulation.

Theorem. If $f : \mathbb{R} \to \mathbb{C}$ is a continuous 2π-periodic function with piecewise continuous derivative, the Fourier series of f is pointwise convergent to $f(x)$ at every $x \in \mathbb{R}$.

When students meet Dirichlet’s theorem in an Analysis course, they are often given a more general version, with a proof based on Hilbert space techniques. After that, applications that are intended as simple are introduced in examples and exercises. For instance, students may be asked to compute the Fourier series of $f(x) = x^2$, extended periodically from $[-\pi, \pi]$ to $\mathbb{R}$, and then use Dirichlet’s theorem to conclude that the Fourier series of f converges to f for all $x \in \mathbb{R}$. They may also be asked to develop the case $x = 0$:

$$0 = f(0) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \text{so that} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}. \qquad (**)$$

In this approach the praxis block (P_{DT} : computation of the Fourier series for a given function and use of Dirichlet’s theorem on such a case) does not support the logos block Λ_{DT} involved in the general proof of the theorem; it just applies the statement, which itself could have come out of nowhere (and does in many courses).

Plan B based on Calculus

Note that the value of the infinite sum (**) obtained above could be derived by other means, as a variant of the famous Basel problem (see e.g. Shoenthal, 2014 and Kondratieva, 2016). One such means is at the root of the design presented below. The aim of our design is to highlight the fact that the general proof (Λ_{DT}) is essentially linked to familiar praxis blocks from Calculus. Thus, we propose that before meeting Λ_{DT} and P_{DT} students could work with an activity which we now describe.

Part 1 of the activity begins with posing the problem of determining the value of $S \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. The praxis blocks acquired in Calculus courses do not provide ready-made techniques to solve this problem. Instead, students are invited to do so through several preliminary problems. Below, in addition to stating the problems for students, we note and name the praxis blocks from Calculus which are needed to solve these problems:

1.1 Compute the integral $\int_{-\pi}^{\pi} x^2 \cos mx\, dx$ for any natural number m (Π_1 : integration rules).
1.2 Show that $\frac{1}{2} + \sum_{n=1}^{m} \cos nx = \frac{\sin(m+1/2)x}{2\sin(x/2)}$ (Π_2 : trigonometric formulae).
1.3 Show that $u(x) \stackrel{\text{def}}{=} \begin{cases} x^2/\sin x & x \neq 0 \\ 0 & x = 0 \end{cases}$ defines a continuous function on $[0, \pi/2]$.
(Π_3 : techniques to compute limits, including the special result $\lim_{x\to 0} x^{-1} \sin x = 1$).
1.4 Find u' and show that this function is bounded on $[0, \pi/2]$
(Π_3, and Π_4: differentiation from first principles).
1.5 Show $\int_{-\pi}^{\pi} x^2 \frac{\sin(m+1/2)x}{2\sin x/2} dx = 8\int_0^{\pi/2} y^2 \frac{\sin(2m+1)y}{\sin y} dy = 8\int_0^{\pi/2} \frac{\cos(2m+1)y}{2m+1} u'(y)\, dy$ (Π_1).
1.6 Show that the integral in 1.5 converges to 0 as $m \to \infty$ (Π_1 and Π_3).
1.7 Finally, combine the results above to find S (Π_1).

In *Part 2* of the activity, students are asked to solve another couple of tasks, which essentially present a simple proof of Dirichlet's theorem in the special case $f(x) = x^2$. That proof runs as follows:

I. First, rewrite the Nth partial Fourier sum, given by $F_N[f](x) \stackrel{\text{def}}{=} \frac{1}{2}a_0 + \sum_{n=1}^{N} a_n \cos nx + \sum_{n=1}^{N} b_n \sin nx$, as: $F_N[f](x) = \int_{-\pi}^{\pi} f(x+y) K_N(y) dy$, where the *Dirichlet kernel* is defined as $K_N(x) \stackrel{\text{def}}{=} \frac{1}{\pi}(\frac{1}{2} + \sum_{n=1}^{N} \cos nx) = \frac{1}{\pi}\frac{\sin(N+1/2)x}{2\sin x/2}$, and $\int_{-\pi}^{\pi} K_N(x) dx = 1$.

II. Show that for all x the "tail" of the Fourier series $\Delta_N(x) \stackrel{\text{def}}{=} F_N[f](x) - f(x) = \int_{-\pi}^{\pi} [f(x+y) - f(x)] K_N(y) dy$ tends to zero as N approaches infinity.

Accomplishing I and II in the specific case $f(x) = x^2$ is very similar to tasks 1.2 -1.6 from part 1 described above. The last exercises of Part 2 ask students to construct the Fourier series for $f(x) = x^2$ and use its convergence in order to find the sum S as in (**). Through a final reflection, the students are expected to realize that the proof (from Λ_{DT}) amounts to little more than a generalization of the sequence of Calculus techniques drawn upon in part 1. Indeed, the proof Λ_{DT} has both a "strategy" and an "implementation" aspect. The strategy follows from the notion of series convergence: whenever the partial sum s_N of an infinite series s_∞ can be written as $s_N = \phi + \Delta_N$, where ϕ is a number and the "tail" $\Delta_N = s_N - \phi$ vanishes as $N \to \infty$, we say that s_∞ converges to ϕ. The pointwise convergence of Fourier series is a very straightforward generalization of the notion of a number series convergence, with $s_N = F_N[f](x)$ and $\phi = f(x)$ because at each point x a Fourier series is a number series. The implementation of the strategy in the proof Λ_{DT} may be reduced to Calculus-like computations in some very special cases.

Our design takes advantage of considering one such case. When students understand the main strategy and the structure of the proof in the simple case, they have a better chance to incorporate the elements needed to achieve greater generality. In the general proof of Dirichlet's theorem, step II above cannot be done by direct computation. Instead, a more sophisticated argument is used (Folland, 1992, 45-46): one shows that $\Delta_N(x)$ can be rewritten as the sum of Nth Fourier coefficients of two functions, each of which is square integrable and 2π-periodic. It follows from a separate result (Bessel's inequality) that the Fourier coefficients a_n and b_n tend to zero as n tends to infinity (ibid., 30-31). Thus, it is concluded that $\Delta_N(x) \to 0$ for $N \to \infty$. This general argument, together with certain technicalities related to the possible non-continuity of f, seems to give students the impression that the proof is way beyond simple techniques from Calculus. Our hope is that the activity described above establishes a strong relation $\Lambda_{DT} \leftrightarrow \bigcup_{k=1}^{4} \Pi_k$ between logos and praxes.

Informal trials

The above design was trialled with five students interested and capable in mathematics who have completed at least 3 years of the undergraduate program at Memorial University of Newfoundland. They were asked to do parts 1 and 2 of the activity (with no firm restrictions in time or access to any materials), followed by 30-60 minutes interviews that were recorded and outlined below, as in Case 1. The interviews were semi-structured, based on the following questions:

- Did you find the individual problems from part 1 familiar/accessible/engaging? Did you know the sum of this infinite series before? Which methods (if any) to prove this result are known to you?
- Do you see any similarities between problems in part 1 and part 2? Do you know a statement about Fourier series convergence and its proof? Do you think that solving problems from part 2 helps to understand this result better?
- What is your overall experience in taking Calculus and Analysis courses? Do you see if and when an activity similar the above could be helpful in making connections between the subjects?

All students said that the tasks 1.1-1.6 were relying on familiar techniques from their Calculus courses, but that they had not been combined in the way required in task 1.7. They found this combination engaging and thought that the "entire part 1 would be accessible and desirable for a Calculus student who is interested in mathematics". Here is a student's comment on problem 1.7:

> As I was going through I was fairly confident that I was going to derive the result. From Problem 1.1 I know this (points to line 1 in Fig. 7). So I just rearranged to isolate the value that

is in the series. As soon as I got that (points to line 2 in Fig. 7) I knew I am on the right track. Then I took the finite sum of both sides and integrated. Then I knew from 1.2 that the sum of cosines and a half equals to that ratio of sinuses (line 6 in Fig. 7).... Then I took the limit and got this value here [points to $\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{x^2}{2}dx$], and that other term is zero from 1.6."

Here we see that the student does exactly what we were hoping for: he considers a partial sum of the series (line 3 in Fig. 7) and rewrites it as a sum of a number and a "tail" (last line in Fig. 7), where the "tail" vanishes in the limit $N \to \infty$. Note that the design of the problems helps to reinforce this approach to series convergence. For instance, problem 1.2 deals with a finite sum related to the partial sum of the series; problem 1.6 gives the necessary property of the series' "tail". When it comes to part 2, students see not only calculations similar to part 1, but they also realise that the concept of convergence of Fourier series could be treated in a similar fashion. According to the same student's comments on part 2, "task 1.2 is used in the step I again, while the result of task 1.6 is directly related to step II" (in the special case $f(x) = x^2$). When asked for direct analogy between part 1 and part 2, another student wrote:

In part 1, $\sum_{n=1}^{m}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} + \Delta_m$; in part 2, $F_m[x^2] = x^2 + \Delta_m(x)$; in both cases $\Delta_m \to 0$ as $m \to \infty$.

This passage indicates the student' grasp of the structure of the proofs and of their similarity in the two cases.

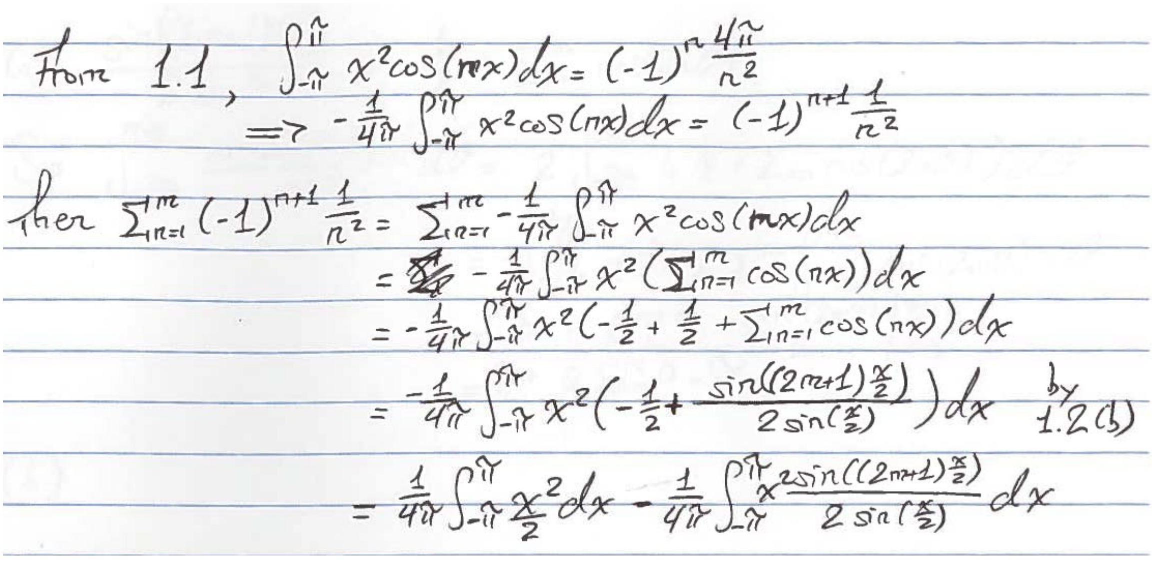

Figure 7. A fragment of a student's solution of 1.7.

However, these students never studied a formal proof of Dirichlet's theorem (only its statement and applications). So, we also interviewed a more advanced student (AS2) who had recently studied Dirichlet's theorem and its proof. AS2 actually knew the value of the sum $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ before doing the activity, and AS2 believes to have seen a derivation in Analysis using "high-powered stuff from Analysis, not using these like (…) lower level Calculus tools", and he thinks, “this was an interesting alternative”. Later, we ask AS2 about the activity described above, and just done by AS2:

> IN: Based on your experience, do you think this project might be accessible at the earlier stages or…?
>
> AS2: (…) it could be a good project but I think it would not be a quick one (…) The whole thing is doable, sure. Once you get into some of the more Analysis stuff… I mean I don’t know, I just know when I walked into Analysis on my first day, I was like: what the heck is going on? [laughing] It took me about half a semester to really understand what the heck was going on in Analysis, personally… so maybe some more examples (…) I had a hard time like grasping how to like prove (…)

AS2 thinks that the activity could be useful in the beginning of Real Analysis, even if more time consuming to students at that level than it was to him:

> AS2: This would be a great project in the beginning of an Analysis course. (…) In particular, I think, if you’re at a university that was big enough to have a Fourier Analysis course, I think this would be a great, like, first assignment. (…) there can be a lot of physicists (…) who have never really seen proof based mathematics (…) so I think this would be a good bridge for something like that for sure.

In the experience of AS2, assignments in Analysis classes are altogether of a different nature:

> AS2: The assignments would be in abstract, typically, so like prove that this class of functions does this thing on this interval (…) the exams would mostly take the general things we proved in class and apply it to more specific functions.

Thus, while theory is covered extensively in classes, the practices ultimately engaged in by students are typically much simpler (applications of general theorems to specific cases, or sometimes construction of short proofs). Part 2 alone is by itself similar to such tasks. When asked about the connection between Part 1 and Part 2, AS2 explains:

> AS2: A lot of the stuff I did in part 1, was stuff I would have to play around with to really prove this limit (points to the expression for $\Delta_N(x) \to 0$ as $\to \infty$).

Here again we have evidence that the student articulates the main strategy of the proof in part 2 and refers to concrete results from part 1 to fulfil this strategy. The complete 50-minute conversation with the more advanced student backs, in these and other ways, our hypothesis that the transition from Calculus to Analysis is not adequately supported by the typical first Analysis courses, and that activities such as the one trialled here can possibly fill a gap. Indeed, helping students to identify shared Calculus-Analysis logos blocks, such as the concept of series convergence, makes the proof of Dirichlet theorem accessible (in the special case) to students possessing praxis skills learned in Calculus. For them, concrete calculations do appear as “technical details” in the general proof. Clearly, Analysis instructors want their students to be able to work also with general arguments which do not rely on Calculus type calculations. So, at some point, students need to be shown the advantages and realize the power of general argumentation in Analysis-level proofs, but there is no need to present it as a completely different kind of mathematical reasoning.

> IN: Looking back at your undergraduate education (…) did it come together in the end?
>
> AS2 : I think in the early courses it (…) was much more compartmentalized. But once I got to the higher level classes, like, they were constantly talking to each other, for example(…) you can prove some results in number theory, using abstract algebra, and (…) like Analysis, it's all Topology (…) maybe not, like, by the explicit efforts of the professors, but, just, me personally, like, I noticed a lot of deeper connections between the courses.

Thus, successful students in current undergraduate mathematics programs may indeed end up with some experience close to Klein's Plan B, basically by their own informal efforts. Such experiences may be fuelled in particularly by establishing explicit links between different solutions to the same *interconnecting* problem (Kondratieva, 2011), as it was the case with the activity we trialled.

Conclusions

While Calculus courses include praxis blocks Π_i compatible with those of professional mathematicians, their theoretical components are more informal and focused on algebraic computation rules. Moreover, the praxis blocks are often isolated from each other, as they occur within separate sections of textbooks and courses, and students typically are not offered opportunities to apply them in combinations. When students move towards Analysis courses, they experience a sharp focus on theoretical blocks Λ_i. While these are certainly related to the praxis blocks Π_i within the field of professional mathematics, the connections $\Pi_i \leftrightarrow \Lambda_i$ between praxis and logos are not necessarily obvious for the learners, and thus need to be explicitly established (let alone less evident "cross cutting relations" of the form $\Pi_i \leftrightarrow \Lambda_j$).

In this paper, we considered two cases where one could help students to establish such connections. In the first case, the fundamental notion of angle and its measure (in trigonometry) can be revisited from the viewpoint of the general construct of curve length and natural parametrization of a curve in $\mathbb{R}^m$. Judged by the interview, students did not succeed in doing so based on the activities proposed in the Analysis course. Thus, we propose to redevelop the praxis block Π_{CL} on curve length (informed by the logos block Λ_{CL}) in order to recover a direct link and to support the theory Λ_{AT} related to the notion of angle. In this way, a relation $\Pi_{CL} \leftrightarrow \Lambda_{AT}$ could be established (see Fig. 8, right).

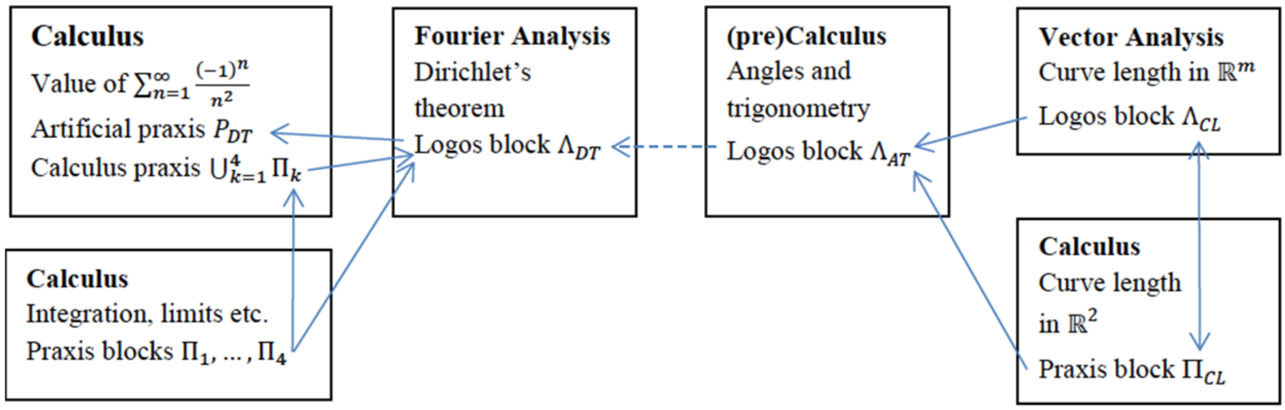

Figure 8: Relations between praxis and logos blocks established in two cases.

In the second case, it is an explicit derivation of the value for an infinite series in Calculus which is employed to emphasize how Calculus techniques are crucial in the proof of Dirichlet's theorem. Specifically, several praxis blocks from Calculus are combined to produce an "infinite series summation" block of praxes $\bigcup_{k=1}^{4}\Pi_k$, which is then related to the logos block Λ_{DT} via a proof of this theorem in the special case $f(x) = x^2$. This way, a relation $\Lambda_{DT} \leftrightarrow \bigcup_{k=1}^{4}\Pi_k$ is established (see Fig. 8, left), not only as a fingertip "application" of the result ($\Lambda_{DT} \rightarrow P_{DT}$), but as a link between the *proof* and previous praxis blocks. According to our small scale trial, this project seems to be not only feasible but also enjoyable and beneficial for students equipped with Calculus techniques.

Finally, we emphasise that the goal of this paper is to call attention to the issue of knowledge *desynchretisation* in undergraduate mathematics in general, and to propose reviving Klein's idea of "Plan B" in order to provide early *resynchretization* experiences. We have illustrated this with situations which could enable students to reconstruct "cross cutting relations" $\Pi_i \leftrightarrow \Lambda_j$, as precise interpretations of Klein's "Plan B". At the same time, constructing integrated praxis blocks such as Π_{CL} or $\bigcup_{k=1}^{4}\Pi_k$ described above, constitutes an essential complement to "Plan A" type courses. The point of this paper is that the construction and implementations of situations which enable links of these types is a promising but demanding direction for enhancing the teaching of Calculus and Analysis.

To construct such situations clearly necessitates a careful analysis of (central) theory blocks of more advanced courses, and of resources found in praxis blocks of previous courses; small scale trials, such as the ones reported here, can clearly be helpful to support and correct the details of such constructions. An implementation of the situations in actual teaching naturally constitutes a second set of challenges which we have not addressed here. While doing so, one would—in most universities—have to confront the institutional mechanisms that have led to the desynchretisation of university mathematics curricula. Dealing with such challenges in teaching practice clearly goes beyond both Klein's idealistic vision for mathematics teaching, and the praxeological analysis provided in this paper.

Note

This paper draws on our contributions for the conferences INDRUM2016 (Montpellier, 2016) and CERME10 (Dublin, 2017), listed in the references.

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